# Beam Ellipses

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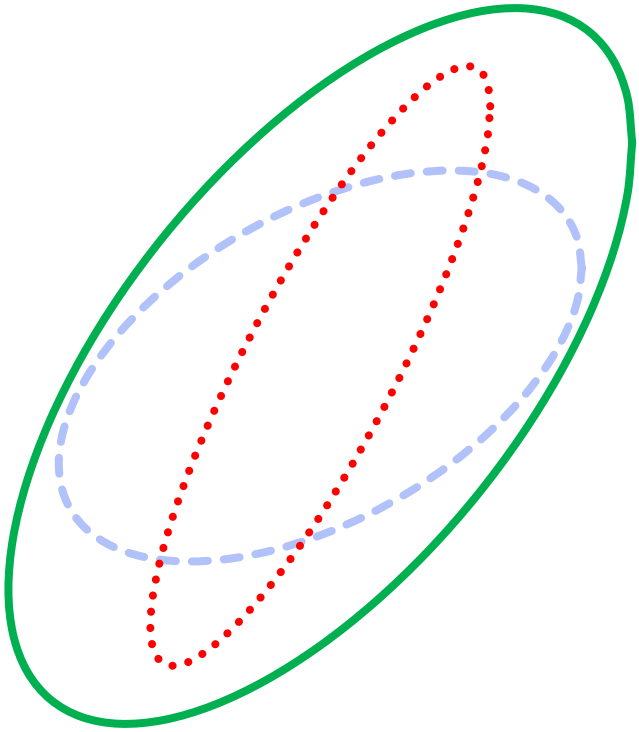


24.11.2013

# Introduction

In 1980 I wrote an internal SIN report on Beam Ellipses wit the titel

«Representation of Beam Ellipses for TRANSPORT calculations»

SIN-Report TM-11-14 , 8.5.1980

At that time our research institute was called SIN (for Swiss Institute for Nuclear Research), which in 1988 was integrated into the new PSI (for Paul Scherrer Institute).

Later a wrote an article called «Fun with Formulas».

I presented it (but did not published it) at a seminar talk given at the CERN Accelerator School on

Synchrotron Radiation and Free Electron Lasers,

Brunnen, Switzerland, 2-9 July 2003

In this article some foils are devoted to beam ellipses.

I think it is thus helpful, if I add these foils here as an addendum to my previous report on beam ellipses.

