Peculiarities of RG Flows in 2D $O(n)$ Models

with B. Zan (2005.07708)

+ WIP with Trufanov, Qiao, Zan, Zhabin

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Disclaimer:

I am not an expert on the stat-mech models I am going to talk about.

I am a high energy theorist.

I study these theorieos because the challenge some intuition about RG flows in continuum field theories that I have.

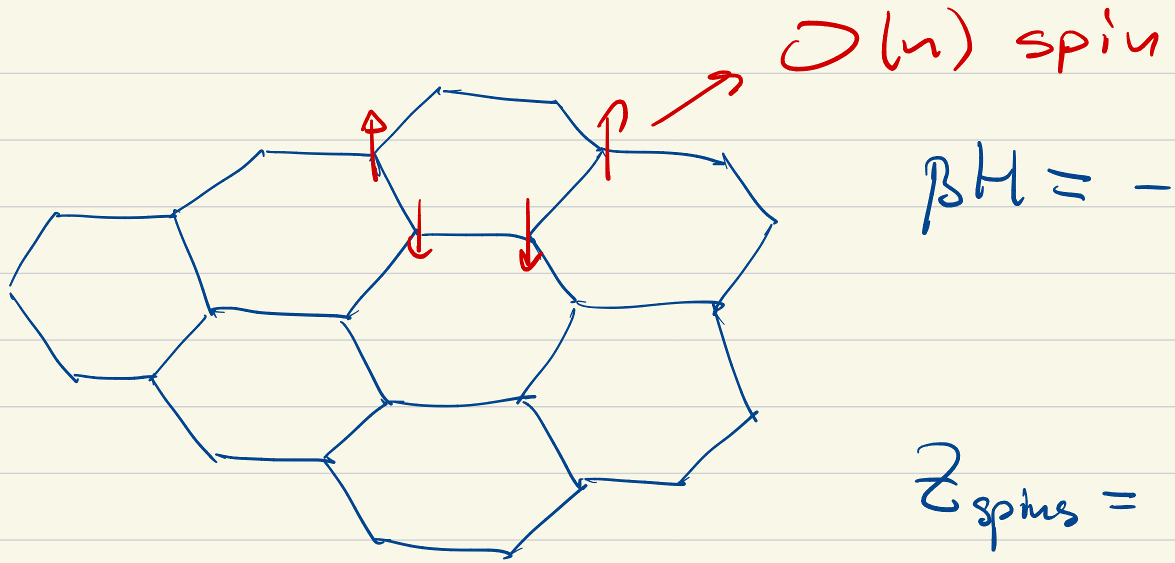
Also I will not try to give all the refs.

Summary:

- O(n) model as a loop model and the structure of its critical points
- A puzzle with RG flows
- Logarithmic CFTs, negative norm states, bootstrap and O(n) representations (technical)
- Resolution of the puzzle
- New puzzle and some speculations
  ↳ [based on numerical data]

$O(n)$ model

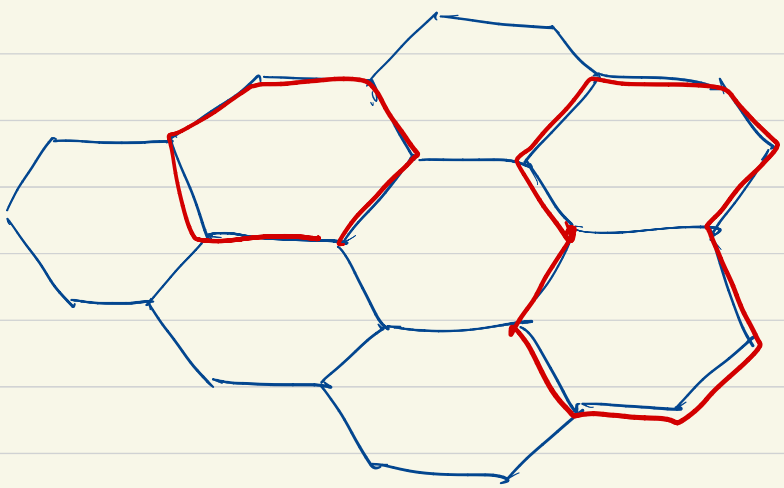
- We start by defining the $O(n)$ model on a honeycomb lattice for arbitrary n.
- It has $O(n)$ symmetry (it can be defined rigorously: Binder, Rychkov)
    - each operator ~ irrep of $O(n)$
    - products of op's ~ products of irreps
    - preserved under RG
- The models are local but non-unitary [not reflection-positive]



