UPPSALA
UNIVERSITET

Nonlinear Beam-Dynamics with Hamiltonians, Lie Maps, and Normal Forms

Volker Ziemann
Uppsala University

UPPSALA UNIVERSITET

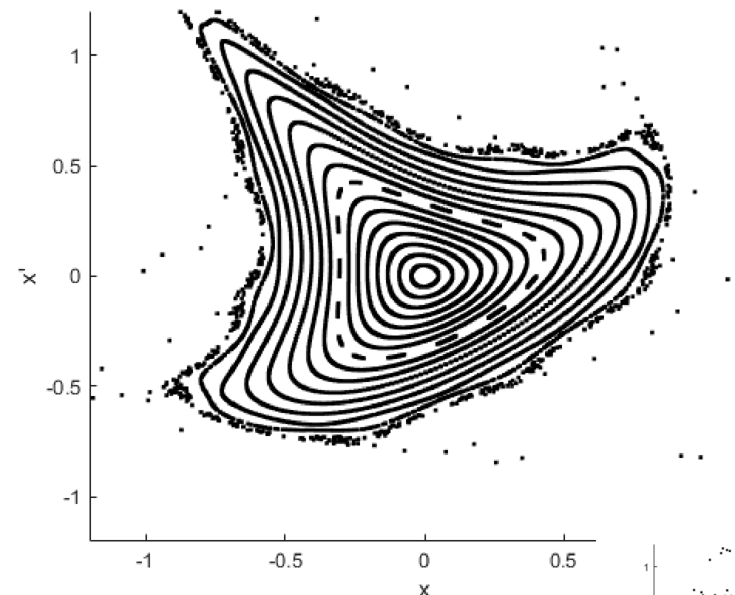
Outline

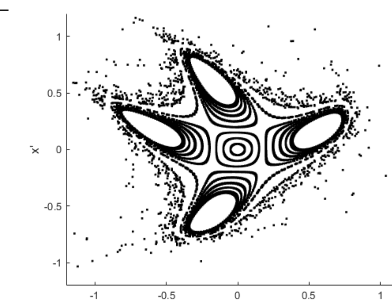
- Motivation
- Hamiltonian Mechanics and Maps
- Examples of Lie maps
- Pushing Lie maps around
- Concatenating Lie-maps
- Resonance driving terms
- Several applications
- Normal Forms

Non-linearities

- Unwanted non-linearities from magnet fringe fields or magnet errors
- Sextupoles are needed for chromaticity correction if the quadrupoles are very strong
- Octupoles needed to add tune spread to provide Landau-damping against instabilities
- Beam stability at large amplitudes $\rightarrow$ Dynamic Aperture
- Can we place multipoles such that they 'cancel'?

$$\begin{pmatrix} \hat{x}_{n+1} \\ \hat{x}'_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} \hat{x}_n \\ \hat{x}'_n - \hat{x}_n^2 \end{pmatrix}$$





- Distortions
- Dynamic aperture
- Sometimes islands

Power Series

$$\begin{pmatrix} \hat{x}_{n+1} \\ \hat{x}'_{n+1} \end{pmatrix} = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} \hat{x}_n \\ \hat{x}'_n - \hat{x}_n^2 \end{pmatrix}$$

- Linear transport described by transfer matrices
- Sextupole kicks: $x' \leftarrow x' - k_2 L\, x^2/2$
- Inserting polynomials into polynomials into polynomials into polynomials into polynomials......
- Differential Algebra codes (M. Berz's COSY-∞)
- Huge (automatized) book-keeping exercise up to a given order
- Power series truncation breaks symplecticity of the map
- Redundant representation, 2x2-TM has 3 independent components, but requires 4 stored numbers
- Does not directly provide 'understanding' of cancellations

UPPSALA UNIVERSITET

Hamiltonians

$$H = h_1 x^2 + h_2 x x' + h_3 x'^2$$

- Hamiltonians to the rescue
- Consider that there are 3 independent monomials of 2nd order: x^2, xx', x'^2
- Coefficients h_i describe **aberrations**
  - **Non-redundant** representation
- Remember quantum mechanics where the hamiltonian is the generator of the motion in time
  - It pushes the wave-function or state-vector forward in time

Hamilton's Equations

- In mechanical systems Hamilton's equations determine trajectory *x(t)*

$$\frac{dx}{dt} = \frac{\partial H}{\partial x'} \quad , \quad \frac{dx'}{dt} = -\frac{\partial H}{\partial x}$$

- Consider the rate of change of a function *f(x,x')*

$$\frac{df}{dt} = \frac{\partial f}{\partial q}\frac{dq}{dt} + \frac{\partial f}{\partial p}\frac{dp}{dt} = \frac{\partial f}{\partial q}\frac{\partial H}{\partial q} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial p} = [f, H] = [-H, f] =: -H : f$$

- Other Nomenclature: $\frac{d}{dt} f(x, x') = [-H, f] =: -H : f$
  - A Lie-operator is a Poisson bracket waiting to happen

Finite steps and Lie-Maps

- Powers of PB

$$:-H:^0 f = f, \quad :-H:^1 f = [-H, f], \quad :-H:^2 f = [-H, [-H, f]]$$

- Allows to write Taylor-series

$$f(t+\Delta t) = \sum_{n=0}^{\infty} \frac{\Delta t^n}{n!} \frac{d^n f}{dt^n} = \sum_{n=0}^{\infty} \frac{\Delta t^n}{n!} :-H:^n f = e^{:-H:\Delta t} f$$

- that describes transport over finite time step

UPPSALA UNIVERSITET

Hamiltonians for Multipoles

- Magnetic fields for **thin-lens multipoles** can be derived from a complex potential

$$F(z) = -B_0 R_0 \sum_{m=1}^{\infty} \frac{b_m + i a_m}{m} \left(\frac{z}{R_0}\right)^m = -\frac{B\rho}{L} \sum_{m=1}^{\infty} \frac{k_{m-1} L}{m!} z^m$$

- Consistent with notation and *w(z)=i dF/dz*

$$i w(z) = B_y + i B_x = -\frac{dF}{dz} = B_0 \sum_{m=1}^{\infty} (b_m + i a_m) \left(\frac{z}{R_0}\right)^{m-1}$$

- Integrate of length and scale with momentum

$$\hat{H}_S = H_S(x, x', y, y')\Delta s = \mathrm{Re}[-F_S(x+iy)L/B\rho] = (k_2 L/6)(x^3 - 3xy^2)$$

Example: Thin Sextupole

- Hamiltonian: $H = (K_2 L/6)(x^3-3xy^2)$
- Map $M = e^{:-H:} = (1+ :-H: + :-H:^2/2! + ...)$

$$:-H: x^m = [x^m, H] = \frac{\partial x^m}{\partial x}\frac{\partial H}{\partial x'} - \frac{\partial x^m}{\partial x'}\frac{\partial H}{\partial x} + (y - terms) = 0$$

$$:-H: x' = \frac{\partial x'}{\partial x}\frac{\partial H}{\partial x'} - \frac{\partial x'}{\partial x'}\frac{\partial H}{\partial x} + \frac{\partial x'}{\partial y}\frac{\partial H}{\partial y'} - \frac{\partial x'}{\partial y'}\frac{\partial H}{\partial y} = -\frac{\partial H}{\partial x} = -\frac{k_2 L}{2}(x^2 - y^2)$$

$$:-H:^2 x' =: -H: \left(-\frac{k_2 L}{2}(x^2 - y^2)\right) = 0$$

- Exponential series truncates
- Complete map

$$\mathcal{M}x = 0 \quad , \quad \mathcal{M}x' = x' - \frac{k_2 L}{2}(x^2 - y^2)$$

$$\mathcal{M}y = 0 \quad , \quad \mathcal{M}y' = y' + k_2 Lxy$$

- Well-known kicks

Drift space and Quadrupole

- Equation of motion $x'' + kx = 0$
- Derive from Hamiltonian $H = \frac{1}{2}\left(x'^2 + kx^2\right)$
- Calculate PB for *x* and *x'*