this file is available on the WEB with

www.google.ch: „Joho PSI Beam Ellipses“ or directly with

http://gfa.web.psi.ch/publications/presentations/WernerJoho/

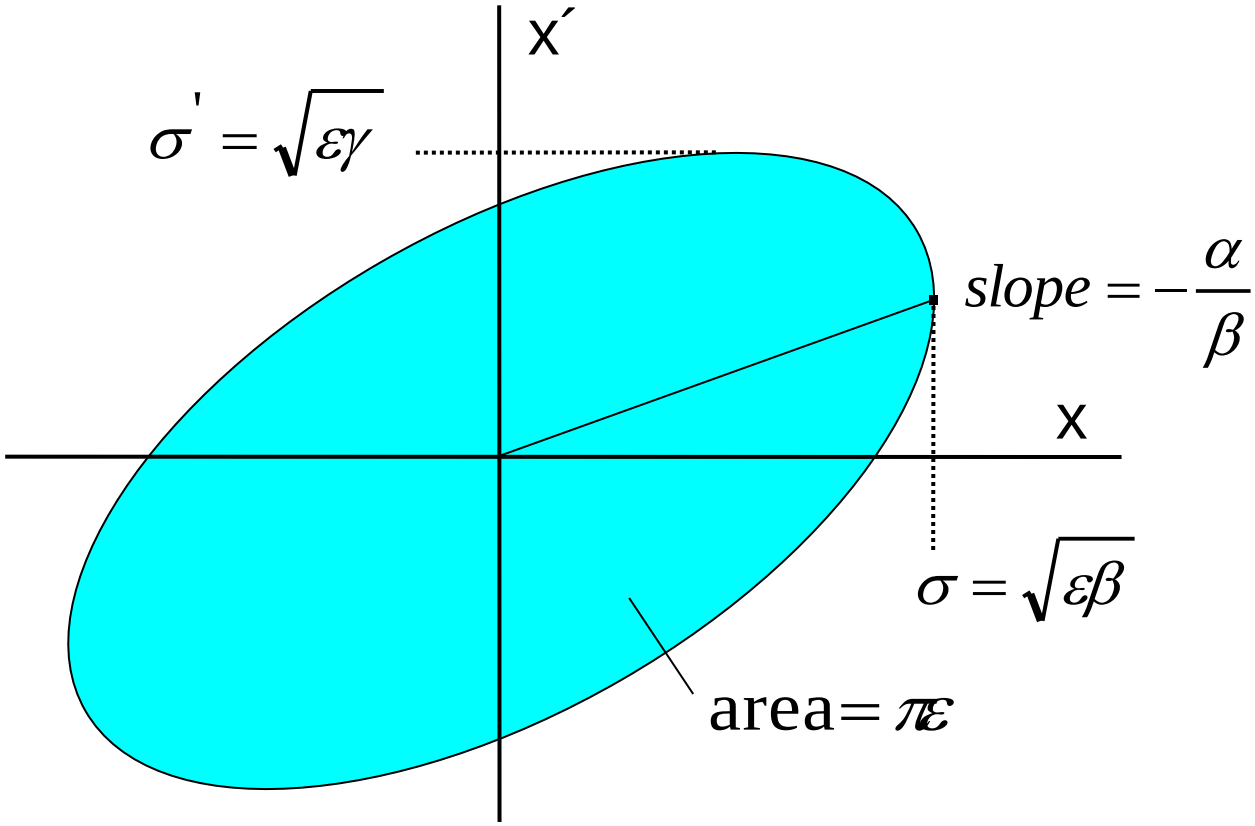
**Courant-Snyder** representation of the rms beam ellipse in phase space (x, x')

$$\varepsilon = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

$\varepsilon^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle x\, x' \rangle^2$ , ($\varepsilon$=emittance)

$\sigma^2 \equiv \langle x^2 \rangle$, $\sigma'^2 \equiv \langle x'^2 \rangle$

$\beta = \sigma^2/\varepsilon$ , $\gamma = \sigma'^2/\varepsilon$ , $\alpha = -\langle xx' \rangle/\varepsilon$ , $1+\alpha^2 = \beta\gamma$




not easy to plot and akward to remember !

# **parametric representation** of the rms beam ellipse in phase space (x, x')

$x = \sigma \cos \varphi$

$x' = \sigma' \sin (\varphi + \chi)$

($0 \leq \varphi \leq 2\pi$ = "running parameter'

but what is phase shift $\chi$ ?

$\sin \chi \equiv r_{12} \equiv \frac{\langle xx' \rangle}{\sigma\sigma'}$ = correlation parameter