$$\beta H = -\sum_{\langle ij \rangle} \log(1 + K\, S_i \cdot S_j)$$

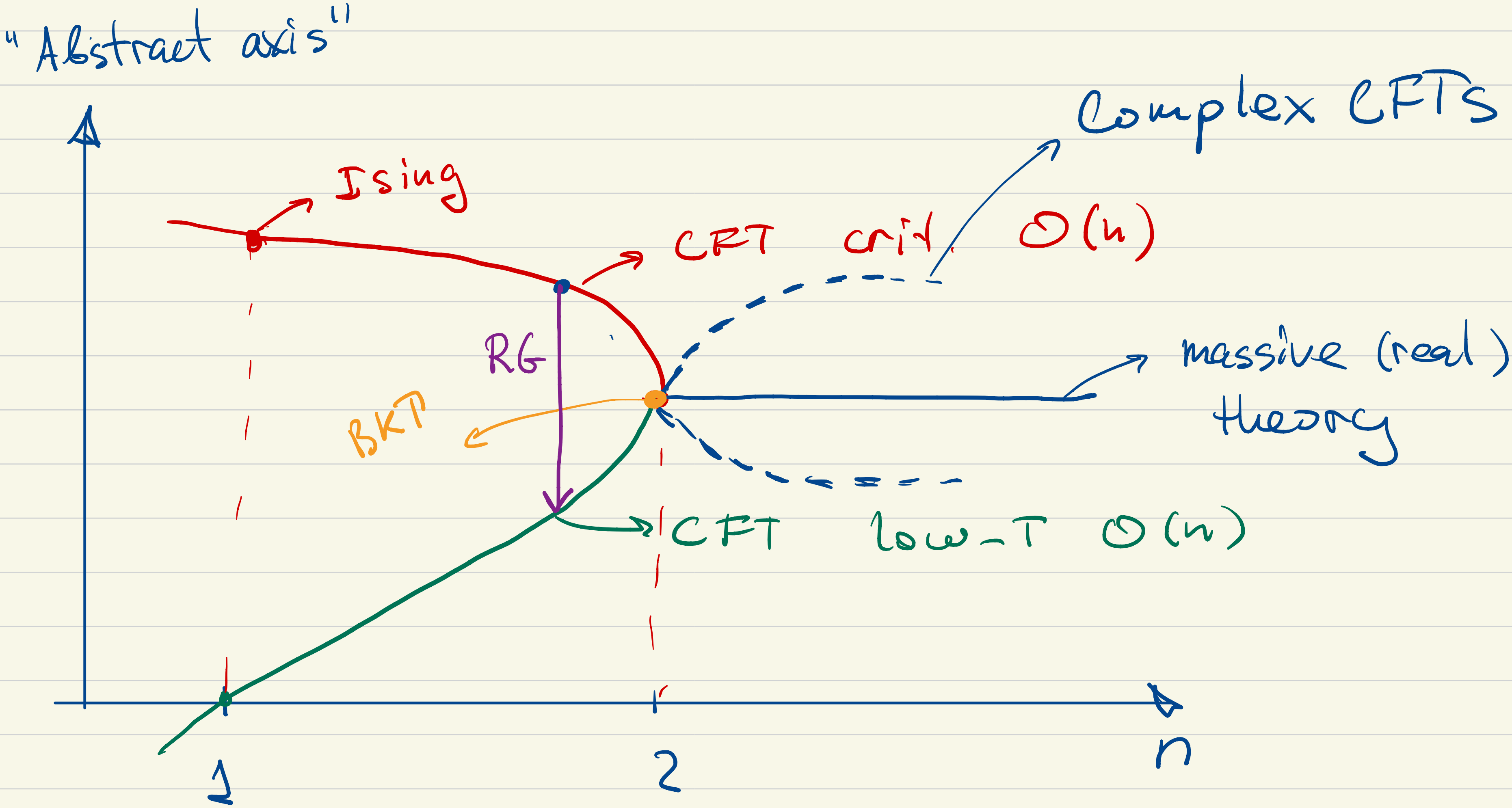
$$Z_{\text{spins}} = \int \prod_i dS_i \, e^{-\beta H}$$

$$\Bigg\|$$ integer n



$$Z_{\text{loops}} = \sum_{\text{loops}} K^{N_{\text{links}}} n^{N_{\text{loops}}}$$

- In the continuum limit the model gives rise to two families of CFTs



- In this talk we focus on $n \leq 2$

$n \leq 2$ Torus partition function known:

$$Z_{O(n)} = (q\bar{q})^{-\frac{c}{24}} \operatorname{Tr} q^{L_0} \bar{q}^{\bar{L}_0}.$$

$$e_0 = g - 1 \qquad g = \arccos\left(\frac{n}{2}\right) \qquad x_{em}, \bar{x}_{em} = \frac{1}{4}(e/\sqrt{g} \pm m\sqrt{g})^2$$

two branches correspond to two CFTs

Polynomials of n

$$Z_{O(n)} = \frac{1}{\eta\bar{\eta}} \sum_{P \in \mathbb{Z}} (q\bar{q})^{x_{e_0+2P,0}} + \frac{1}{\eta\bar{\eta}} \sum_{\substack{M,N=1 \\ N \text{ divides } M}}^{\infty} \Lambda(M,N) \sum_{\substack{P \in \mathbb{Z} \\ P \wedge N = 1}} q^{x_{2P/N,M/2}} \bar{q}^{\bar{x}_{2P/N,M/2}}. =$$

$$= (q\bar{q})^{-\frac{c}{24}} \sum_{\text{primaries}} \overbrace{N_{h\bar{h}}(n)}^{\text{multiplicity of primaries}} \chi_{h\bar{h}}(q,\bar{q})$$

$$N_{h\bar{h}}(n) = \operatorname{Dim}(\operatorname{Rep.}(O(n)))$$

- Some light scalar primaries and multiplicities:

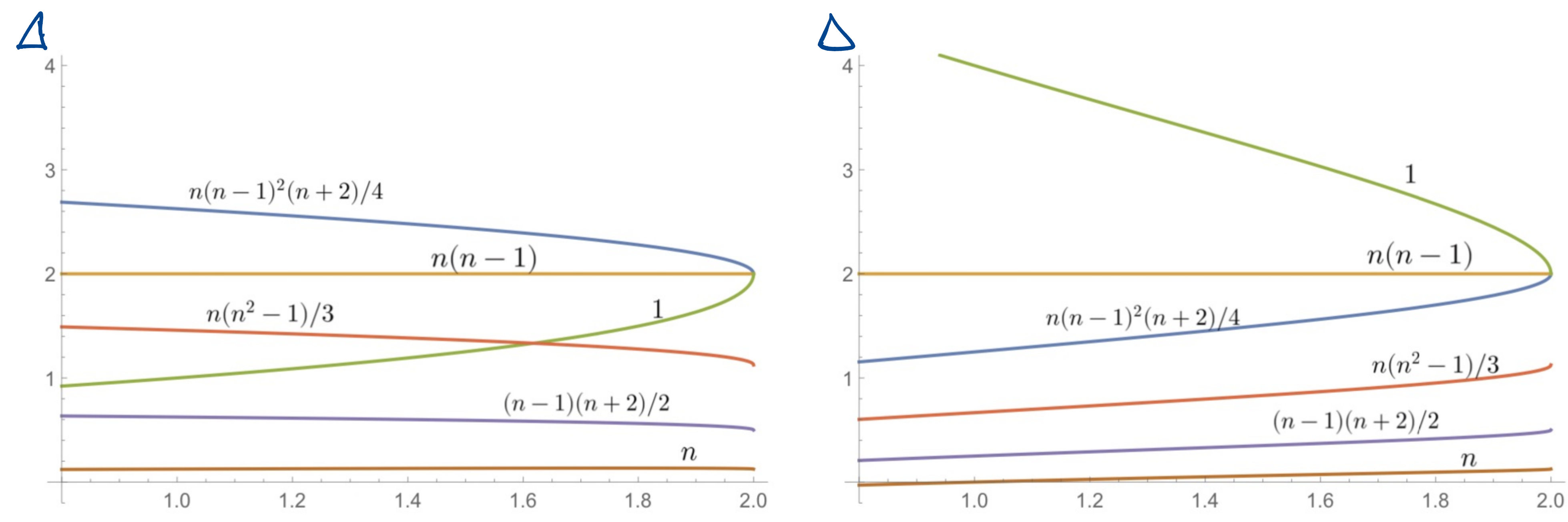


Figure 2: Dimensions and multiplicities of light scalar operators as functions of n in the critical (left) and low-T fixed points (right).

see 2111.01106 for a very detailed understanding of reps and multiplicities

BKT Fixed Point

At $n=2$ both theories approach the $c=1$ BKT CFT (how this happens is subtle)

Free boson $R = \frac{1}{\sqrt{2}}$

$O(2) \times O(2)$ symmetry

$$S = \frac{1}{8\pi} \int d^2x \, (\partial \varphi)^2$$

Operator spectrum is known, OPE's calculable

$$V_{n,m} = e^{n\varphi + m\bar{\varphi}} + \\ + P(\partial\varphi, \bar{\partial}\varphi)$$

$$Z_{O(n)} \xrightarrow[n\to 2]{} Z_{BKT}$$

In particular, there are 5 marginal operators in BKT: $\partial\varphi\bar{\partial}\varphi$ + 4 vertex

$$1 + n(n-1) + \frac{1}{4}n(n-1)^2(n+2)\Big|_{n=2} = 5\,.$$

# of close to marginal Op's: seems smooth

There is a puzzle:

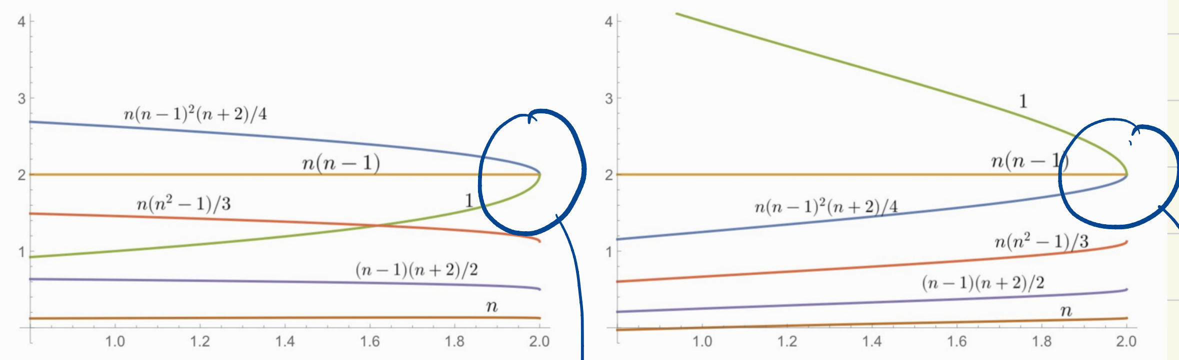
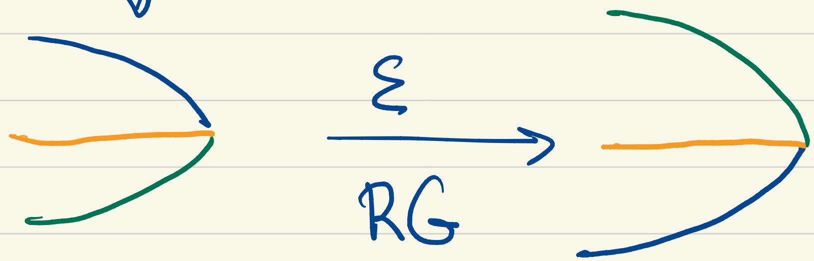


Figure 2: Dimensions and multiplicities of light scalar operators as functions of n in the critical (left) and low-T fixed points (right).



This RG flow puts constraints on OPE's and there turns out to be no consistent match of 5 operators to the BKT spectrum!

Logarithmicity

In a non-unitary theory D may not be diagonalizable. Instead it can have a Jordan block structure:

$$D \sim \begin{pmatrix} \Delta & 1 \\ 0 & \Delta \end{pmatrix}$$

In the symplest case this leads to the following structure:

$$\langle A A \rangle = \frac{\log |z|^2}{|z|^{2\Delta}}$$

$$\langle B B \rangle = 0 \longrightarrow \text{zero norm state}$$

$$\langle B A \rangle = -\frac{1}{|z|^{2\Delta}}$$

- We will use Bootstrap + consistency of $n \to 2$ limit to uncover this structure in the $O(n)$ model.

Let us study in details the currents multiplet:

$$\underbrace{\frac{n(n-1)}{2}\ J\text{'s}}_{\text{Adj}}^{(0,1)} + \frac{n(n-1)}{2}\ \bar{J}\text{'s}^{(1,0)}$$

Basic idea: $\bar{\partial} J \neq \partial \bar{J} \neq 0$

$\bar{\partial} J - \partial \bar{J} = \partial_\mu J^\mu = 0 \Rightarrow$ one global $O(n)$ sym.

$\bar{\partial} J + \partial \bar{J} = B$ [marginal operator!]

This forms a more complicated rep. called "staggered module":

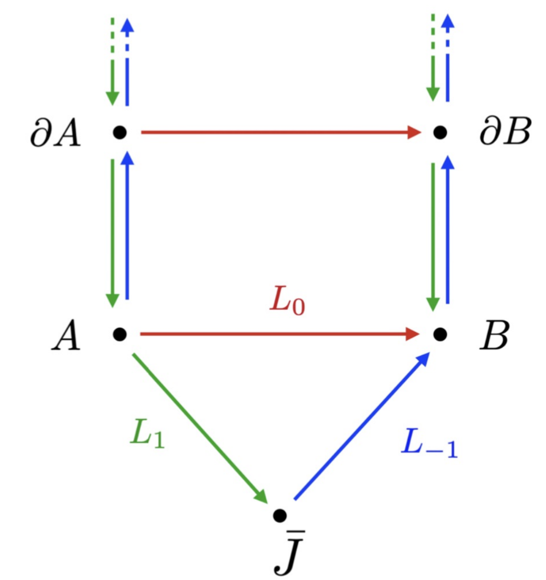


Figure 3: The structure of the staggered module for the currents (colors online). For simplicity we only indicate the action of L_0 and $L_{\pm 1}$. B is both a descendant and a primary, while A is neither a descendant nor a primary.

To test this picture we use analytic bootstrap.

Study $\langle J J \varepsilon \varepsilon \rangle$ correlator.

ε is degenerate at level 3: (for any n)

$$\left(\mathcal{L}_{-3} - \frac{2(m+1)}{3m+1}\mathcal{L}_{-1}\mathcal{L}_{-2} + \frac{(m+1)^2}{2m(3m+1)}\mathcal{L}_{-1}^3\right)\langle \varepsilon(z)\phi_1(z_1)\ldots\phi_n(z_n)\rangle = 0$$

$$\mathcal{L}_{-k} = \sum_{i=1}^{n}\left(\frac{(k-1)h_i}{(z_i-z)^k} - \frac{1}{(z_i-z)^{k-1}}\partial_{z_i}\right).$$