$$:-H:x = [-H, x] = x'$$

$$:-H:x' = [-H, x'] = -kx$$

- and for powers of

$$:-H:^2 x = [-H, [-H, x]] = -kx$$

UPPSALA UNIVERSITET

Drifts and quadrupoles 2

$$:-H:x=[-H,x]=x'$$

$$:-H:x'=[-H,x']=-kx$$

- Calculate for finite step size *s*

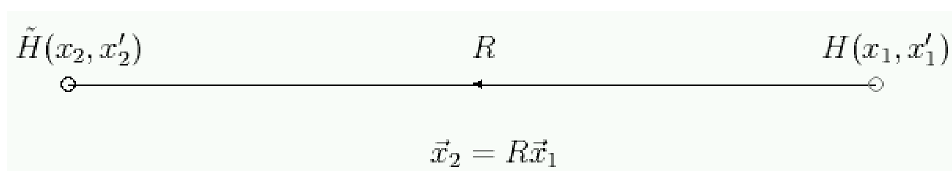
$$\begin{aligned}
e^{:-H:s}x &= x+s:-H:x+\frac{s^2}{2!}:-H:^2x+\frac{s^3}{3!}:-H:^3x+\ldots\\
&= x+sx'-\frac{ks^2}{2!}x-\frac{ks^3}{3!}x'+\ldots\\
&= x\left(1-\frac{ks^2}{2!}+\frac{k^2s^4}{4!}+\ldots\right)+x'\left(s-\frac{ks^3}{3!}+\ldots\right)\\
&= x\cos(\sqrt{k}s)+\frac{x'}{\sqrt{k}}\sin(\sqrt{k}s)
\end{aligned}$$

- same as first line of transfer matrix for quad
- Drift matrix for k → 0

UPPSALA UNIVERSITET

Pushing Hamiltonians around...

- Nothing really gained yet, just shown that hamiltonians yield well-known maps → No new functionality, yet!
- **Problem:** If two elements (magnets) live at different places in a beamline, their Hamiltonians depend on different variables
- **Solution:** Push all Hamiltonians to a reference point, normally at the end of the beam line (Idea due to J. Irwin, SLAC)



$$M = R\, e^{:-H(x1):} = (R\, e^{:-H(x1):} R^{-1})\, R = \text{(non-trivial)} = e^{:-H(Rx1):}\, R = e^{:-H(x2):}\, R$$

- Push a hamiltonian to the reference point with a **similarity transform** by changing its variables to those of the reference point. This makes the effect of the Hamiltonians commensurate.

Aside: Pushing with Software

- 1st order: $$y_i = R_{ij}x_j \quad \rightarrow \quad x_i = R^{-1}_{ij}y_j$$

$$H^{(1)} = h^{(1)}_i x_i = h^{(1)}_i R^{-1}_{ij} y_j = \tilde{h}^{(1)}_j y_j$$

$$\tilde{h}^{(1)}_j = R^{-1}_{ij} h^{(1)}_i \quad \rightarrow \quad \tilde{h}^{(1)} = \left(R^{-1}\right)^T h^{(1)} = S^{(1)} h^{(1)}$$

- 2nd order: $$y_i y_j = R_{ik}R_{jl}x_k x_l$$

$$\left(\begin{array}{c} y_1^2 \\ y_1 y_2 \\ y_2^2 \end{array}\right) = \left(\begin{array}{ccc} R_{11}^2 & 2R_{11}R_{12} & R_{12}^2 \\ R_{11}R_{21} & R_{11}R_{22} + R_{12}R_{21} & R_{11}R_{21} \\ R_{21}^2 & 2R_{21}R_{22} & R_{22}^2 \end{array}\right) \left(\begin{array}{c} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{array}\right) \qquad (\vec{yy}) = RR(\vec{xx})$$

$$H^{(2)} = \sum_i h^{(2)}_i (xx)_i = \sum_i \sum_j h^{(2)}_i (RR^{-1})_{ij} (yy)_j = \sum_j \tilde{h}^{(2)}_j (yy)_j$$

$$\tilde{h}^{(2)}_j = \sum_i h^{(2)}_i (RR^{-1})_{ij} \quad \rightarrow \quad \tilde{h}^{(2)} = \left(RR^{-1}\right)^T h^{(2)} = S^{(2)} h^{(2)}$$

- Analogous in higher orders, coded up to 5th (decapole) order

UPPSALA UNIVERSITET

...and it in practice

$$\tilde{H}(x_2,x_2') \xleftarrow{\quad R \quad} H(x_1,x_1') \qquad \vec{x}_2 = R\vec{x}_1$$

- Drift space with length L

$$\begin{pmatrix} \bar{x} \\ \bar{x}' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} \quad \to \quad \begin{cases} x = \bar{x} - L\bar{x}' \\ x' = \bar{x}' \end{cases}$$

- First order (corrector magnets)

$S^{(1)}$

$$\begin{aligned} h_a x &= h_a(\bar{x} - L\bar{x}') = \tilde{h}_a\bar{x} + \tilde{h}_b\bar{x}' \\ h_b x' &= h_b\bar{x}' = \tilde{h}_a\bar{x} + \tilde{h}_b\bar{x}' \end{aligned} \quad \to \quad \begin{pmatrix} \tilde{h}_a \\ \tilde{h}_b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -L & 1 \end{pmatrix} \begin{pmatrix} h_a \\ h_b \end{pmatrix}$$

- Third order (sextupoles)

$S^{(3)}$

$$\begin{aligned} h_1 x^3 &= h_1(\bar{x} - L\bar{x}')^3 \\ &= h_1(\bar{x}^3 - 3L\bar{x}^2\bar{x}' + 3L^2\bar{x}\bar{x}'^2 - L^3\bar{x}'^3) \\ &= \tilde{h}_1\bar{x}^3 + \tilde{h}_2\bar{x}^2\bar{x}' + \tilde{h}_3\bar{x}\bar{x}'^2 + \tilde{h}_4\bar{x}'^3 \\ h_2 x^2 x' &= \ldots \end{aligned}$$

$$\begin{pmatrix} \tilde{h}_1 \\ \tilde{h}_2 \\ \tilde{h}_3 \\ \tilde{h}_4 \end{pmatrix} = \begin{pmatrix} 1 & * & * & * \\ -3L & * & * & * \\ 3L^2 & * & * & * \\ -L^3 & * & * & * \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}$$

- Automatic with help of two book-keeping arrays
  - ii=MM(n+10*m), position of h_{ii} with $x^n\, x'^m$
  - n=MO(jj,1), power n of x^n in monomial jj

...and in Matlab

UPPSALA UNIVERSITET

Set up book-keeping arrays

```
% returns the array MO, MM, CN
% MO(ii,1) is the power of x in monomial ii
% MO(ii,2) is the power of xp in monomial ii
% ii=MM(n+10*m) position ii with x^n*xp^m
% CN strings with the monomials, nice to have for display
function [MO,MM,CN]=hamini
N=2; NM=14;
MO=zeros(NM,N);
MM=-1000*ones(40,1);
%....................first
ii=0;
for i1=1:N
  ii=ii+1;
  MO(ii,:)=0;
  MO(ii,i1)=1;
  MM(MO(ii,1)+10*MO(ii,2))=ii;
end
%....................second
for i1=1:N
  for j1=i1:N
    ii=ii+1;
    MO(ii,:)=0;
    MO(ii,i1)=MO(ii,i1)+1;
    MO(ii,j1)=MO(ii,j1)+1;
    MM(MO(ii,1)+10*MO(ii,2))=ii
  end
end
%....................third
for i1=1:N
  for j1=i1:N
```

