A Self-Consistent 3D Finite Element Vlasov-Maxwell Solver with Particles

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PSI

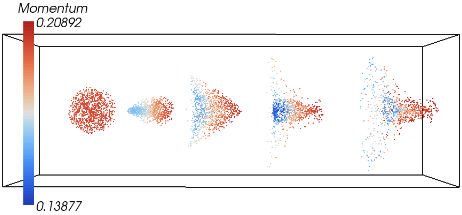
OUTLINE

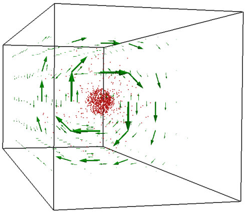
- **Self-consistent Formalism**
  - Motivate Vlasov-Maxwell equations
  - Charge Discretization
  - Field Discretization
  - Time Stepping
  - Combining Particles and Fields
  - Integration Scheme
- **Implementation**
- **Tests and Results**
  - Cavity Eigenmodes
  - Poisson Problem

VLASOV-MAXWELL

$\mathbf{x} \in \Omega \subset \mathbb{R}^3, \quad \gamma m\mathbf{v} = \mathbf{p} \in \Pi = \mathbb{R}^3$

$\partial\Omega$: perfect electric conductor





HAMILTONIAN EQUATIONS OF MOTION (SELF-CONSISTENCY ≡ SOLVING THIS COUPLED NONLINEAR SYSTEM)

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + q(\mathbf{E} + c^{-1}\mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{p}} f = 0$$

$$\dot{\mathbf{B}} = -\nabla \times \mathbf{E} \qquad \nabla \cdot \epsilon_0 \mathbf{E} = \rho$$

$$\epsilon_0 \dot{\mathbf{E}} = \nabla \times \mu_0^{-1}\mathbf{B} - \mathbf{J} \qquad \nabla \cdot \mathbf{B} = 0$$

Phase space distribution $f(\mathbf{x}, \mathbf{p}, t) > 0$

$$\rho(\mathbf{x}, t) = q \int_{\mathbb{R}^3} f \, d\mathbf{p}$$

$$\mathbf{J}(\mathbf{x}, t) = q \int_{\mathbb{R}^3} \mathbf{v} f \, d\mathbf{p}$$

- plasma physics $\longrightarrow$ self-interaction of ρ
- no collisional interaction $\longrightarrow$ mean field
- is usually appropriate for beam dynamics
- no analytical solution known in 3d
- solving full system w.r.t. f is impossible
- particle methods are stable

GOAL: SELF-CONSISTENT TREATMENT

- $f(\mathbf{x}, \mathbf{p}, t) \longrightarrow \sum_i \Big(\delta(\mathbf{x} - \mathbf{x}_i(t)), \delta(\mathbf{p} - \mathbf{p}_i(t)) \Big)$
- $\mathbf{E}, \mathbf{B} \longrightarrow$ Finite Elements

PARTICLE DISCRETIZATION

(RELATIVISTIC PARTICLES)

$$H(f, \mathbf{E}, \mathbf{B})$$

$$f \rightarrow \sum_{i=1}^{N} \text{particle}_i$$

$$H_\delta(\vec{\mathbf{x}}, \vec{\mathbf{p}}, \mathbf{E}, \mathbf{B})$$

$$\dot{f} = \{\{f, H\}\}^1$$

$$\begin{aligned} \dot{\mathbf{x}}_i &= \nabla_{\mathbf{p}_i} H_\delta \\ \dot{\mathbf{p}}_i &= -\nabla_{\mathbf{x}_i} H_\delta \end{aligned}$$

PDE

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + q(\mathbf{E} + c^{-1}\mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{p}} f = 0$$

