Entanglement Scaling and Criticality with Tensor Networks
at Bernouilli Center,

Nov 28—Dec 2, 2022

Finite-Size "Level Spectroscopy" Approach with Tensor Network Renormalization

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Tensor Networks:

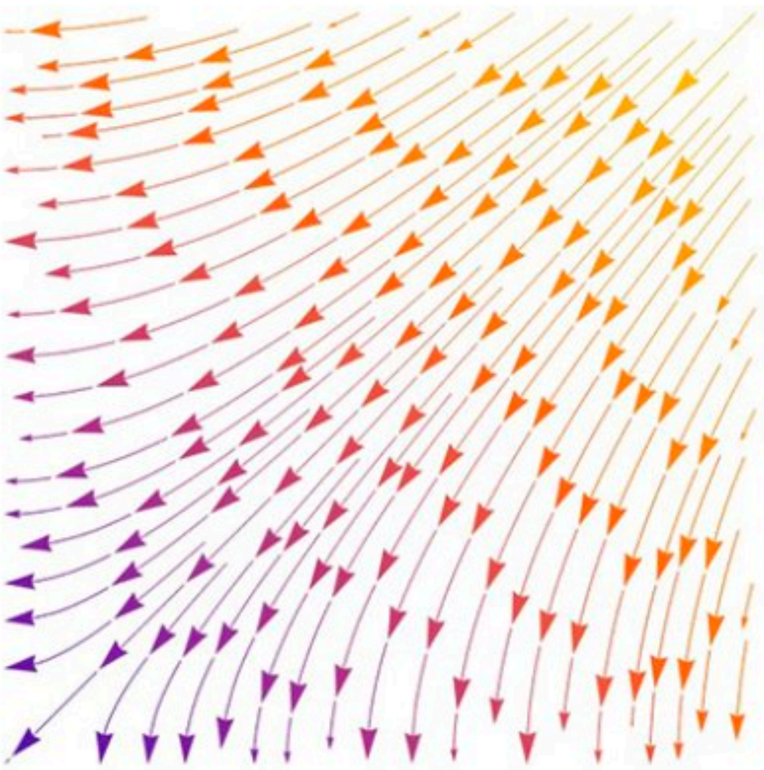
very powerful numerical/conceptual tool to study quantum many-body systems & stat mech

Finite "bond dimension":

TN can describe limited entanglement/correlation length

how to study critical phenomena?

- change the topology of the Tensor Networks (MERA, etc.)
- **finite-size scaling,**
  in particular of Transfer Matrix spectra using Tensor Networks



Atsushi Ueda

EDITORS' SUGGESTION

Resolving the Berezinskii-Kosterlitz-Thouless transition in the two-dimensional XY model with tensor-network-based level spectroscopy

The discovery of the Berezinskii-Kosterlitz-Thouless transition some fifty years ago was a subject of the 2016 Nobel Prize in Physics. However, quantitative study of the transition in the two-dimensional XY spin model still suffers from significant finite-size effects. The authors implement the level-spectroscopy method, originally developed for quantum systems. They utilize the modern tensor-network renormalization scheme. This allows for an extremely accurate determination of the critical point as well as for a visualization of the celebrated Kosterlitz renormalization-group flow.

Atsushi Ueda and Masaki Oshikawa

Phys. Rev. B **104**, 165132 (2021)

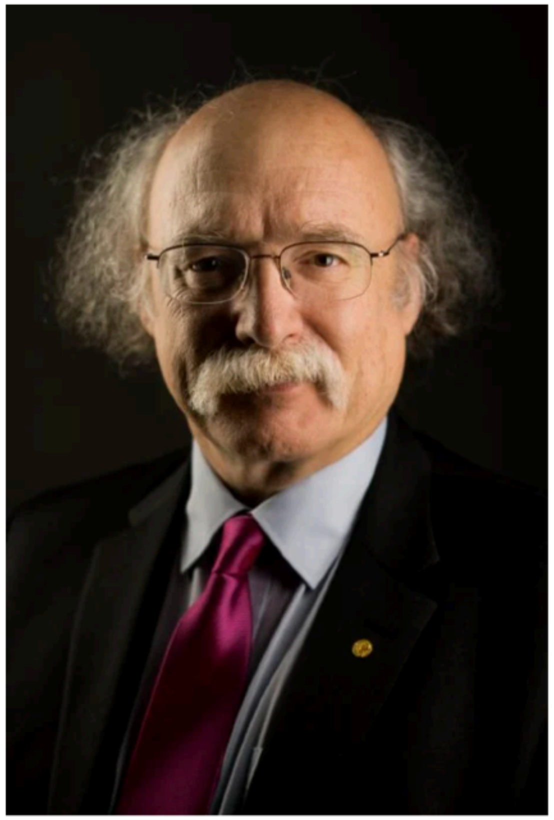
+ Phys. Rev. E **106**, 014104 (2022)

+ in preparation

The Nobel Prize in Physics 2016

David J. Thouless

Prize share: 1/2



F. Duncan M. Haldane

Prize share: 1/4

J. Michael Kosterlitz

Prize share: 1/4

The Nobel Prize in Physics 2016 was awarded with one half to David J. Thouless, and the other half to F. Duncan M. Haldane and J. Michael Kosterlitz "for theoretical discoveries of topological phase transitions and topological phases of matter."