Construct the matrices S(1),S(2),...

```
% constructs the matrices to propagate the monomials
function [S1,S2,S3,S4]=adjoint2(R)
global MO MM CN
N=2; N1=N; N2=N*(N+1)/2; N3=N2*(N+2)/3; N4=N3*(N+3)/4;
IR=zeros(1,N);
%...............first order
S1=R;
%..............second order
S2=zeros(3);
ii=0;
for i1=1:N
  for j1=i1:N
    ii=ii+1;
    for i2=1:N
      for j2=1:N
        IR(:)=0;
        IR(i2)=IR(i2)+1;
        IR(j2)=IR(j2)+1;
        jj=MM(IR(1)+10*IR(2))-N1;
        S2(ii,jj)=S2(ii,jj)+S1(i1,i2)*S1(j1,j2);
      end
    end
  end
end
%..............third order
S3=zeros(4);
ii=0;
for i1=1:N
  for j1=i1:N
    for k1=j1:N
      ii=ii+1;
      for i2=1:N
```

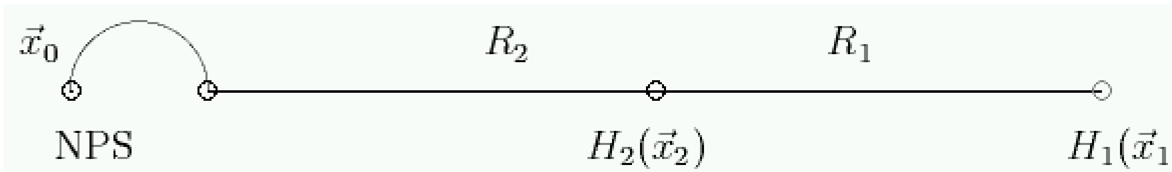
Propagate Hamiltonian

```
% Propagates a Hamiltonian through transfer matrix R
function H1=propham(R,H0)
N=2; N1=N; N2=N*(N+1)/2; N3=N2*(N+2)/3; N4=N3*(N+3)/4;
NM=length(H0);
H1=zeros(NM,1);
[S1,S2,S3,S4]=adjoint2(R);
H1(1:N1)=S1'*H0(1:N1);
H1(N1+1:N1+N2)=S2'*H0(N1+1:N1+N2);
H1(N1+N2+1:N1+N2+N3)=S3'*H0(N1+N2+1:N1+N2+N3);
H1(N1+N2+N3+1:N1+N2+N3+N4)=S4'*H0(N1+N2+N3+1:N1+N2+N3+N4);
```

UPPSALA UNIVERSITET

Pushing to a Reference Point

- Consider beamline with two elements



- NPS=normalized phase space

$$\mathrm{NPS} = \begin{pmatrix} 1/\sqrt{\beta} & 0 \\ -\alpha/\sqrt{\beta} & \sqrt{\beta} \end{pmatrix} \longleftarrow \quad \text{absorbed in} \, R_2$$

- Map

$$\begin{aligned}
\mathcal{M} &= R_2 e^{:-H_2(\vec{x}_2):} R_1 e^{:-H_1(\vec{x}_1):} \\
&= R_2 \; e^{:-H_2(\vec{x}_2):} \overbrace{R_2^{-1} R_2}^{=1} R_1 \; e^{:-H_1(\vec{x}_1):} \overbrace{R_1^{-1} R_2^{-1} R_2 R_1}^{=1} \\
&= \underbrace{R_2 \; e^{:-H_2(\vec{x}_2):} R_2^{-1}}_{e^{:-\tilde{H}_2(\vec{x}_0):}} \underbrace{R_2 R_1 \; e^{:-H_1(\vec{x}_1):} R_1^{-1} R_2^{-1}}_{e^{:-\tilde{H}_1(\vec{x}_0):}} R_2 R_1 \\
&= e^{:-\tilde{H}_2(\vec{x}_0):} e^{:-\tilde{H}_1(\vec{x}_0):} R_2 R_1
\end{aligned}$$

- Both multipoles pushed to the end plus linear transport
  - exact representation; only linear change of variables; generalize to more elements; all have the same independent variables; caveat about ordering of Lie maps

Concatenation

UPPSALA UNIVERSITET

- Concatenate with **Campbell-Baker-Hausdorff** (CBH) formula

$$e^{:H:}e^{:K:} = e^{:H:+:K:+(1/2)[:H:,:K:]+(1/12)[:H:-:K:,[:H:,:K:]]+\dots}$$

- Interpretation: *H* is traversed before *K*, left to right, different from matrices!
- Step through beam line and concatenate the next element to what is already there
- It is mandatory that *H* and *K* depend on the same variables, otherwise: what does *[H,K]* mean?
- Contains effect of three interacting elements consistently
- Symplectic representation of the full map:

$$M = e^{:-H:} R \quad \rightarrow \quad \text{Super-duper-pop-up kick + linear map}$$

UPPSALA UNIVERSITET

...and in practice

- Poisson bracket of two monomials

$$
\begin{aligned}
f &= h_{ii} x^{i_1} x'^{i_2} \\
g &= h_{jj} x^{j_1} x'^{j_2}
\end{aligned}
$$

$$
\begin{aligned}
[f,g] &= \frac{\partial f}{\partial x}\frac{\partial g}{\partial x'} - \frac{\partial f}{\partial x'}\frac{\partial g}{\partial x} \\
&= h_{ii} h_{jj} \left(i_1 x^{i_1-1} x'^{i_2} x^{j_1} j_2 x'^{j_2-1} - x^{i_1} i_2 x'^{i_2-1} j_1 x^{j_1-1} x'^{j_2} \right) \\
&= h_{ii} h_{jj} (i_1 j_2 - i_2 j_1) x^{i_1+j_1-1} x'^{j_1+j_2-1}
\end{aligned}
$$

- Easy to code with the help of MM and M0

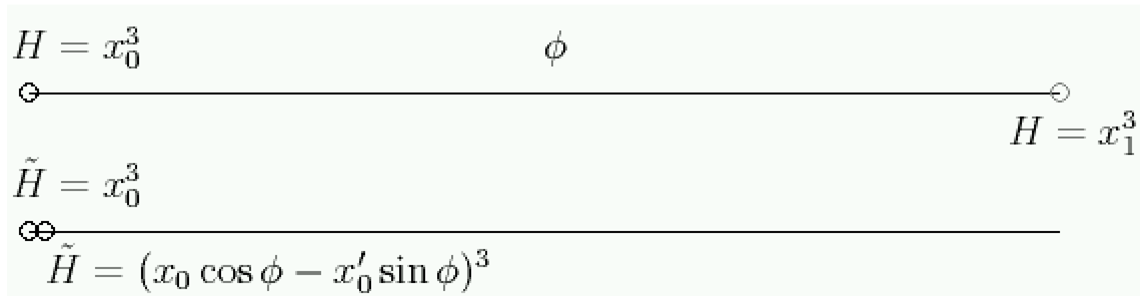
```
% calculate the Hamiltonian H0 for the beamline
% calcmat must be called beforehand for the transfer matrices
function H0=fulham(beamline)
nlines=size(beamline,1);  H0=zeros(14,1);
for k=nlines:-1:1
  if (beamline(k,1)==1000)         % its a nonlinearity
    Htmp=thamlie(beamline(k,4),beamline(k,5)); % dispham(Htmp,'Htmp =')
    R=TM(k,nlines);                % TM to the end
    Htmp=propham(sinv(R),Htmp);    % propagate hamiltonian
    H0=CBH(Htmp,H0);               % concatenate with what is already there
  end
end
```

```
% Poisson bracket
function H3=PB(H1,H2)
global M0 MM
NM=length(H1); H3=zeros(NM,1);
for ii=1:NM
  if abs(H1(ii))<1e-10, continue; end
  i1=M0(ii,1); i2=M0(ii,2);
  for jj=1:NM
    if abs(H2(jj))<1e-10, continue; end
    j1=M0(jj,1); j2=M0(jj,2);
    x12=H1(ii)*H2(jj); l1=i1*j2-i2*j1;
    if (l1==0), continue; end
    k1=i1+j1-1; if (k1<0 || k1>4), continue; end
    k2=i2+j2-1; if (k2<0 || k2>4), continue; end
    if (k1+k2>4), continue; end  % limit to octupole order
    kk=MM(k1+k2*10);
    H3(kk)=H3(kk)+x12*l1;
  end
end
```