ODE

$$\begin{aligned} \dot{\mathbf{x}}_i &= \frac{1}{\gamma_i m_i} \mathbf{p}_i \\ \dot{\mathbf{p}}_i &= q_i \left[\mathbf{E}(\mathbf{x}_i, t) + \frac{\mathbf{p}_i}{\gamma_i m_i c} \times \mathbf{B}(\mathbf{x}_i, t) \right] \end{aligned}$$

$$\begin{aligned} \rho(\mathbf{x}, t) &= q \int f \, \mathrm{d}\mathbf{p} \\ \mathbf{J}(\mathbf{x}, t) &= q \int \mathbf{v} f \, \mathrm{d}\mathbf{p} \end{aligned}$$

$$\begin{aligned} \rho_\delta(\mathbf{x}, t) &= \sum_i q_i \cdot \delta(\mathbf{x} - \mathbf{x}_i(t)) \\ \mathbf{J}_\delta(\mathbf{x}, t) &= \sum_i q_i \mathbf{v}_i(t) \cdot \delta(\mathbf{x} - \mathbf{x}_i(t)) \end{aligned}$$

[1] J.B. Marsden, A. Weinstein, *The Hamiltonian Structure of the Maxwell-Vlasov Equations*, Physica 4D (1982)

GALERKIN DISCRETIZATION

$$H_\delta(\vec{\mathbf{x}}, \vec{\mathbf{p}}, \mathbf{E}, \mathbf{B})$$

$$\dot{\mathbf{E}} = \{\{\mathbf{E}, H_\delta\}\}$$
$$\dot{\mathbf{B}} = \{\{\mathbf{B}, H_\delta\}\}$$

NÉDÉLEC ELEMENTS

$$\mathbf{E}(\mathbf{x}, t) \approx \sum_k^{N_1} e_k(t) \cdot \mathbf{w}_k(\mathbf{x})$$
$$\mathbf{w}_k \in \mathcal{P}_{\text{curl}} \subset H(curl; \Sigma)$$
$$e_k \in \mathcal{D}_{\text{curl}} = \mathcal{P}^*_{\text{curl}} \text{ (dual space)}$$
$$\mathbf{B}(\mathbf{x}, t) \approx \sum_k^{N_2} b_k(t) \cdot \mathbf{f}_k(\mathbf{x})$$
$$\mathbf{f}_k \in \mathcal{P}_{\text{div}} \subset H(div; \Sigma)$$
$$b_k \in \mathcal{D}_{\text{div}} = \mathcal{P}^*_{\text{div}} \text{ (dual space)}$$

$$\dot{\mathbf{B}} = -\nabla \times \mathbf{E}$$
$$\epsilon\dot{\mathbf{E}} = \nabla \times \mu^{-1}\mathbf{B} - \mathbf{J}_\delta$$

Galerkin discretization →

$$\dot{\mathbf{b}} = -\mathbf{K}_{12}\mathbf{e}$$
$$\mathbf{M}_1^{BC}\dot{\mathbf{e}} = \mathbf{K}_{12}^T\mathbf{M}_2\mathbf{b} - \mathbf{L}_1(\mathbf{J}_\delta)$$

$\mathbf{K}_{12}$	:	Topological Curl-matrix
$\mathbf{M}_1^{BC}$	:	Curl-mass-matrix with boundary cond.
$\mathbf{M}_2$	:	Div-mass-matrix

$$\mathbf{L}_1(\mathbf{J}_\delta)_k = \int_\Sigma \mathbf{J}_\delta \cdot \mathbf{w}_k(\mathbf{x})\, d\mathbf{x}$$

DIVERGENCE CONSTRAINTS

EXACT SEQUENCE DIAGRAM

$$\underset{\mathbf{\Phi}}{\overset{\text{0-form}}{H^1(\Omega)/\mathbb{R}}} \xrightarrow[\nabla]{\mathrm{d}} \underset{\mathbf{E},\mathbf{A}}{\overset{\text{1-form}}{H(\text{curl};\Omega)}} \xrightarrow[\nabla\times]{\mathrm{d}} \underset{\mathbf{B},\mathbf{J}}{\overset{\text{2-form}}{H(\text{div};\Omega)}} \xrightarrow[\nabla\cdot]{\mathrm{d}} \underset{\rho}{\overset{\text{3-form}}{L^2(\Omega)}} \xrightarrow[\nabla]{\mathrm{d}} 0$$

$\longrightarrow$ decreasing continuity $\longrightarrow$

DIVERGENCE CONSTRAINTS REMAIN CORRECT TO MACHINE PRECISION

$$\partial_t \quad \nabla \cdot \mathbf{B} = -\nabla \cdot (\nabla \times \mathbf{E}) = 0$$

$$\partial_t \quad \nabla \cdot \epsilon_0 \mathbf{E} = \nabla \cdot \nabla \times \mu^{-1} \mathbf{B} - \nabla \cdot \mathbf{J} = \dot{\rho}$$