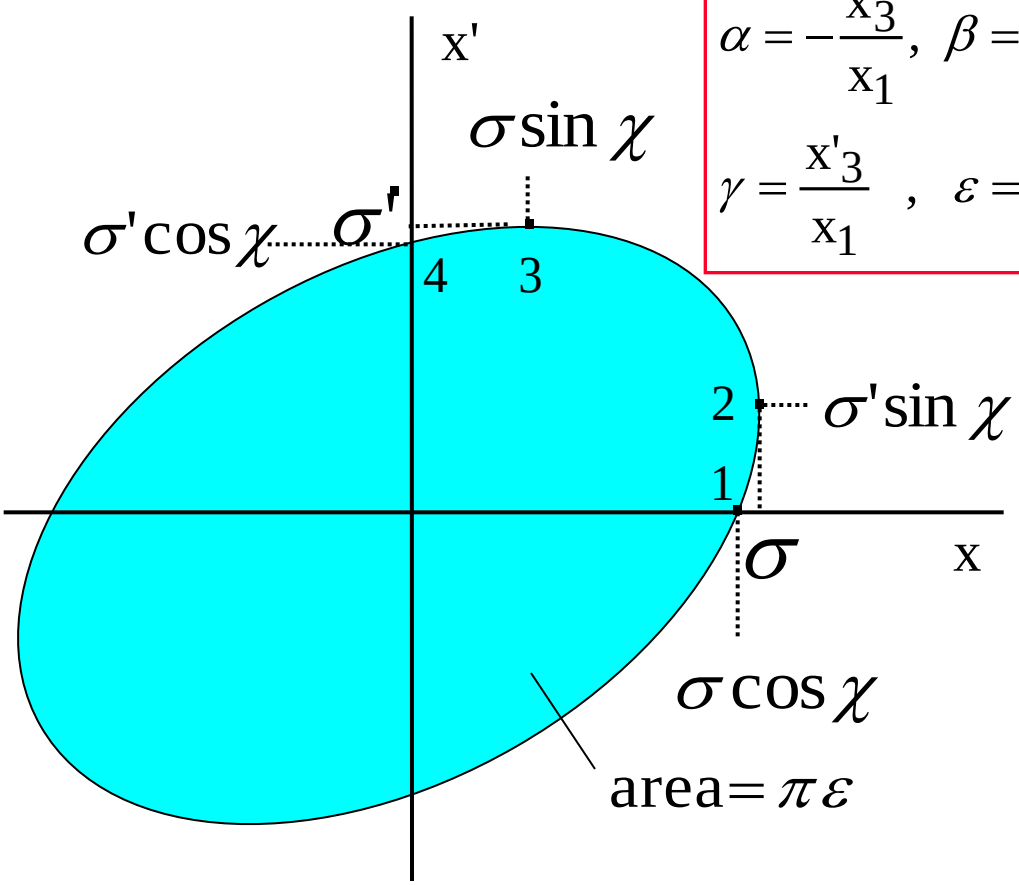
this representation of the rms-ellipse is easy to remember

and easy to plot! ($\chi = 0$ gives a circle)

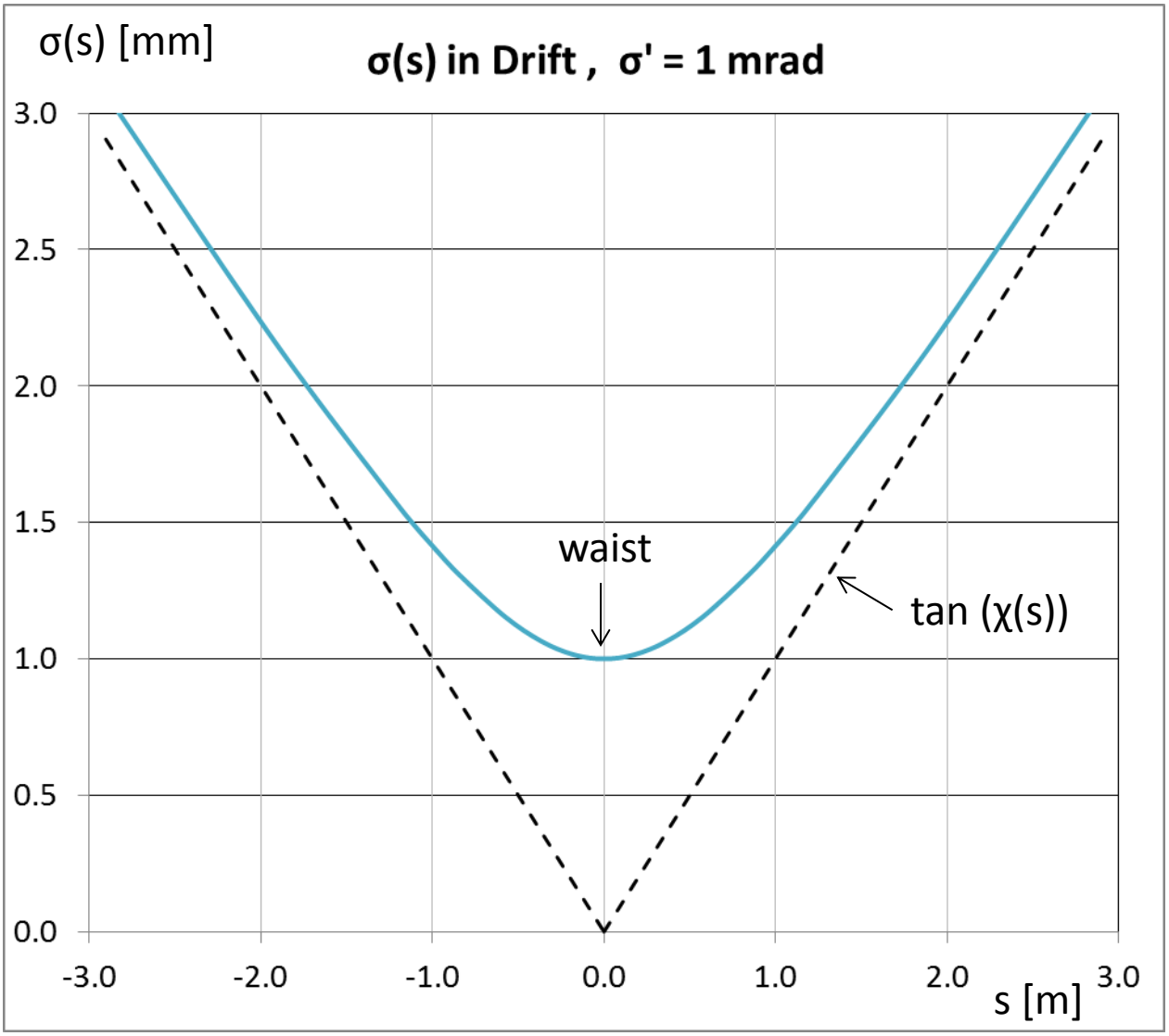
connection with Courant-Snyder: **α =** . . . . $\chi$