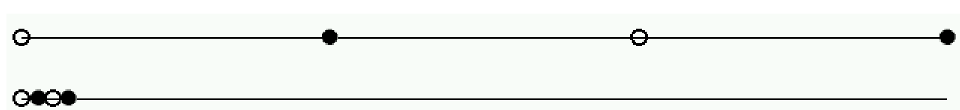
```
% Campbell-Baker-Haussdorff
function H3=CBH(H1,H2)
Haux=PB(H1,H2);
H3=H1+H2+0.5*Haux+PB(H1-H2,Haux)/12;
```

UPPSALA UNIVERSITET

Example: Placement of Sextupoles



- Two sextupoles with equal strength at places with equal beta function
- Push both to the end of the beam line
- $H_{both} = x_0^3 + (x_0 \cos\varphi - x'_0 \sin\varphi)^3 + 1/2[x_0^3, (x_0 \cos\varphi - x'_0 \sin\varphi)^3] + \ldots$
- Cancels to **all orders,** if φ=180 degrees phase advance
- What happens with interleaved sextupoles (as in SLC-FF)?



- Sextupole order cancels pairwise, but octupole-order aberrations appear by PB of the empty and full dots.
  - and you can explicitely calculate what octupole aberrations appear

Resonance Driving Terms

$M = e^{:-H:} R$

- $H(x_0, x'_0, y_0, y'_0)$ is given in variables of normalized phase space
- Introduce action-angle variables

$$x_0 = \sqrt{2J_x}\cos(\psi_x) \quad , \quad x_0' = \sqrt{2J_x}\sin(\psi_x)$$
$$y_0 = \sqrt{2J_y}\cos(\psi_y) \quad , \quad y_0' = \sqrt{2J_y}\sin(\psi_y)$$

$$H = H(J_x, J_y, \psi_x, \psi_y)$$

- Collect terms proportional to $\cos/\sin(m\psi_x \pm n\psi_y)$
- Example: 1-D sextupole already at the end of beam line

$$\begin{aligned} H &= \frac{k_2 l}{6} x^3 = \frac{k_2 l}{6}\beta_x^{3/2} x_0^3 = \frac{k_2 l}{6}\beta_x^{3/2}(2J_x)^{3/2}\cos^3\psi_x \\ &= (2J_x\beta_x)^{3/2}\frac{k_2 l}{6}\left(\frac{1}{4}\cos(3\psi_x) + \frac{3}{4}\cos(\psi_x)\right) \\ &= (2J_x)^{3/2}\left(\underbrace{\beta_x^{3/2}\frac{k_2 l}{24}\cos(3\psi_x)}_{\text{driving term of} 3Q_x} + \underbrace{\beta_x^{3/2}\frac{k_2 l}{8}\cos(\psi_x)}_{\text{driving term of} Q_x}\right) \end{aligned}$$

Can be done for resonances of any order

H has information about all resonances in the beam line

UPPSALA UNIVERSITET

Example: Global Knobs

- Hamiltonian representation has the advantage that each coefficient represents an independent aberration. There is **no redundancy** among coefficients (unlike Taylor-maps).
- To first order in CBH, coefficients in the Hamiltonian are linear in the magnet excitations $k_n L$. Linear combinations of magnet excitations that control a single coefficient of the hamiltonian only, are easily constructed by matrix inversion

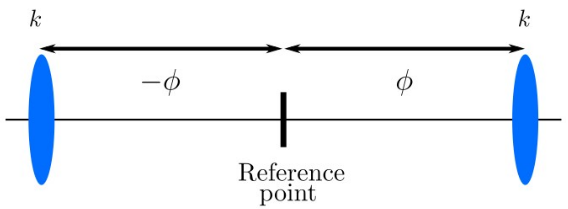
  $\rightarrow$ Knobs

- Numerology for the geometric aberrations in 2D
  - 1st order: 4 aberrations $\rightarrow$ 4 steering dipoles
  - 2nd order: 10 = 2x3+4 $\rightarrow$ 4 skew quads
  - 3rd order: 2 x 10, half upright, half skew sextupoles
  - 4th order: 35 aberrations, octupoles
  - 5th order: N ((N+1)/2) ((N+2)/3) ((N+3)/4) ((N+4)/5) = 56 aberrations

Compensating amplitude-dependent tune-shift without driving fourth-order resonances, J. Ögren, VZ, NIM A869 (2017) 1

- **Question:** Can we place octupoles to only cause amplitude-dependent tune-shift, but no other aberrations? [*sin* or *cos*$(2\,Q_x+2\,Q_y)$]

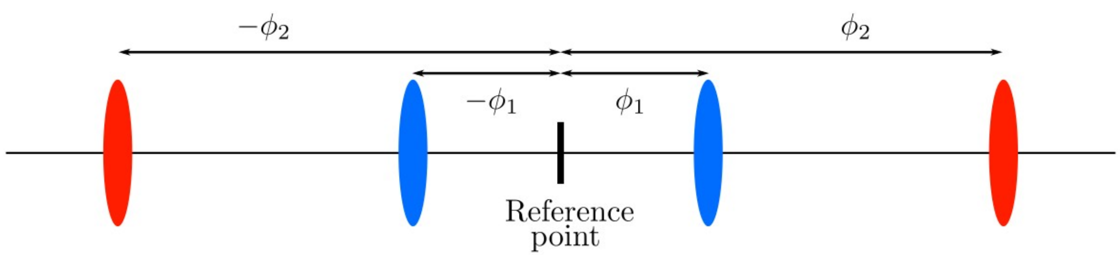
- Two octupoles, 1D



$$\tilde{H} = k(x\cos\phi + x'\sin\phi)^4 + k(x\cos\phi - x'\sin\phi)^4$$
$$= 2k\left\{x^4\cos^4\phi + 6x^2x'^2\cos^2\phi\sin^2\phi + x'^4\sin^4\phi\right\}$$

We want the Hamiltonian to only depend on $2J=x^2+x'^2$.

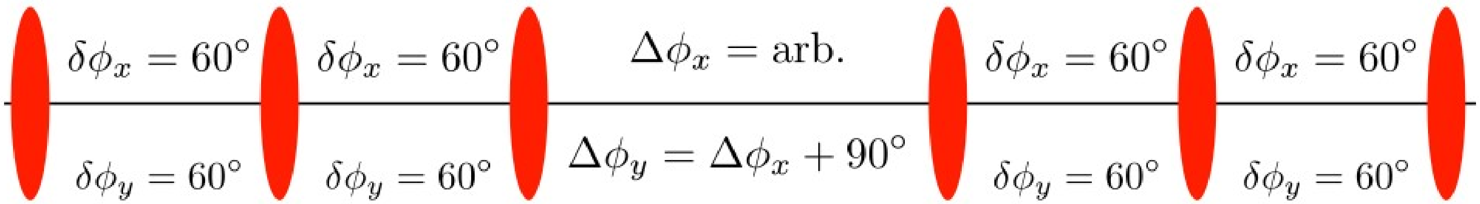
- Four octupoles, 1D



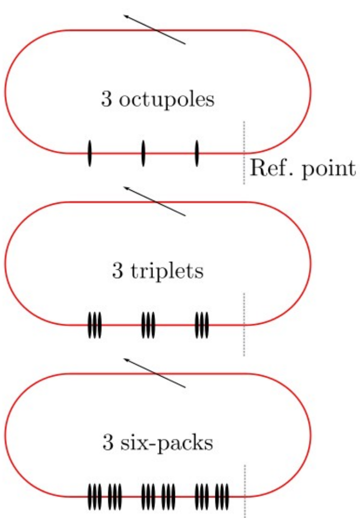
Can achieve this with **four** octupoles equally excited, provided 45° inbetween.