Cont.Eq.: $\nabla \cdot \mathbf{J} = -\dot{\rho}$

TEMPORAL DISCRETIZATION

$$\begin{aligned} \dot{\mathbf{x}}_i &= \frac{1}{\gamma_i m_i}\mathbf{p}_i \\ \dot{\mathbf{p}}_i &= q_i\left[\mathbf{E}(\mathbf{x}_i,t) + \frac{\mathbf{p}_i}{\gamma_i m_i c}\times\mathbf{B}(\mathbf{x}_i,t)\right] \end{aligned}$$

$$\begin{aligned} \dot{\mathbf{b}} &= -\mathbf{K}_{12}\mathbf{e} \\ \mathbf{M}_1^{BC}\dot{\mathbf{e}} &= \mathbf{K}_{12}^T\mathbf{M}_2\mathbf{b} - \mathbf{L}_1(\mathbf{J}_\delta) \end{aligned}$$

$\mathbf{x}_i \rightarrow q$
$\mathbf{p}_i \rightarrow p$

$\mathbf{e} \rightarrow q$
$\mathbf{b} \rightarrow p$

GENERALIZED LEAP-FROG (SYMPLECTIC)

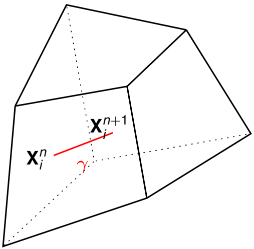
$$\begin{aligned} p^{n+\frac{1}{2}} &= p^n - \frac{\Delta t}{2}\nabla_q H(p^{n+\frac{1}{2}}, q^n) \\ q^{n+1} &= q^n + \frac{\Delta t}{2}\left[\nabla_p H(p^{n+\frac{1}{2}}, q^n) + \nabla_p H(p^{n+\frac{1}{2}}, q^{n+1})\right] \\ p^{n+1} &= p^{n+\frac{1}{2}} - \frac{\Delta t}{2}\nabla_q H(p^{n+\frac{1}{2}}, q^{n+1}) \end{aligned}$$

COMBINING FIELDS AND PARTICLES

$$\mathbf{p}_i^{n+\frac{1}{2}} = \mathbf{p}_i^n + \frac{\Delta t}{2} q_i \left[\mathbf{E}_{\text{loc}}^n(\mathbf{x}_i^n) + \frac{\mathbf{p}_i^{n+\frac{1}{2}}}{\gamma_i^{n+\frac{1}{2}} m_i c} \times \mathbf{B}_{\text{loc}}^n(\mathbf{x}_i^n) \right]$$

$$\mathbf{M}_1^{\text{BC}} \mathbf{e}^{n+1} = \mathbf{M}_1^{\text{BC}} \mathbf{e}^n + \Delta t \left(\mathbf{K}_{12}^T \mathbf{M}_2 \mathbf{b}^{n+\frac{1}{2}} - \mathbf{L}_1^{n+\frac{1}{2}} (\mathbf{J}_\delta) \right)$$

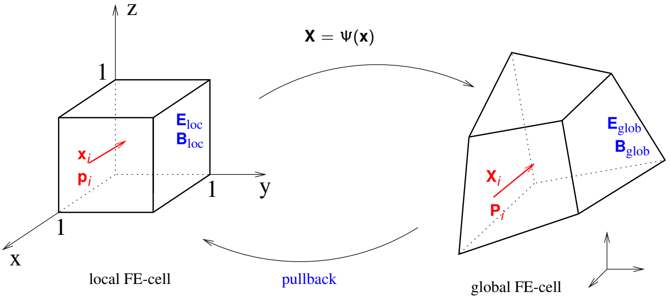
No interpolation to grid points as in FDTD