$$m(n) = \frac{\pi}{\cos^{-1}\left(\frac{n}{2}\right)},$$

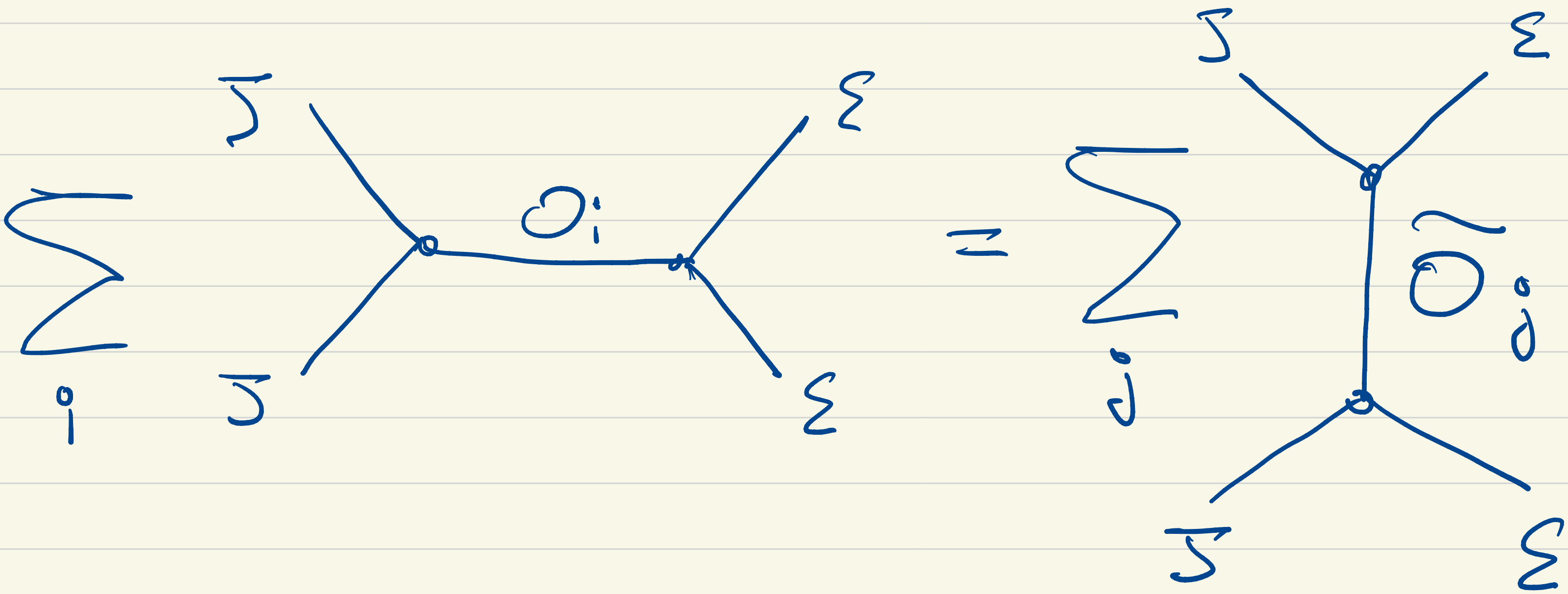
$n \to 2$

$m \to \infty$

(there is the same anti-holomorphic eqn.)

This eqn. has 3 solutions, that depend on h_i's

We then look for the solution as a series expansion in two channels:



(BP2)

These equations allow to find the free parameters in the equations.

For a generic n some solutions have $\log(z)$'s $\rightarrow$ this corresponds to logarithmic multiplets

$n \to \infty$

$u \to 2$

we find, e.g.

$$\lambda_{JJ\varepsilon} = \frac{2}{\sqrt{3}\, m} + O\left(\frac{1}{m^2}\right)$$

$$\lambda_{\delta \bar{J} \varepsilon} = \pm \frac{1}{\sqrt{3}} + O\left(\frac{1}{m^2}\right)$$

$$\lambda_{AS\varepsilon} = -\sqrt{\frac{m}{6}} + O(1)$$

critical 2d $O(n)$ model is logarithmic for a generic n

Back to our puzzle. Let's define operators

$$\psi_1 = \frac{\sqrt{m}}{r} B - \frac{r}{\sqrt{m}} A \qquad \psi_2 = \frac{\sqrt{m}}{r} B + \frac{r}{\sqrt{m}} A$$

$$\langle \psi_1 \psi_1 \rangle = \frac{1}{|z|^4} \qquad \langle \psi_2 \psi_2 \rangle = -\frac{1}{|z|^4}$$

$$\langle \psi_1 \psi_2 \rangle = 0$$

[at leading order in $\frac{1}{m}$]

reminder:

$$\langle A A \rangle = \frac{\log |z|^2}{|z|^{2\Delta}} \qquad \langle B A \rangle = -\frac{1}{|z|^{2\Delta}} \qquad \langle B B \rangle = 0$$

We can also compute OPE's!