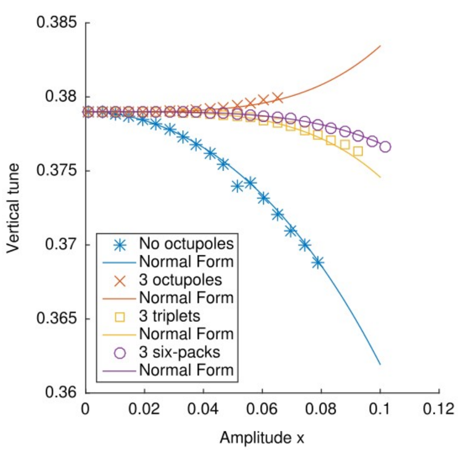
But $\varphi_1=0$ also works with **three** equally excited octupoles

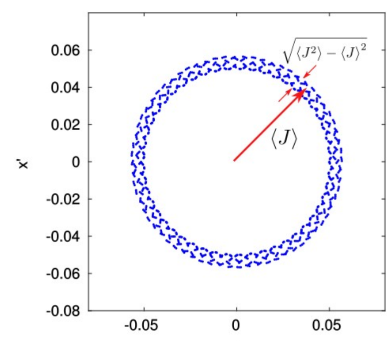
2D octupoles



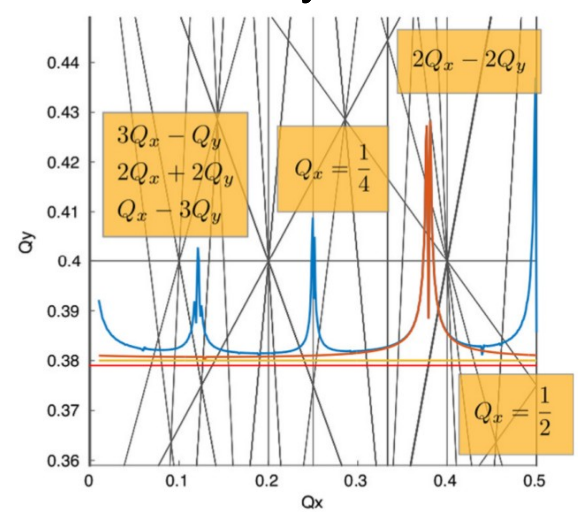
- One 60° triplet: $\tilde{H} = \frac{9}{2}\left[k_x J_x^2 + k_y J_y^2 - 4k_{xy}J_x J_y - 2k_{xy}J_x J_y \cos(2\psi_x - 2\psi_y)\right]$
- Two triplets = six-pack $\tilde{H} = 9k\left[\beta_x^2 J_x^2 + \beta_y^2 J_y^2 - 4\beta_x\beta_y J_x J_y\right]$
- Need three six-packs with different β_x/β_y







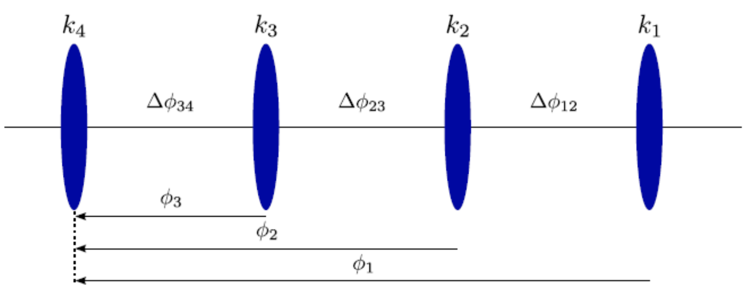
Use Smear as f.o.m



UPPSALA UNIVERSITET

Optimum resonance control knobs for sextupoles, J. Ögren, VZ, NIM A894 (2018) 111

- **Task:** Find sextupole knobs to address third-order geometric aberrations with the least excitation. Avoid fighting correction elements!



$$H = k\beta^{3/2}(x\cos\phi - x'\sin\phi)^3$$

$$H = \frac{k}{4}\left[\cos(3\phi)(2J\beta)^{3/2}\cos(3\psi) + \sin(3\phi)(2J\beta)^{3/2}\sin(3\psi) + 3\cos(\phi)(2J\beta)^{3/2}\cos(\psi) + 3\sin(\phi)(2J\beta)^{3/2}\sin(\psi)\right]$$

$$\begin{bmatrix} C\{\frac{1}{4}(2J\beta)^{3/2}\cos(3\psi)\} \\ C\{\frac{1}{4}(2J\beta)^{3/2}\sin(3\psi)\} \\ C\{\frac{3}{4}(2J\beta)^{3/2}\cos(\psi)\} \\ C\{\frac{3}{4}(2J\beta)^{3/2}\sin(\psi)\} \end{bmatrix} = \begin{bmatrix} \cos(3\phi_1) & \cos(3\phi_2) & \cos(3\phi_3) & 1 \\ \sin(3\phi_1) & \sin(3\phi_2) & \sin(3\phi_3) & 0 \\ \cos(\phi_1) & \cos(\phi_2) & \cos(\phi_3) & 1 \\ \sin(\phi_1) & \sin(\phi_2) & \sin(\phi_3) & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix}$$

Democratic treatment of all resonance driving terms, if condition number ($\lambda_{max}/\lambda_{min}$) of matrix is unity.

$\rightarrow$ Parsimonious knobs

45° phase advance between sextupoles achieves that

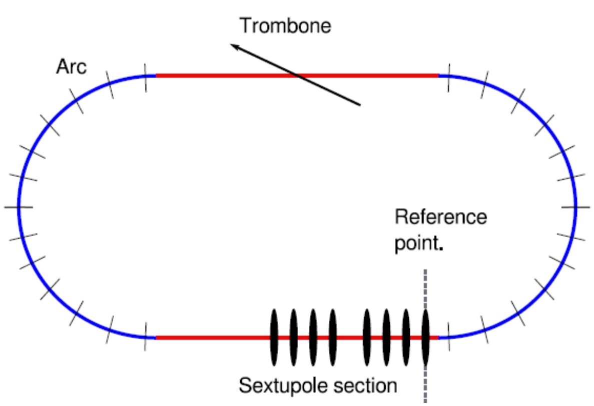
UPPSALA UNIVERSITET

Two-dimensional sextupole knobs

$$\vec{C} = \begin{bmatrix} C\{-\frac{3}{4}(2J_x\beta_x)^{1/2}2J_y\beta_y\cos(\psi_x-2\psi_y)\} \\ C\{-\frac{3}{4}(2J_x\beta_x)^{1/2}2J_y\beta_y\sin(\psi_x-2\psi_y)\} \\ C\{-\frac{3}{4}(2J_x\beta_x)^{1/2}2J_y\beta_y\cos(\psi_x+2\psi_y)\} \\ C\{-\frac{3}{4}(2J_x\beta_x)^{1/2}2J_y\beta_y\sin(\psi_x+2\psi_y)\} \\ C\{\frac{1}{4}(2J_x\beta_x)^{3/2}\cos(3\psi_x)\} \\ C\{\frac{1}{4}(2J_x\beta_x)^{3/2}\sin(3\psi_x)\} \\ C\left\{\begin{matrix} -\frac{3}{2}(2J_x\beta_x)^{1/2}2J_y\beta_y\cos(\psi_x) \\ \frac{3}{4}(2J_x\beta_x)^{3/2}\cos(\psi_x) \end{matrix}\right\} \\ C\left\{\begin{matrix} -\frac{3}{2}(2J_x\beta_x)^{1/2}2J_y\beta_y\sin(\psi_x) \\ \frac{3}{4}(2J_x\beta_x)^{3/2}\sin(\psi_x) \end{matrix}\right\} \end{bmatrix} = M\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_N \end{bmatrix}$$