$$\begin{aligned}
\mathbf{L}_{1,k}^{n+\frac{1}{2}} &= \int_\Omega \mathbf{J}_\delta \cdot \mathbf{w}_k \, \mathrm{d}\mathbf{x} \\
&= \int_\Omega \sum_i q_i \mathbf{v}_i(t) \, \delta\big(\mathbf{x} - \mathbf{x}_i(t)\big) \cdot \mathbf{w}_k \, \mathrm{d}\mathbf{x} \\
&= \sum_i q_i \mathbf{v}_i \cdot \int_{\gamma_i} \mathbf{w}_k \, \mathrm{d}\mathbf{x} \\
&= \sum_i \frac{q_i}{\Delta t} \left(\mathbf{X}_i^{n+1} - \mathbf{X}_i^n \right) \cdot \int_{\gamma_i} \mathbf{w}_k \, \mathrm{d}\mathbf{x}
\end{aligned}$$

- Exact integral w.r.t. FE-discretization (using Gauss quadrature)
- A particle induces contribution of Dof's of corresponding FE-cell

PARTICLE TRACKING / LOCAL COORDINATES



FETCH FIELDS FOR PARTICLES

- $\dot{\mathbf{p}}_i = q_i \left[\mathbf{E}(\mathbf{x}_i, t) + \frac{\mathbf{p}_i}{\gamma_i m_i c} \times \mathbf{B}(\mathbf{x}_i, t) \right]$
- store particles w.r.t one FE-cell
- pullback of fields

FE STATE-OF-THE-ART

- integration on reference element
- transformation via pullback

$$H(curl): \quad \mathbf{E}_{loc}(\mathbf{x}) = (\Psi'_{\mathbf{x}})^T \mathbf{E}_{glob}(\Psi(\mathbf{x}))$$
$$H(div): \quad \mathbf{B}_{loc}(\mathbf{x}) = \det(\Psi'_{\mathbf{x}})(\Psi'_{\mathbf{x}})^{-1} \mathbf{B}_{glob}(\Psi(\mathbf{x}))$$

FULL INTEGRATION SCHEME

$$\mathbf{p}_i^{n+\frac{1}{2}} = \mathbf{p}_i^n + \frac{\Delta t}{2} q_i \left[\mathbf{E}_{\text{loc}}^n(\mathbf{x}_i^n) + \frac{\mathbf{p}_i^{n+\frac{1}{2}}}{\gamma_i^{n+\frac{1}{2}} m_i c} \times \mathbf{B}_{\text{loc}}^n(\mathbf{x}_i^n) \right]$$

$$\mathbf{x}_i^{n+1} = \mathbf{x}_i^n + \frac{\Delta t}{\gamma^{n+\frac{1}{2}} m} \mathbf{p}_i^{n+\frac{1}{2}}$$

$$\mathbf{j}^{n+\frac{1}{2}} = \sum_{i,k} \mathbf{L}_{i,k}^{n+\frac{1}{2}} = \sum_{i,k} \frac{q_i}{\Delta t} (\mathbf{X}_i^{n+1} - \mathbf{X}_i^n) \cdot \int_{\mathbf{x}_i^n}^{\mathbf{x}_i^{n+1}} \mathbf{w}'^k_{\text{glob}}(\mathbf{x})\, \mathrm{d}\mathbf{x}$$

$$\mathbf{b}^{n+\frac{1}{2}} = \mathbf{b}^n - \frac{\Delta t}{2} \mathbf{K}_{12} \mathbf{e}^n$$

$$\mathbf{M}_1^{\text{BC}} \mathbf{e}^{n+1} = \mathbf{M}_1^{\text{BC}} \mathbf{e}^n + \Delta t \left(\mathbf{K}_{12}^T \mathbf{M}_2 \mathbf{b}^{n+\frac{1}{2}} - \mathbf{j}^{n+\frac{1}{2}} \right)$$

$$\mathbf{b}^{n+1} = \mathbf{b}^{n+\frac{1}{2}} - \frac{\Delta t}{2} \mathbf{K}_{12} \mathbf{e}^{n+1}$$

$$\mathbf{p}_i^{n+1} = \mathbf{p}_i^{n+\frac{1}{2}} + \frac{\Delta t}{2} q_i \left[\mathbf{E}_{\text{loc}}^{n+1}(\mathbf{x}_i^{n+1}) + \frac{\mathbf{p}_i^{n+\frac{1}{2}}}{\gamma_i^{n+\frac{1}{2}} m_i c} \times \mathbf{B}_{\text{loc}}^{n+1}(\mathbf{x}_i^{n+1}) \right]$$

$\rightarrow$ calculate quantities (energy, momentum, ...)