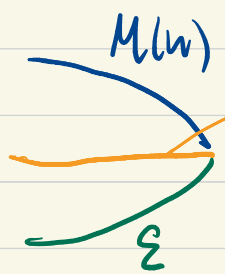
$$\lambda_{\psi_1 \sigma \varepsilon} = \frac{2}{\sqrt{3}} + O\left(\sqrt{n-2}\right)$$

$$\lambda_{\psi_2 \sigma \varepsilon} = O\left(\sqrt{n-2}\right)$$

$\longrightarrow$ ψ_2 decouples from σ, ε

In fact, close to $n=2$ there are more than 5 close to marginal operators "hiding":

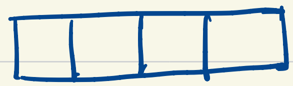
$$\overset{\varepsilon}{\uparrow}\!\!\!\;1 + \overset{A,\,B}{\overbrace{n(n-1)}} + \overset{M(n)}{\overbrace{\frac{1}{4}n(n-1)^2(n+2)}}\Big|_{n=2} = 5\,.$$

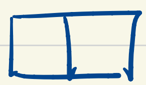
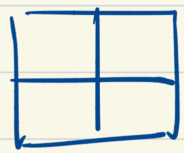
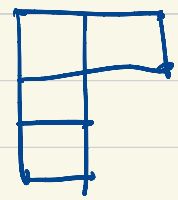


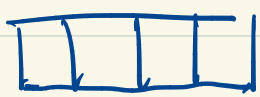
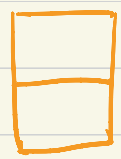
$A, B \longrightarrow \psi_1, \psi_2$

$$M(n) = \underset{\displaystyle 2}{\underset{\big\downarrow\, n\to 2}{\frac{1}{4}n(n-1)^2(n+2)}} = \underset{\displaystyle 3}{\underset{\big\downarrow\, n\to 2}{M_+(n)}} + \underset{\displaystyle -1}{\underset{\big\downarrow\, n\to 2}{M_-(n)}}$$

We found a unique consistent decomposition of $M(n) = M_+(n) + M_-(n)$:

$M_+(n) = \bullet +$  $\xrightarrow[n\to 2]{} 3$

$M_-(n) =$  $+$  $+$  $\xrightarrow[n\to 2]{} -1$

$5 = \bullet +$  $+$  $+ \overset{\varepsilon}{\bullet}$

ψ_2

ψ_1 and $M_-(n)$ operators decouple

$$\mathcal{A}(n) \longrightarrow \mathcal{A}_{BKT} + \overline{\mathcal{A}}$$

$\downarrow$

Algebra of all operators

$$\psi_1, \; \mu_- \in \overline{\mathcal{A}}$$

$\mathcal{A}_{BKT}$ is a closed subalgebra:

$$\mathcal{O} \times \mathcal{O} \to \mathcal{O}$$

$$\overline{\mathcal{O}} \times \overline{\mathcal{O}} \to \mathcal{O}, \; \overline{\mathcal{O}}$$

$$\overline{\mathcal{O}} \times \mathcal{O} \to \mathcal{O}, \; \overline{\mathcal{O}}$$

$$Z_{A(2)} = Z_{BKT} \Rightarrow$$

$$Z_{\overline{A}} = 0$$

- Now with all the operators included the RG flow is consistent!

- It has some interesting features...

- The most important conclusion is that one "blue" operator is a $O(n)$ singlet (we call it S)

- This leads to the new puzzle [known before, but not formulated in this language]

Summary:

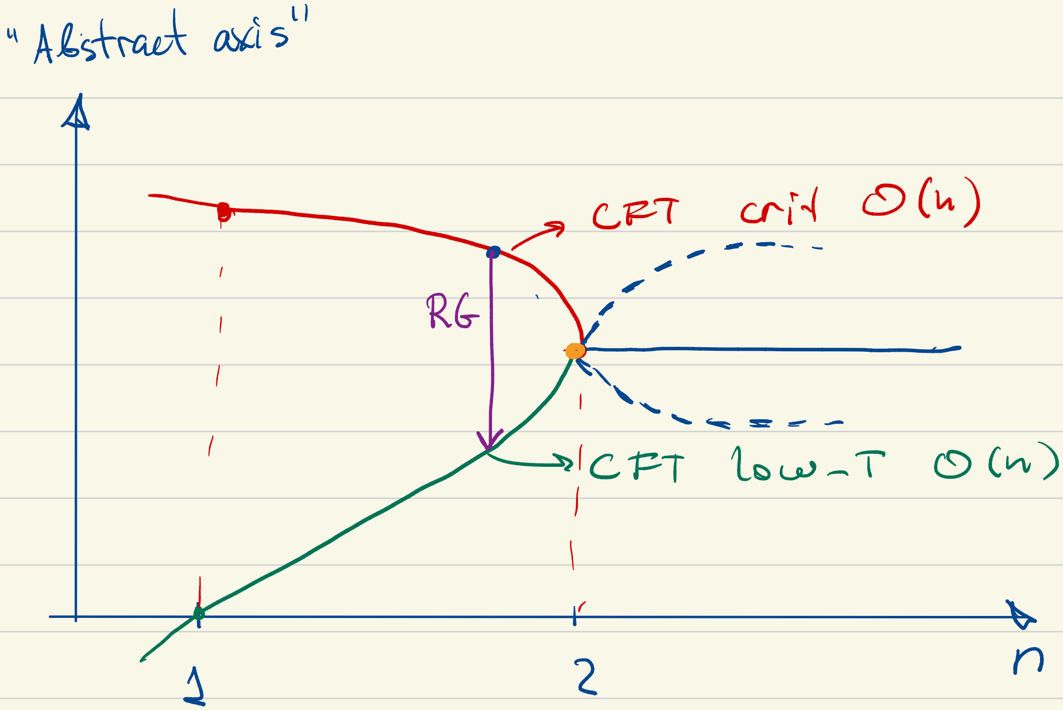
✓ $O(n)$ model as a loop model and the structure of its critical points