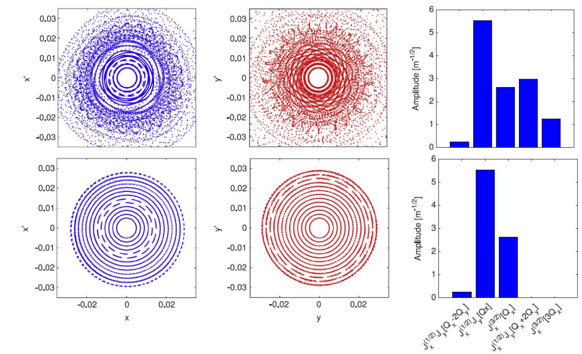
Sextupoles	$\Delta\phi_x$ [degr.]	$\Delta\phi_y$ [degr.]
1–2	135°	45°
2–3	135°	45°
3–4	135°	45°
4–5	180°	90°
5–6	135°	45°
6–7	135°	45°
7–8	135°	45°

works in TME cells

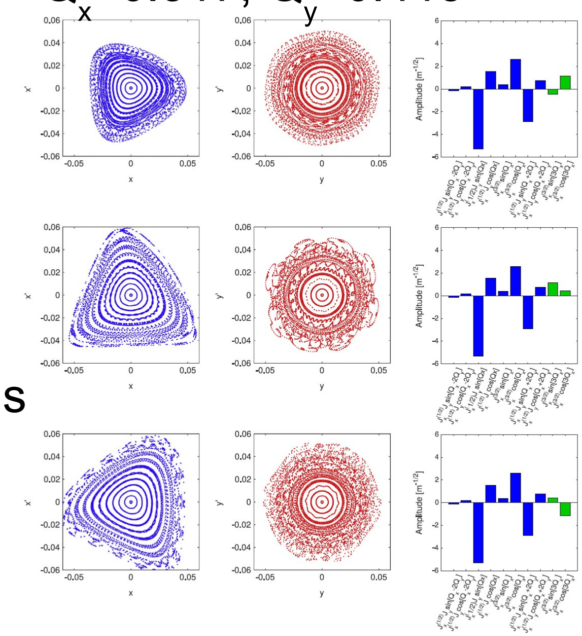


$$M = \begin{bmatrix} \cos(\phi_{x1}-2\phi_{y1}) & \cos(\phi_{x1}-2\phi_{y1}) & \dots & \cos(\phi_{xN}-2\phi_{yN}) \\ \sin(\phi_{x1}-2\phi_{y1}) & \sin(\phi_{x2}-2\phi_{y2}) & \dots & \sin(\phi_{xN}-2\phi_{yN}) \\ \cos(\phi_{x1}+2\phi_{y1}) & \cos(\phi_{x1}+2\phi_{y1}) & \dots & \cos(\phi_{xN}+2\phi_{yN}) \\ \sin(\phi_{x1}+2\phi_{y1}) & \sin(\phi_{x2}+2\phi_{y2}) & \dots & \sin(\phi_{xN}+2\phi_{yN}) \\ \cos(3\phi_{x1}) & \cos(3\phi_{x2}) & \dots & \cos(3\phi_{xN}) \\ \sin(3\phi_{x1}) & \sin(3\phi_{x2}) & \dots & \sin(3\phi_{xN}) \\ \cos(\phi_{x1}) & \cos(\phi_{x2}) & \dots & \cos(\phi_{xN}) \\ \sin(\phi_{x1}) & \sin(\phi_{x2}) & \dots & \sin(\phi_{xN}) \end{bmatrix}$$

Turn off one resonance near Q_x+2Q_y



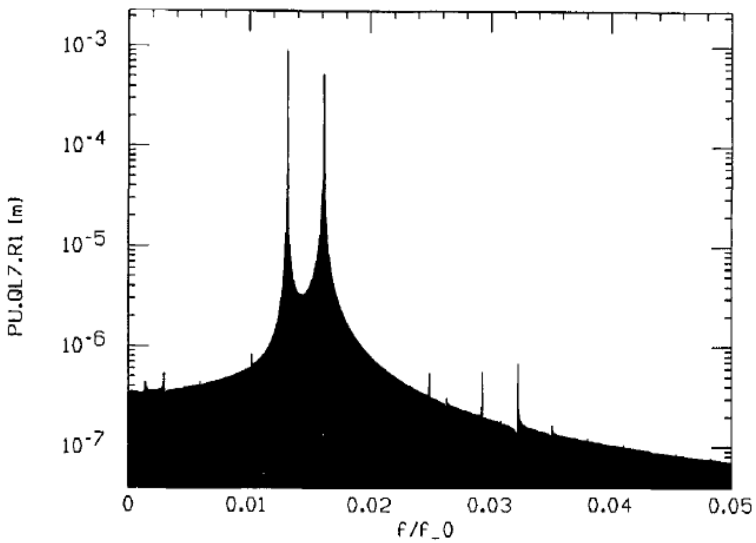
Rotate horiz-phase space $Q_x=0.317$, $Q_y=0.415$

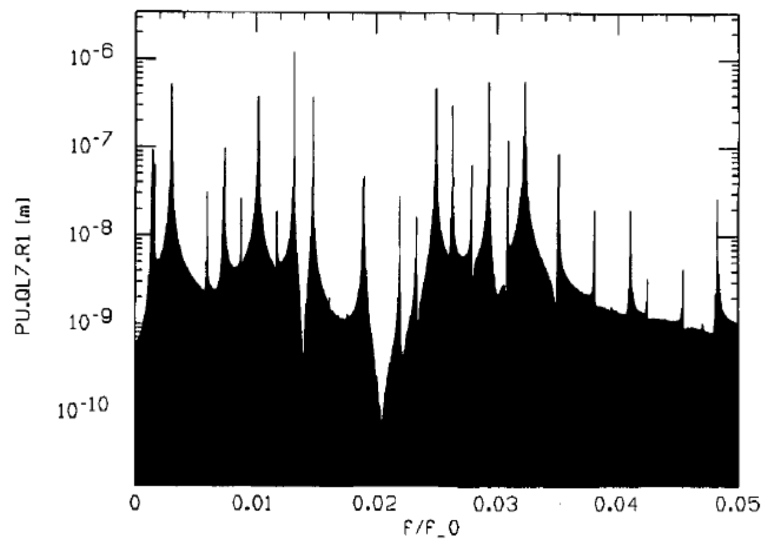


Parsimonious knobs minimize contributions to higher orders!

Measuring Hamiltonian Coefficients... VZ, Part. Acc. 55 (1996) 419

- Idea: low-frequency (~f_0/100) wobble (2h+2v) steerers and observe mixing frequencies on (2h+2v) BPMs.
- Harmonic distortion of closed orbit (simulations done with LEP lattice)
- Need to remove fundamental driving frequency (notch f.)





different frequencies are observed: 0, $2f_1, f_1+f_2, f_1-f_2, f_1+f_3, f_1-f_3, f_1+f_4, f_1-f_4, 2f_2, f_2+f_3, f_2-f_3, f_2+f_4, f_2-f_4, 2f_3, f_3+f_4, f_3-f_4, 2f_4$ at each of the 4 BPM.