IMPLEMENTATION

FEM-FIRE

- Time stepping
- Grid manager $\longrightarrow$ Dune
- Solver $\longrightarrow$ Trilinos
- Particles (IPPL)
- Boundary condition (Dirichlet, periodic, ABC)

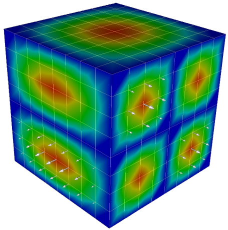
FEMSTER

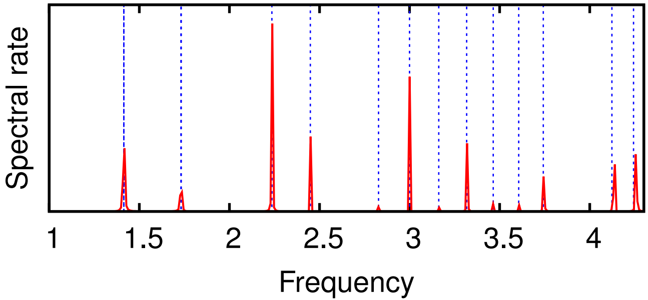
- local finite elements
- (hexahedral) basis functions of arbitrary polynomial degree

Language: C++
Future: powerful, parallel framework (Dune and Trilinos)

FIELD INTEGRATION TESTS

- symplectic integrator ⟶ conservation of energy
- Comparison of exact cavity eigenfrequencies with the spectrum of random field
  - → adjusting the grid size w.r.t emitting spectrum of the particles





Test case:
8^3 hex grid
order basis fct$=$ 2
$\Delta t = 0.005$
FFT of field comp at pos.
40'000 time steps

Initial Conditions

- Integrating Vlasov-Maxwell (no analytical solution for 3d)
- We need Maxwell conform initial conditions

given: ρ_{init}, $\mathbf{J}_{\text{init}}$ calculate: $\mathbf{E}_{\text{init}}$, $\mathbf{B}_{\text{init}}$

$$\begin{aligned} \mathbf{E}_{\text{init}} &= -\nabla\Phi - \dot{\mathbf{A}} \\ \mathbf{B}_{\text{init}} &= \nabla \times \mathbf{A} \end{aligned}$$

Coulomb Gauge: $\nabla \cdot \mathbf{A} = 0$

$$\begin{aligned} \Delta\Phi &= \rho \\ \Delta\mathbf{A} - \frac{1}{c^2}\ddot{\mathbf{A}} &= \mathbf{J}_t \end{aligned}$$

$$\mathbf{J} = \underbrace{\mathbf{J}_l}_{\nabla\times\mathbf{J}_l=0} + \underbrace{\mathbf{J}_t}_{\nabla\cdot\mathbf{J}_t=0}$$

Nature of $\mathbf{J}_{\text{init}}$

- no current: $\mathbf{J}_{\text{init}} = 0 \implies \mathbf{A} = \mathbf{B} = 0$
- magnetostatic: $\partial_t \mathbf{J}_{\text{init}} = 0 \implies \Delta\mathbf{A} = \mathbf{J}_t$

  moving bunch $\not\Rightarrow$ magnetostatic (bunch surface)
- dynamic: $\mathbf{J}_{\text{init}} = \mathbf{J}_\delta \implies \Delta\mathbf{A} - \frac{1}{c^2}\ddot{\mathbf{A}} = \mathbf{J}_t$

  hard to solve (as particles radiate in a certain bandwith)

  (eigenmode expansion, FFT)

$\mathbf{J}_{\text{INIT}} = 0$ – No initial Current

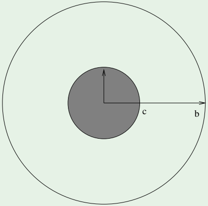
$$\begin{aligned} \Delta\Phi &= \rho \quad \text{in } \Omega \\ \Phi &= 0 \quad \text{on } \partial\Omega \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \mathbf{S}_0^{BC}\vec{\varphi} &= \mathbf{L}_0(\rho_\delta) \\ \mathbf{e} &= -\mathbf{K}_{01}\vec{\varphi} \end{aligned}$$