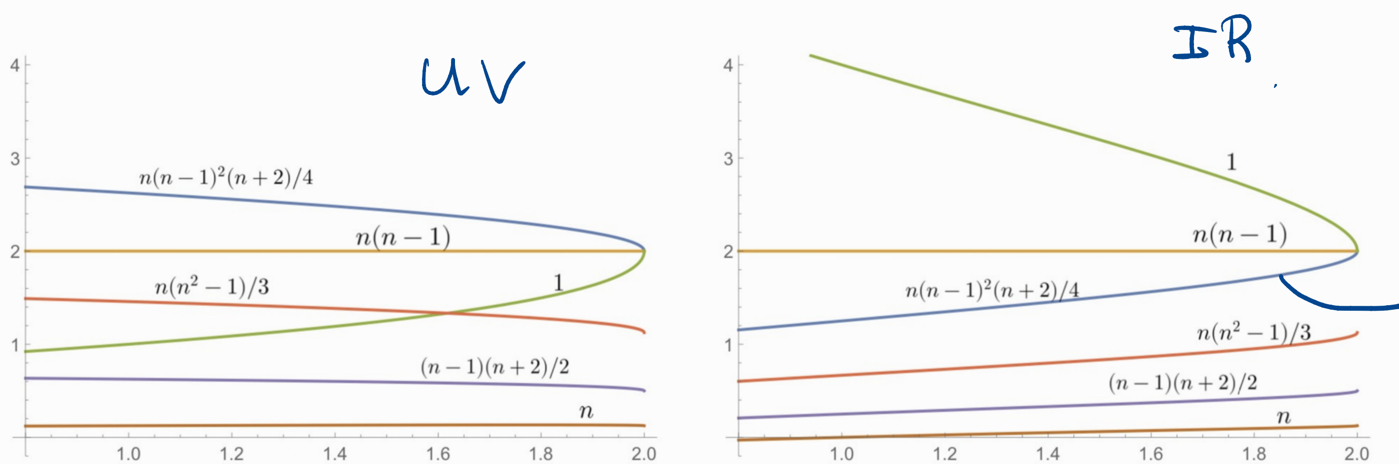
✓ A puzzle with RG flows

✓ Logarithmic CFTs, negative norm states, bootstrap and $O(n)$ representations (technical)

✓ Resolution of the puzzle

→ New puzzle and some speculations





relevant singlet operator!

Figure 2: Dimensions and multiplicities of light scalar operators as functions of n in the critical (left) and low-T fixed points (right).

- No tuning is needed in the UV to reach the low-T fixed point

- This seems to be true for many non-intersecting lattice loop models on various lattices (not just the RG from CFT_{crit})

H. W. J. Blote and B. Nienhuis, "Critical behaviour and conformal anomaly of the o(n) model on the square lattice," *Journal of Physics A: Mathematical and General* **22** no. 9, (May, 1989) 1415–1438. https://doi.org/10.1088%2F0305-4470%2F22%2F9%2F028.

W. Guo and H. W. J. Blöte, "Crossover phenomena involving the dense o(n) phase," *Phys. Rev. E* **83** (Feb, 2011) 021115. https://link.aps.org/doi/10.1103/PhysRevE.83.021115.

→ studies non-integrable versions.

J. Jacobsen, N. Read, and H. Saleur, "Dense loops, supersymmetry, and Goldstone phases in two-dimensions," *Phys. Rev. Lett.* **90** (2003) 090601, arXiv:cond-mat/0205033.

H. W. Blote and B. Nienhuis, "The phase diagram of the o(n) model," *Physica A: Statistical Mechanics and its Applications* **160** no. 2, (1989) 121 – 134.

- The lore is that any relevant singlet coupling has to be tuned...