$$\alpha = -\frac{x_3}{x_1}, \quad \beta = \frac{x_2}{x'_4}$$

$$\gamma = \frac{x'_3}{x_1}, \quad \varepsilon = x_1 x'_3 = x_2 x'_4$$



# Beamsize in Drift




In waist at $s = 0$:

$$\sigma_0 = 1\text{mm}, \quad \sigma' = 1\text{mrad}, \quad \beta_0 = \frac{\sigma_0}{\sigma'} = 1m$$

emittance $\varepsilon = \sigma_0 \sigma' = 1\text{mm mrad}$

After drift s: $\tan\chi = \dfrac{s}{\beta_0} \quad (= -\alpha)$

$$\sigma(s) = \frac{\sigma_0}{\cos\chi}$$

$$\beta(s) = \sqrt{\varepsilon\sigma} \text{ or } \beta(s) = \beta_0 + \frac{s^2}{\beta_0}$$

# Dictionary for Beam Parameters

*ome useful quantities are easy to guess from the factors*

*sin$\chi$ or cos$\chi$ ($\chi$ is 0 at a waist!) and dimensional arguments*

*(using m, mm and mrad)*

emittance: $\varepsilon = \sigma\sigma' \cos\chi$ [mm mrad]

slope of envellope: $d\sigma/ds = \sigma' \sin\chi$ [mrad]

virtual waist size: $x_w = \sigma \cos\chi$ [mm]

$\beta$-function at virtual waist: $\beta_{min} = (\sigma/\sigma') \cos\chi$ [m]

distance from virtual waist: $L_w = (\sigma/\sigma') \sin\chi$ [m]

$= \beta_{min} \tan\chi$

phase advance from virtual waist: $\psi = \chi$

the dictionary between the 2 representations is:

$$\alpha = -\tan\chi = -\frac{\langle xx' \rangle}{\varepsilon} \quad (= -x_3/x_1 = -x_2'/x_4') \quad [1]$$

$$\beta = \frac{\sigma^2}{\varepsilon} = \frac{\sigma}{\sigma' \cos\chi} \quad (= x_2/x_4') \quad [m]$$

$$\gamma = \frac{\sigma'^2}{\varepsilon} = \frac{\sigma'}{\sigma \cos\chi} \quad (= x_3'/x_1) \quad [m^{-1}]$$

(as a check : $\beta\gamma = 1/\cos^2\chi = 1 + \tan^2\chi = 1 + \alpha^2$ )

# Convolution of two ellipses

Example: **convolution** of the electron beam ellipse ($x_1$, $x_1'$), with parameter $\sigma_1, \sigma_1', \chi_1$ and the diffraction limited photon beam ($x_2$, $x_2'$), with parameter $\sigma_2, \sigma_2', \chi_2$ from an undulator.