UPPSALA UNIVERSITET

Theory

$$H(x,x',y,y') = h_1x^3 + h_2x^2x' + h_3x^2y + h_4x^2y' + h_5xx'^2 + h_6xx'y + h_7xx'y' + h_8xy^2 + h_9xyy' + h_{10}xy'^2 + h_{11}x'^3 + h_{12}x'^2y + h_{13}x'^2y' + h_{14}x'y^2 + h_{15}x'yy' + h_{16}x'y'^2 + h_{17}y^3 + h_{18}y^2y' + h_{19}yy'^2 + h_{20}y'^3 \,.$$

- One-turn effect of perturbation

$$\vec{x}_{\text{final}} = e^{:-H:}R(\vec{x}_{\text{initial}} + \vec{\varepsilon})$$

- Periodic solution to first order in the Hamiltonian

$$\vec{x} = (1 - R)^{-1}\left[R\vec{\varepsilon} - :H:(R(\vec{x} + \vec{\varepsilon}))\right]$$

- Parametrize effect of Hamiltonian with $a_{\alpha jk}$, $z=m^{(2)}$

$$:H: x_\alpha = [H, x_\alpha] = \sum_{j=1}^{20}\sum_{k=1}^{10} a_{\alpha jk} h_j z_k$$

- Solve perturbatively

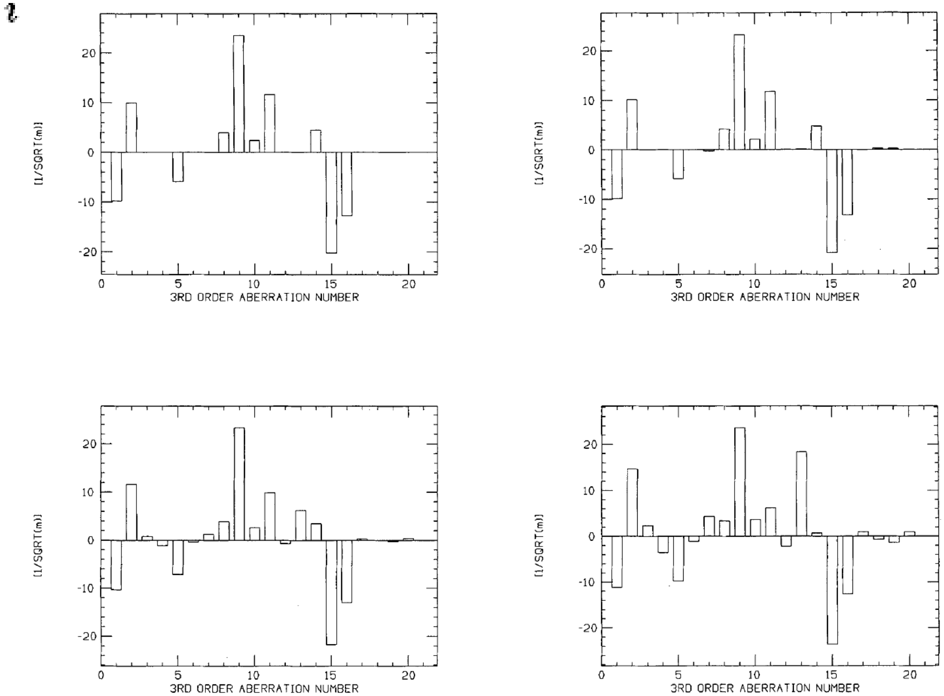
$$\vec{x} = \vec{x}_0 + \sum_{i=1}^{4} \vec{x}_{1,i} \sin\omega_i t + \sum_{j=1}^{16} \vec{x}_{2,j} \cos\tilde{\omega}_j t$$

signal amplitude at harmonic

mixing frequencies

Results

- Linear dependence of 4x17 BPM signals s_i on the 20 Hamiltonian coefficients h_j: $$s_i = \sum_j T_{ij} h_j$$
- Invert $h_j = \left((T^t T)^{-1} T^t\right)_{ji} s_i$
- Only upright sextupoles
  - 10 aberrations
- BPM errors *(0-30μm)*
- *1/sqrt(Nturn)*

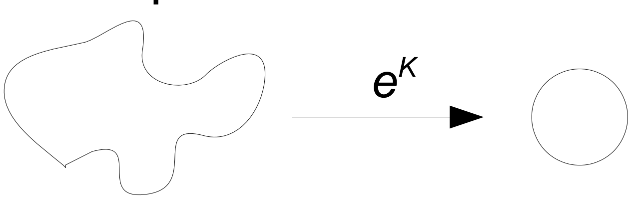


Normal Forms, Motivation

- Start with map M represented as $e^{-H}\, R$ (:::no more colons:::)
- and assume that the map goes from NPS to NPS
  - Then R is a rotation matrix
- Task: express M in terms of physically relevant quantities
- Require representation/decomposition of M in the form

$$e^{-H}\, R = e^{-K}\, e^{-C}\, R\, e^{K} \qquad \text{“diagonalization”}$$

  - C is required to depend only on the action variables $J_x=(x^2+x'^2)/2$ and $J_y=(y^2+y'^2)/2$ and is called the non-linear tune-shift hamiltonian
  - Action dependent tune-shift
  - e^K maps into the generalized normalized phase space
  - Non-resonant normal forms
  - There is also a resonant normal form



UPPSALA UNIVERSITET

Non-resonant Normal Form

- Rewrite normal form condition:

$$e^{-H}R = e^{-K}e^{-C}Re^{K}$$

- Put e^{-K} on other side and multiply by $1=R^{-1}R$

$$e^{-H}\underbrace{Re^{-K}R^{-1}}R = e^{-K}e^{-C}R \longrightarrow e^{-H}Re^{-K}R^{-1} = e^{-K}e^{-C}$$

- Use: $R\, e^{-K}R^{-1} = e^{-SK}$
- Solve order by order

$$e^{-H}e^{-SK} = e^{-K}e^{-C}$$

$$\begin{aligned} H &= H^{(3)} + H^{(4)} + H^{(5)} \\ K &= K^{(3)} + K^{(4)} + K^{(5)} \\ SK &= S^{(3)}K^{(3)} + S^{(4)}K^{(4)} + S^{(5)}K^{(5)} \end{aligned}$$

- Solve for the tuneshift polynomial C and the transformation K

UPPSALA UNIVERSITET

Third order

- Just keep terms of third order

$$e^{-H^{(3)}}e^{-S^{(3)}K^{(3)}} = e^{-K^{(3)}}e^{-C^{(3)}}$$

- Application of CBH on left and right side yields the exponents

$$H^{(3)} + S^{(3)}K^{(3)} = K^{(3)} + C^{(3)} + \text{higher orders}$$

- Solving for $K^{(3)}$ results in

$$(1 - S^{(3)})K^{(3)} = H^{(3)} - C^{(3)} = H^{(3)}$$

- because the is no tune shift term $C^{(3)}$ in third order
- We find for the polynomial K in third order

$$K^{(3)} = (1 - S^{(3)})^{-1}H^{(3)}$$

UPPSALA UNIVERSITET

Fourth order

- Write down the equation for *C* and *K* to fourth order

$$e^{-H^{(3)}-H^{(4)}}e^{-S^{(3)}K^{(3)}-S^{(4)}K^{(4)}} = e^{-K^{(3)}-K^{(4)}}e^{-C^{(4)}}$$

- apply CBH and collect terms

$$\begin{aligned} H^{(3)} + H^{(4)} + S^{(3)}K^{(3)} + S^{(4)}K^{(4)} \\ +\frac{1}{2}[H^{(3)} + H^{(4)}, S^{(3)}K^{(3)} + S^{(4)}K^{(4)}] + \dots \\ = K^{(3)} + K^{(4)} + C^{(4)} + \frac{1}{2}[K^{(3)} + K^{(4)}, C^{(4)}] + \dots \end{aligned}$$

- Only those are of fourth order

$$H^{(4)} + S^{(4)}K^{(4)} + \frac{1}{2}[H^{(3)}, S^{(3)}K^{(3)}] = K^{(4)} + C^{(4)}$$

- solve for K(4) and C(4)

$$(1 - S^{(4)})K^{(4)} + C^{(4)} = H^{(4)} + \frac{1}{2}[H^{(3)}, S^{(3)}K^{(3)}]$$

Trouble in Paradise (4th order)

- $(1\text{-}S^{(4)})$ is not invertible, because it has three zero eigenvalues with eigenvectors corresponding to the polynomials

$$(x^2+x'^2)^2,\ (y^2+y'^2)^2,\ (x^2+x'^2)(y^2+y'^2)$$

- Invert $(1\text{-}S^{(4)})$ by Singular Value Decomposition which projects out the nullspace, which can be put into $C^{(4)}$. Remember that the tuneshift polynomial C contains action variables which are just those $J_x=(x^2+x'^2)$ and $J_y=(y^2+y'^2)$