$\mathbf{S}_0$: Stiffness matrix of H^1

$\mathbf{K}_{01}$: Topological derivative matrix

Particle Cloud in spherical Cavity

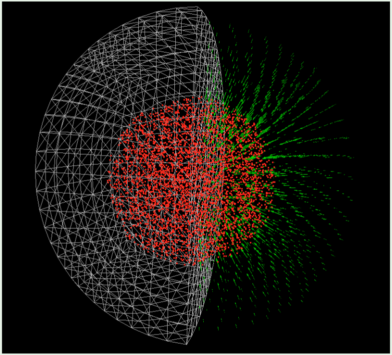
$$\Phi(r) = \frac{Q}{4\pi\epsilon_0} \cdot \begin{cases} \frac{1}{c}\left(\frac{3}{2} - \frac{r^2}{2c^2}\right) - \frac{1}{b} & \text{if } 0 \le r \le c \\ \frac{1}{r} - \frac{1}{b} & \text{if } c < r \le b \end{cases}$$



$$\begin{aligned} W_{\text{el}} &= \frac{Q^2}{40\pi\epsilon_0}\left(\frac{6}{c} - \frac{5}{b}\right) \\ &\approx \frac{1}{2}\mathbf{e}^T \mathbf{M}_1^{BC} \mathbf{e} \end{aligned}$$

Test:
16^3 Grid, $p = 2$, 4000 particles

$$\frac{\Delta W_{\text{el}}}{W_{\text{el}}} = 4.3\,‰$$



CONCLUSION

SELF-CONSISTENT FORMALISM FOR VLASOV-MAXWELL

- Vlasov-Maxwell is discretized by **FE** and **macro particles**
- coupling between fields and particles is **self-consistent**
- Particles in **local FE-cell-coordinates**
- tests show **stable** Poisson convergence and long-time-integration

NEXT STEPS

- **Penning Trap** (Code Validation)
- code is being **parallelized**

ONE HAMILTONIAN – TWO COUPLED SYSTEMS

HAMILTONIAN EQUATIONS OF MOTION

$$\dot{\mathbf{q}} = \nabla_{\mathbf{p}} H(\mathbf{p}, \mathbf{q})$$
$$\dot{\mathbf{p}} = -\nabla_{\mathbf{q}} H(\mathbf{p}, \mathbf{q})$$

$$H(\mathbf{p}, \mathbf{x}, \Phi, \mathbf{A}) := \frac{1}{2} \int_{\Omega\times\Pi} |\mathbf{p} - \mathbf{A}(\mathbf{x}, t)|^2 f(\mathbf{x}, \mathbf{p})\, d\mathbf{x} d\mathbf{p} + \frac{1}{2} \int_{\Omega} \left(|\nabla\Phi(\mathbf{x}, t)|^2 + |\nabla \times \mathbf{A}(\mathbf{x}, t)|^2 \right) d\mathbf{x}$$

GENERALIZED LEAP-FROG (TIME-STEPPING)

$$p^{n+\frac{1}{2}} = p^n - \frac{\Delta t}{2} \nabla_q H(p^{n+\frac{1}{2}}, q^n)$$
$$q^{n+1} = q^n + \frac{\Delta t}{2} \left[\nabla_p H(p^{n+\frac{1}{2}}, q^n) + \nabla_p H(p^{n+\frac{1}{2}}, q^{n+1}) \right]$$
$$p^{n+1} = p^{n+\frac{1}{2}} - \frac{\Delta t}{2} \nabla_q H(p^{n+\frac{1}{2}}, q^{n+1})$$

$$\Downarrow \quad \left[f \to \sum_i \text{particle}_i \right]$$

$$H_{\text{part}}(\vec{\mathbf{p}}, \vec{\mathbf{x}}) = \sum_{i=1}^{N_{\text{part}}} c \sqrt{\left(\mathbf{p}_i - \frac{q_i}{c} \mathbf{A}(\mathbf{x}_i, t)\right)^2 + m_i^2} + q_i \Phi(\mathbf{x}_i, t)$$

Fully relativistic!