Simple recipe:

add variances and correlations linearly

to form the combined ellipse (X,X') with parameter $\Sigma$, $\Sigma'$, $\chi$

$$\langle X^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle$$

$$\langle X'^2 \rangle = \langle x_1'^2 \rangle + \langle x_2'^2 \rangle$$

$$\langle XX' \rangle = \langle x_1 x_1' \rangle + \langle x_2 x_2' \rangle$$

or

$$\Sigma^2 = \sigma_1^2 + \sigma_2^2$$

$$\Sigma'^2 = \sigma_1'^2 + \sigma_2'^2$$

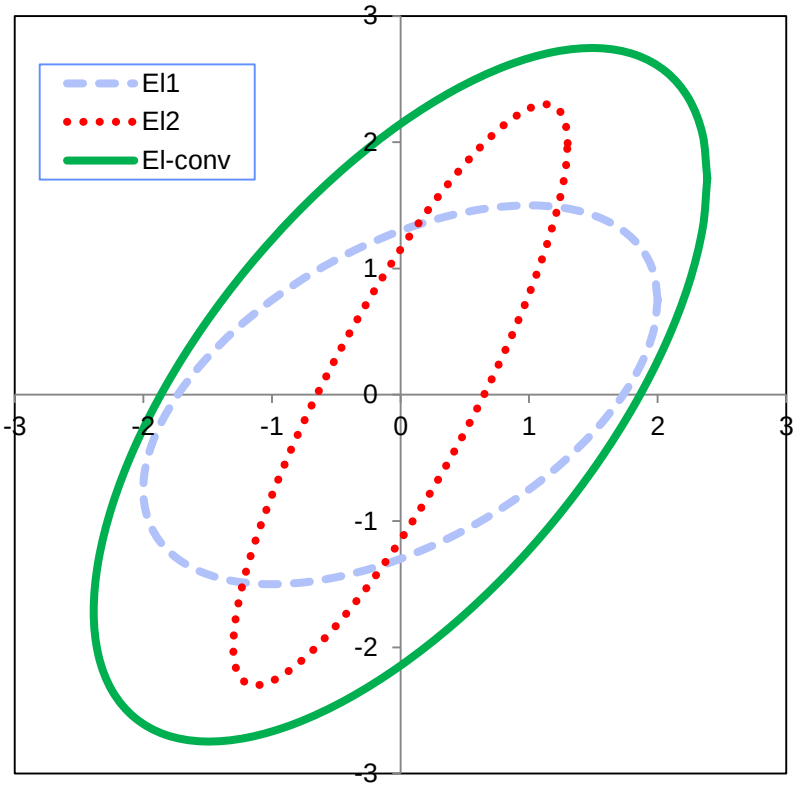
$$\Sigma\, \Sigma' \sin\chi = \sigma_1 \sigma_1' \sin\chi_1 + \sigma_2 \sigma_2' \sin\chi_2$$

the convoluted emittance is

$$\varepsilon = \Sigma\Sigma' \cos\chi \quad (\varepsilon \geq \varepsilon_1 + \varepsilon_2)$$

with the dictionary one can, if necessary, transform these values back to the Courand-Snyder values $\alpha$, $\beta$, $\gamma$ .