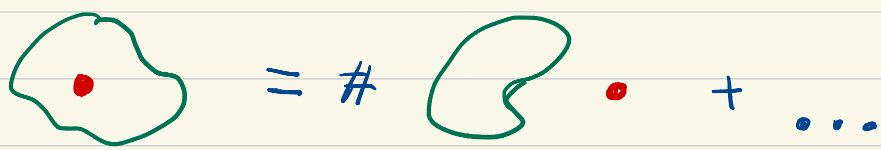
- No global symmetry is seen that can protect generation of S:

1) $\lambda_{SSS} = -\frac{2\sqrt{2}}{\sqrt{3}}$ $\lambda_{SS\varepsilon} = -\frac{2}{\sqrt{3}}$ $\Rightarrow$ no $\mathbb{Z}_2$

2) We only have $\frac{n(n-1)}{2}$ currents, so no larger global symmetry that can mix S with other "blue" operators

- To me the puzzle remains open. Possible resolutions could be:

$\rightsquigarrow$ a Non-invertible categorical symmetry under which S is charged and ε is not



$\rightsquigarrow$ Quantum group structure which protects S and that is not broken by ε.

(Fendley, Saleur, Zamolodchikov '94)

- At least some versions of the $O(n)$ model are invariant under the $SL_q(2)$ symmetry $(2\cos q = n)$
- I expect that in the continuum it acts on the local operators in some non-local way, but still commutes with Virasoro...
- Likely, it is also responsible for degeneracies of primaries inside $M(n)$ and on the other levels

- Whatever the mechanism is it
  - → operate for a large class of loop models without intersections and be broken by intersections
  - → be preserved under RG
  - → remain in the continuum and distinguish local operators $\mathcal{E}$ and S

Conclusions

- We studied the critical points of $O(n)$ model and RG flow between them

- We discovered that the model is logarithmic for any n (so is Potts for any Q)

- $n \to 2$ limit is continuous, but happens in an interesting way: more operators are in the game than naively seems

- One of them is a relevant (in low-T) singlet

The situations when singlet relevant operators are not produced by RG are very rare (I know two examples). I think it is worth investigating such examples both analytically and numerically as thoroughly as possible.

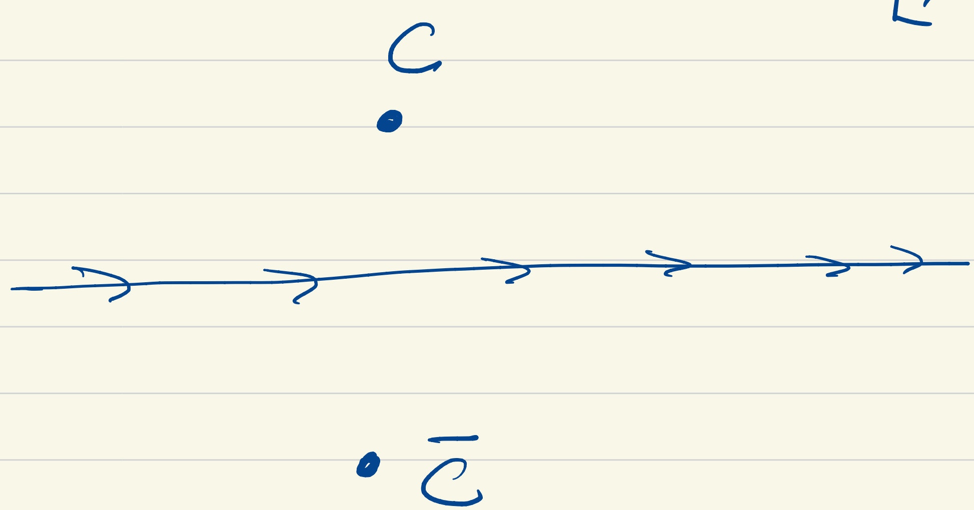
$n > 2$ and complex CFTs

Remember spectrum of operators:

$$g = \arccos\left(\frac{n}{2}\right) \qquad x_{em}, \bar{x}_{em} = \frac{1}{4}\left(e/\sqrt{g} \pm m\sqrt{g}\right)^2$$

$\hookrightarrow \sim \sqrt{2-n} = \delta$

We can analytically continue all CFT data



9. "RG-periodic" S-matrix

L. Cordova and P. Vieira, "Adding flavour to the S-matrix bootstrap," *JHEP* **12** (2018) 063, arXiv:1805.11143 [hep-th].

L. Cordova, Y. He, M. Kruczenski, and P. Vieira, "The O(N) S-matrix Monolith," arXiv:1909.06495 [hep-th].

M. Hortacsu, B. Schroer, and H. Thun, "A Two-dimensional σ Model With Particle Production," *Nucl. Phys. B* **154** (1979) 120–124.

A. Zamolodchikov, "Exact S matrix associated with selfavoiding polymer problem in two-dimensions," *Mod. Phys. Lett. A* **6** (1991) 1807–1814.

$$S_{sing} = -e^{-ik\theta} \prod_{l=1}^{\infty} \frac{\sinh[k(i\theta - 2l\pi)] \sinh[k(i\theta + (2l+1)\pi)]}{\sinh[k(i\theta + 2l\pi)] \sinh[k(i\theta - (2l+1)\pi)]}, \qquad k = \frac{\operatorname{arccosh}\frac{n}{2}}{\pi}.$$

periodic in $\theta \to \theta + 2\pi^2/\operatorname{arccosh}\frac{n}{2}$

$$S \xrightarrow[\theta\to\infty]{} e^{-ik\theta}$$