$$\dot{\mathbf{x}}_i = \frac{1}{\gamma_i m_i} \mathbf{p}_i$$
$$\dot{\mathbf{p}}_i = q_i \left[\mathbf{E}(\mathbf{x}_i, t) + \frac{\mathbf{p}_i}{\gamma_i m_i c} \times \mathbf{B}(\mathbf{x}_i, t) \right]$$

$$\mathbf{p}_i = \gamma_i m_i \mathbf{v}_i + \frac{q_i}{c} \mathbf{A}(\mathbf{x}_i, t), \quad \gamma_i = \sqrt{\frac{1}{1-\beta^2}}, \quad \beta = \frac{v}{c}$$

$$\dot{\mathbf{B}} = -\nabla \times \mathbf{E}$$
$$\epsilon_0 \dot{\mathbf{E}} = \nabla \times \mu^{-1} \mathbf{B} - \sum_i q_i \mathbf{v}_i$$

BC: $\mathbf{E} \perp \partial\Omega, \quad \mathbf{B} \parallel \partial\Omega$

$$\mathbf{E} = -\dot{\mathbf{A}} - \nabla\Phi, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Variational Formulation

$$\begin{aligned}
\dot{\mathbf{B}} &= -\nabla \times \mathbf{E} \\
(\mu\dot{\mathbf{B}}, \mathbf{B}')_\Omega &= -(\mu\nabla \times \mathbf{E}, \mathbf{B}')_\Omega \\
(\mu\mathbf{f}_i, \mathbf{f}_j)_\Omega \dot{\mathbf{b}} &= -(\mu\nabla \times \mathbf{w}_i, \mathbf{f}_j)_\Omega \mathbf{e} \\
\mathbf{M}_2\dot{\mathbf{b}} &= -\mathbf{M}_2\mathbf{K}_{12}\mathbf{e} \\
\dot{\mathbf{b}} &= -\mathbf{K}_{12}\mathbf{e}
\end{aligned}$$

$$\begin{aligned}
\epsilon\dot{\mathbf{E}} &= \nabla \times \mu^{-1}\mathbf{B} - \mathbf{J} \\
(\epsilon\dot{\mathbf{E}}, \mathbf{E}')_\Omega &= (\nabla \times \mu^{-1}\mathbf{B}, \mathbf{E}')_\Omega - (\mathbf{J}, \mathbf{E}')_\Omega \\
(\epsilon\dot{\mathbf{E}}, \mathbf{E}')_\Omega &= (\mu^{-1}\mathbf{B}, \nabla \times \mathbf{E}')_\Omega - (\mu^{-1}\mathbf{B}, \mathbf{E}')_{\partial\Omega} - (\mathbf{J}, \mathbf{E}')_\Omega \\
(\epsilon\mathbf{w}_i, \mathbf{w}_j)_\Omega \dot{\mathbf{e}} &= (\mu^{-1}\mathbf{f}_i, \nabla \times \mathbf{w}_j)_\Omega \mathbf{b} - (\mathbf{J}, \mathbf{w}_i)_\Omega \\
\mathbf{M}_1^{BC}\dot{\mathbf{e}} &= \mathbf{K}_{12}^T\mathbf{M}_2\mathbf{b} - \mathbf{L}_1(\mathbf{J})
\end{aligned}$$

SPATIAL + TEMPORAL DISCRETIZATION

GALERKIN DISCRETIZATION

- multiply by test function
- integrate over domain
- plug in basis of discretized function space

$\mathbf{E}, \mathbf{E}' \in H(curl;\Omega)$
$\mathbf{B}, \mathbf{B}' \in H(div;\Omega)$

$\mathbf{K}_{12}$: Topological Curl-matrix
$\mathbf{M}_1$: Curl-mass-matrix
$\mathbf{M}_2$: Div-mass-matrix

$$\begin{aligned} \dot{\mathbf{B}} &= -\nabla \times \mathbf{E} \\ &\downarrow \\ \dot{\mathbf{b}} &= -\mathbf{K}_{12}\mathbf{e} \end{aligned}$$

$$\begin{aligned} \epsilon\dot{\mathbf{E}} &= \nabla \times \mu^{-1}\mathbf{B} - \mathbf{J} \\ &\downarrow \\ \mathbf{M}_1^{BC}\dot{\mathbf{e}} &= \mathbf{K}_{12}^T\mathbf{M}_2\mathbf{b} - \mathbf{L}_1(\mathbf{J}) \end{aligned}$$