# Two Ellipses convoluted




Ellipses :

$$x = x_m \cos\varphi, \quad y = y_m \sin(\varphi + \chi), \quad (0 \le \varphi \le 2\pi)$$

$$\text{area} = \pi\varepsilon, \quad \varepsilon = x_m y_m \cos\chi$$

convolutedellipse :

$$x_m^2 = x_{m1}^2 + x_{m2}^2$$

$$y_m^2 = y_{m1}^2 + y_{m2}^2$$

$$x_m y_m \sin\chi = x_{m1} y_{m1} \sin\chi_1 + x_{m2} y_{m2} \sin\chi_2$$

| | Xm | Ym | χ | ε |
|---|---|---|---|---|
| Ellipse 1 | 2.0 | 1.5 | $30^0$ | 2.60 |
| Ellipse 2 | 1.3 | 2.3 | $60^0$ | 1.49 |
| El-conv | 2.39 | 2.75 | $38.6^0$ | 5.12 |

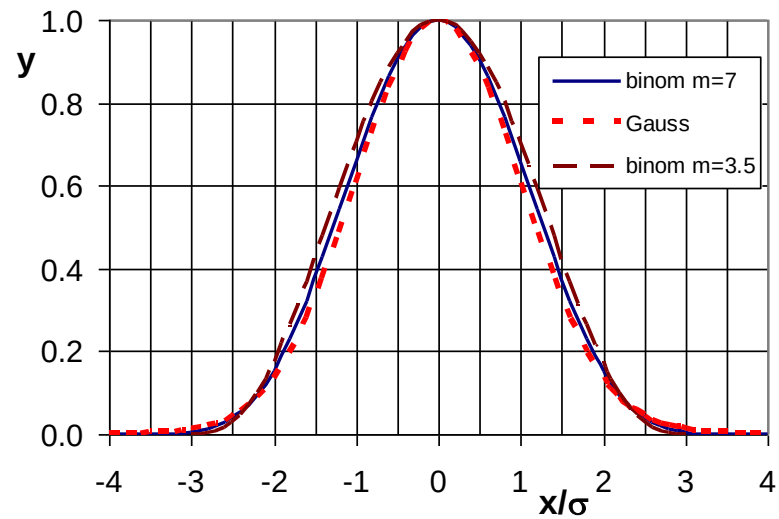
# representations of beam profiles with binomials

1) Gaussian: $y = e^{-1/2(x/\sigma)^2}$

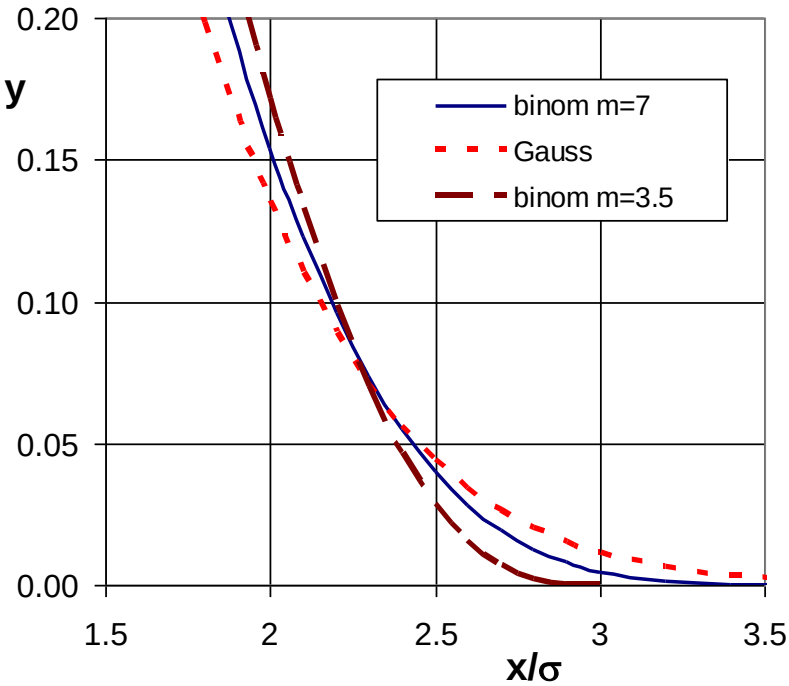
2) Binomials :

$$y = [(1-(\frac{x}{x_L})^2]^{m-1/2}, \quad x \le x_L, \quad x_L = \sqrt{2(m+1)}\,\sigma$$

clipped tails at $x_L$ (e.g. m=3.5 => $x_L$= 3σ ; m=7 => $x_L$= 4σ)