All well in Paradise

UPPSALA UNIVERSITET

- Solve by SVD

$$(1-S^{(4)})K^{(4)} + C^{(4)} = H^{(4)} + \frac{1}{2}[H^{(3)}, S^{(3)}K^{(3)}]$$

- Project out the invariant tuneshift component

$$C^{(4)} = \hat{P}_{\lambda=0}\left(H^{(4)} + \frac{1}{2}[H^{(3)}, S^{(3)}K^{(3)}]\right)$$

Amplitude-dependent tune shift: $C^{(4)}$

- and invert the rest by SVD tricks

$$K^{(4)} = \mathcal{O}_2\,``\Lambda^{-1}``\mathcal{O}_1^T\left(H^{(4)} + \frac{1}{2}[H^{(3)}, S^{(3)}K^{(3)}]\right)$$

- Gauge invariance: adding K' = K'(nullspace) to $K^{(4)}$ does not change anything
  - $K^{(4)}$ is only determined modulo nullspace
  - 'Fix the gauge' by choosing zero projection to nullspace of $K^{(4)}$

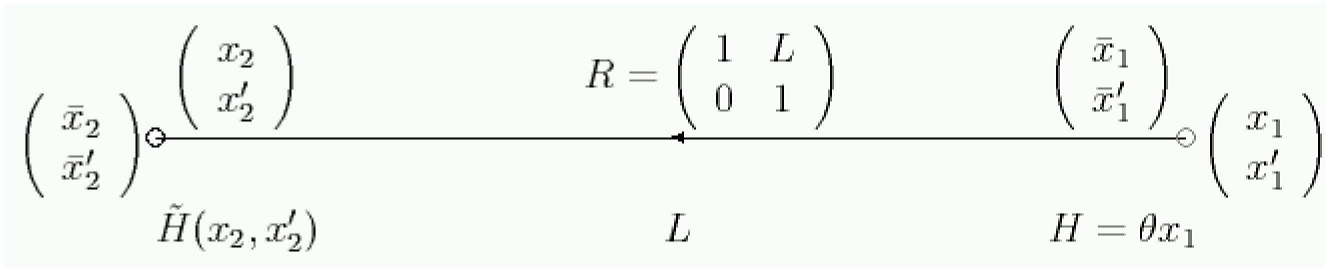
UPPSALA UNIVERSITET

What have we gained in this tour-de-force on Hamiltonians?

- Hamiltonians help us to understand aberrations and their cancellations
- Useful for constructing knobs
- Measuring Hamiltonians
- Normal forms allow us to calculate action dependent tune-shift and phase-space distortion
- Hamiltonians help to 'think beam lines'

Backup slides follow

Example: Pushing a dipole kick



- Hamiltonian kick

$$\begin{aligned} \bar{x}_1 &= e^{:-H:}x_1 = (1- :\theta x_1:)x_1 = x_1 - \theta[x_1, x_1] = x_1 \\ \bar{x}_1' &= e^{:-H:}x_1' = (1- :\theta x_1:)x_1' = x_1 - \theta[x_1, x_1'] = x_1' - \theta \end{aligned}$$

- Traditional way

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_1' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' - \theta \end{pmatrix} = \begin{pmatrix} x_1 + Lx_1' - L\theta \\ x_1' - \theta \end{pmatrix}$$

- Hamiltonian way: need the transformation

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} x_1 + Lx_1' \\ x_1' \end{pmatrix} \to \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = \begin{pmatrix} x_2 - Lx_2' \\ x_2' \end{pmatrix}$$

Pushing a dipole kick 2

- The pushed hamiltonian (just express the old one in the new variables)

$$\tilde{H}(x_2, x_2') = H(x_1, x_1') = \theta x_1 = \theta(x_2 - L x_2')$$

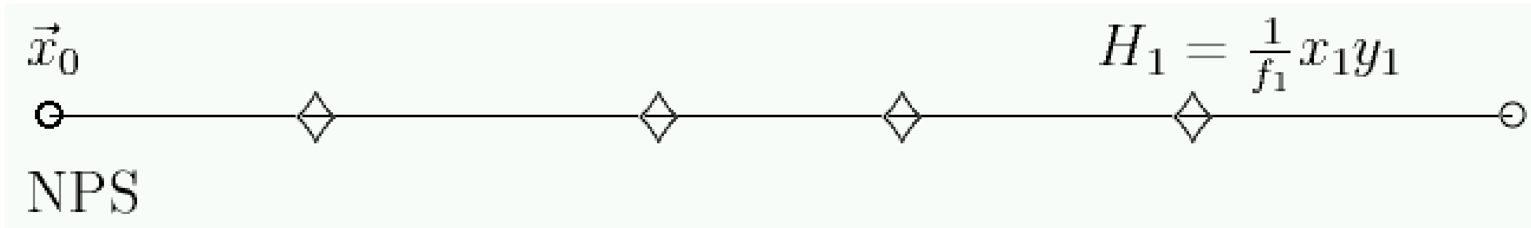
- Check that it does the right thing

$$\begin{aligned} \bar{x}_2 &= e^{:-\tilde{H}:} x_2 = \{1 - :\theta(x_2 - L x_2'):\} x_2 = x_2 - \theta[x_2 - L x_2', x_2] \\ &= x_2 + L\theta[x_2', x_2] = x_2 - L\theta = x_1 + L x_1' - L\theta \end{aligned}$$

$$\begin{aligned} \bar{x}_2' &= e^{:-\tilde{H}:} x_2' = \{1 - :\theta(x_2 - L x_2'):\} x_2' = x_2' - \theta[x_2, x_2'] \\ &= x_2' - \theta = x_1' - \theta \end{aligned}$$

- Agrees with the directly calculated values on the previous slide
- Exchanged the kick and the linear transport!

Example: Coupling



- Consider linear uncoupled beam line with extra skew quadrupoles
- Pushing all skew quads to the left

$$\mathcal{M} = e^{:-\tilde{H}(x_0, x_0', y_0, y_0'):} R$$

- Ten coefficients in the hamiltonian

$$\begin{aligned} -\tilde{H}(x_0, x_0', y_0, y_0') \quad = \quad & h_1 x_0^2 + h_2 x_0 x_0' + h_3 x_0 y_0 + h_4 x_0 y_0' + + h_5 x_0'^2 \\ & h_6 x_0' y_0 + h_7 x_0' y_0' + h_8 y_0^2 + h_9 y_0 y_0' + h_{10} y_0'^2 \end{aligned}$$

  – horizontal coefficients h_1, h_2, h_5 and vertical coefficients h_8, h_9, h_{10} lead to tuneshift and beta-beat
  – four coupling elements h_3, h_4, h_6, h_7 → resonance driving terms for sum and difference resonance $(\sigma_c, \sigma_s, \Delta_c, \Delta_s)$

Coupling 2

- Consider the two of the coupling terms only

$$h_3 x_0 y_0 + h_7 x_0' y_0' = h_3 \frac{1}{2}\sqrt{2J_x 2J_y}\,(\cos(\psi_x - \psi_y) + \cos(\psi_x + \psi_y)) + h_7 \frac{1}{2}\sqrt{2J_x 2J_y}\,(\cos(\psi_x - \psi_y) - \cos(\psi_x + \psi_y))$$

- Resonance driving terms for the sum an difference resonance
- $$2\pi\sigma_c = \frac{1}{2}(h_3 - h_7) \quad , \quad 2\pi\Delta_c = \frac{1}{2}(h_3 + h_7)$$
- and similarly for the other (sine) phase
- Remark: The minimum tune separation ΔQ in a closest-tune scan that is done to measure the coupling is given by

$$\Delta Q = \sqrt{\Delta_c^2 + \Delta_s^2}$$