TIMESTEPPING

$$\begin{aligned} \mathbf{b}^{n+\frac{1}{2}} &= \mathbf{b}^n - \frac{\Delta t}{2}\mathbf{K}_{12}\mathbf{e}^n \\ \mathbf{M}_1^{BC}\mathbf{e}^{n+1} &= \mathbf{M}_1^{BC}\mathbf{e}^n + \Delta t\left(\mathbf{K}_{12}^T\mathbf{M}_2\mathbf{b}^{n+\frac{1}{2}} - \mathbf{j}^{n+\frac{1}{2}}\right) \\ \mathbf{b}^{n+1} &= \mathbf{b}^{n+\frac{1}{2}} - \frac{\Delta t}{2}\mathbf{K}_{12}\mathbf{e}^{n+1} \end{aligned}$$

- EM-FETD is well established
- BC: Dirichlet, periodic, ABC

MOTIVATION OF DIFFERENTIAL FORMS

FARADAY EQUATION

$$
\begin{aligned}
\partial_t \mathbf{B} &= -\nabla \times \mathbf{E} \\
\partial_t \int_F \mathbf{B} \cdot d\mathbf{F} &= -\int_s (\nabla \times \mathbf{E}) \cdot d\mathbf{s} \\
\partial_t \mathcal{B} &= -d\mathcal{E} \\
\partial_t \int_F \mathcal{B} &= -\int_{\partial F} d\mathcal{E}
\end{aligned}
$$

STOKES

$$\int_\Omega d\omega = \int_{\partial\Omega} \omega$$

STRUCTURE OF ELECTROMAGNETISM W.R.T. DIFFERENTIAL FORMS

PROPERTIES OF K-FORMS

- Independent of any metric
- Material transitions are modelled correctly
- k-forms are **isometric** to the Hilbert spaces
  $L^2(\Omega) = \{u \in \mathcal{C}^k, k \in \mathbb{N}_0 | \int_\Omega |u|^2 dV < \infty\}$
  $H^1(\Omega) = \{u \in L^2(\Omega) | \nabla u \in L^2(\Omega)\}$
  $H(curl;\Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 | \nabla \times \mathbf{v} \in (L^2(\Omega))^3\}$
  $H(div;\Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 | \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$

K-FORM OPERATORS

Assume $g \in \Psi^k$, $h \in \Psi^l$

- $\wedge$: $\Psi^k \wedge \Psi^l \rightarrow \Psi^{k+l}$, $\quad g \wedge h = -h \wedge g$
- d : $d\Psi^k \rightarrow \Psi^{k+1}$, $\quad ddg = 0$
- $\star$: $\star\Psi^k \rightarrow \Psi^{3-k}$, $\quad \star\star g = g$

DE RHAM DIAGRAM

$$
\begin{array}{ccccccccccc}
 & & \begin{matrix}0-form\\ H^1(\Omega)/\mathbb{R}\end{matrix} & \xrightarrow[\nabla]{d} & \begin{matrix}1-form\\ H(curl;\Omega)\end{matrix} & \xrightarrow[\nabla\times]{d} & \begin{matrix}2-form\\ H(div;\Omega)\end{matrix} & \xrightarrow[\nabla\cdot]{d} & \begin{matrix}3-form\\ L^2(\Omega)\end{matrix} & \xrightarrow[\nabla]{d} & 0 \\
 & & \downarrow\star & & \downarrow\star & & \downarrow\star & & \downarrow\star & & \\
0 & \xleftarrow[\nabla]{d} & \begin{matrix}3-form\\ L^2(\Omega)\end{matrix} & \xleftarrow[\nabla\cdot]{d} & \begin{matrix}2-form\\ H(div;\Omega)\end{matrix} & \xleftarrow[\nabla\times]{d} & \begin{matrix}1-form\\ H(curl;\Omega)\end{matrix} & \xleftarrow[\nabla]{d} & \begin{matrix}0-form\\ H^1(\Omega)/\mathbb{R}\end{matrix} & &
\end{array}
$$