Profiles




Tails of Profiles

(full width at 10% level :

≈ 4.4 σ for large range of m)

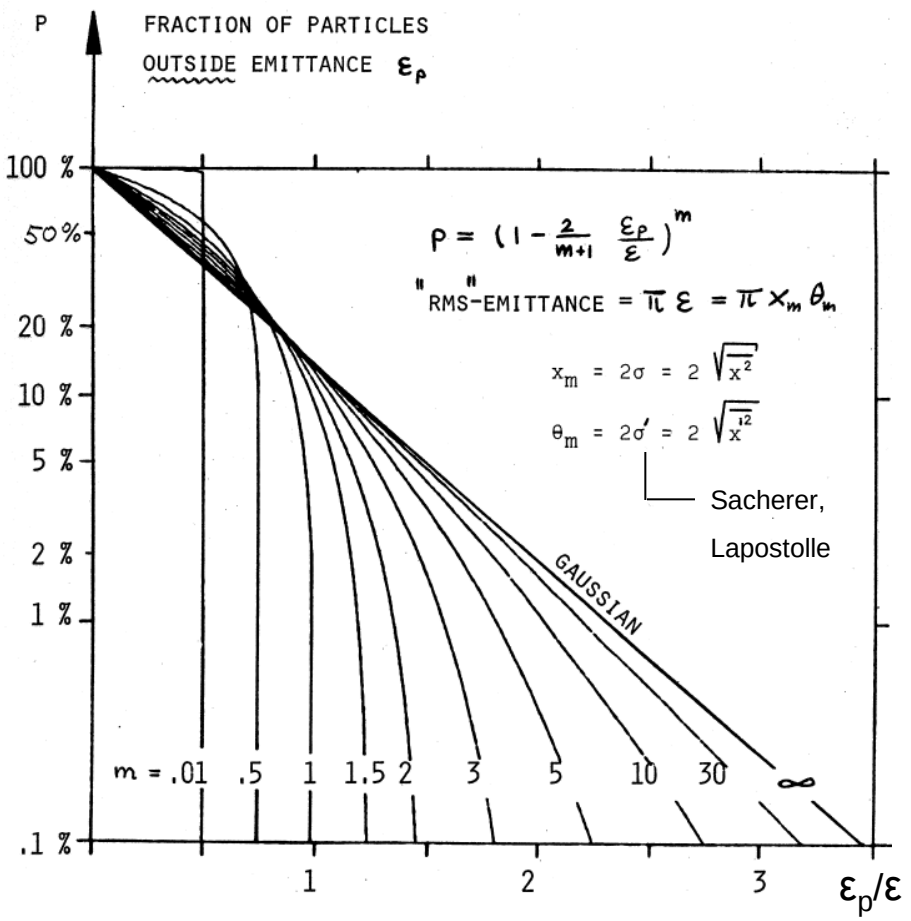
# clipped binomial phase space densities

$$\rho(x,x') = (1-a^2)^{m-1}$$

$$(a^2 \equiv u^2 + v^2 \le 1, \quad u \equiv \frac{x}{x_L}, \quad v \equiv \frac{x'}{x'_L})$$

projected profile : $y(x) = (1-u^2)^{m-1/2}$

we get again a binomial with the exponent reduced by 1/2




**big trick:**

plot fraction which is **outside** of ellipse! (and in logarithmic scale)

for m≥1.5 the curves have a crossing point at $\varepsilon_p \approx \varepsilon$ and p ≈ 13%; i.e. ca. 87% of all particles are inside an ellipse with emittance $\varepsilon=(2\sigma)\cdot(2\sigma')$ , independent of m.

For a Gaussian distribution we have $p=\exp(-2\varepsilon_p/\varepsilon)$, which gives a straight line in this diagram (m=∞).

W.Joho, 1980
SIN report TM 11-4