“Topological Phase Transitions”

Prototype: Berezinskii-Kosterlitz-Thouless Transition
which is also closely related to “Haldane Gap”

Canonical model for the BKT transition:
2D classical XY model

$$H_{XY} = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

Classical stat mech: no negative sign problem, etc.
Easily studied with Monte Carlo, right?

Journal of the Physical Society of Japan **81** (2012) 113001 LETTERS
http://dx.doi.org/10.1143/JPSJ.81.113001

Large-Scale Monte Carlo Simulation of Two-Dimensional Classical XY Model Using Multiple GPUs

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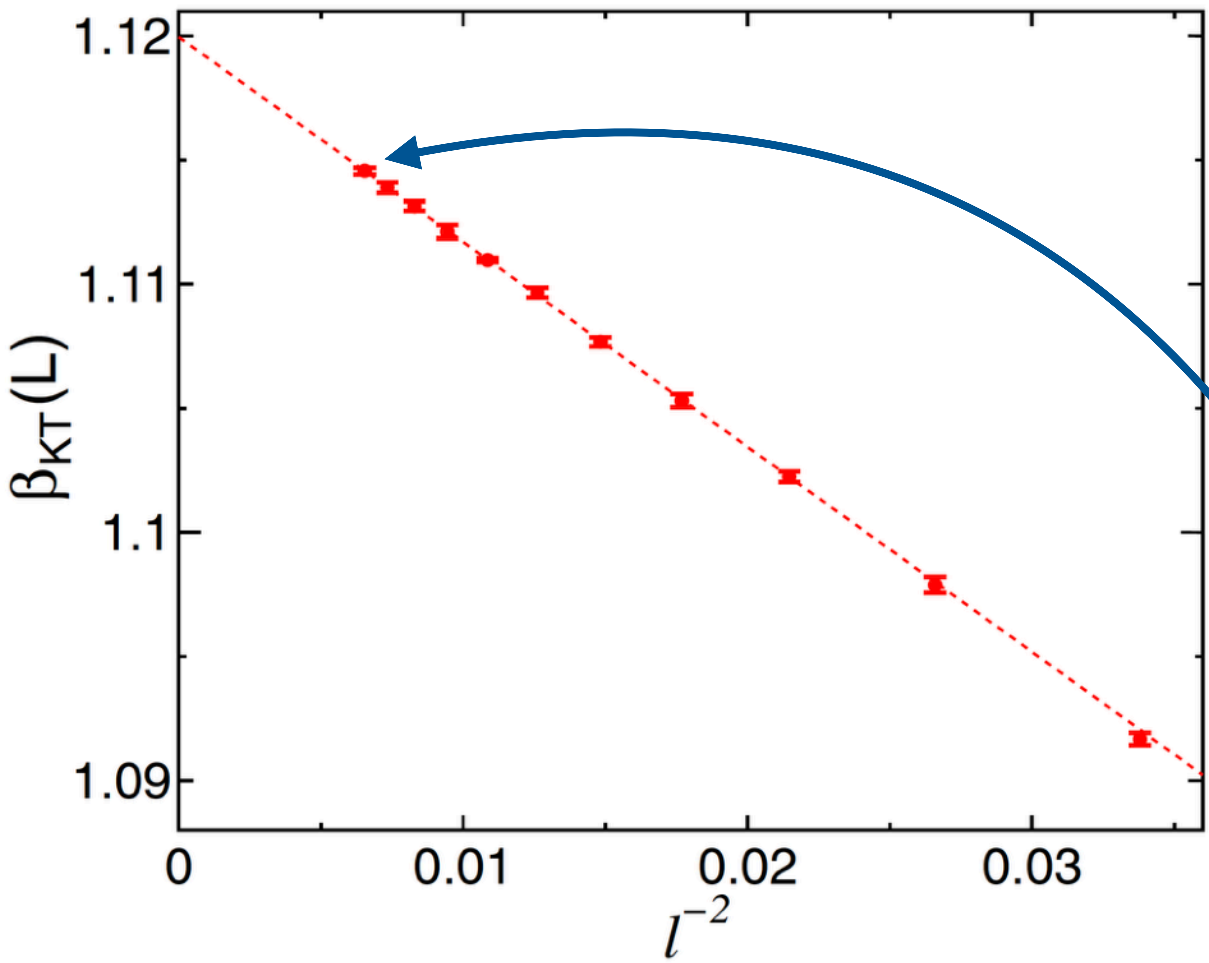
(Received August 27, 2012; accepted September 24, 2012; published online October 12, 2012)



finite-size scaling of T_c

$$l = \ln bL$$

largest system:
L=65536
calculation using 256GPUs

Sine-Gordon Field Theory for BKT

$$\boxed{\mathcal{L} = \frac{1}{2\pi K}(\partial_\mu \phi)^2 - y_K(\partial_\mu \phi)^2 + y_V V}$$

θ angle of the XY spin $\longleftrightarrow$ ϕ

dual (mutually non-local)

$\theta \sim \theta + 2\pi$ $\qquad\qquad$ $\phi \sim \phi + \pi$

$$V = \cos 2\phi = \frac{1}{2}\left(\boxed{e^{2i\phi}} + \boxed{e^{-2i\phi}}\right)$$

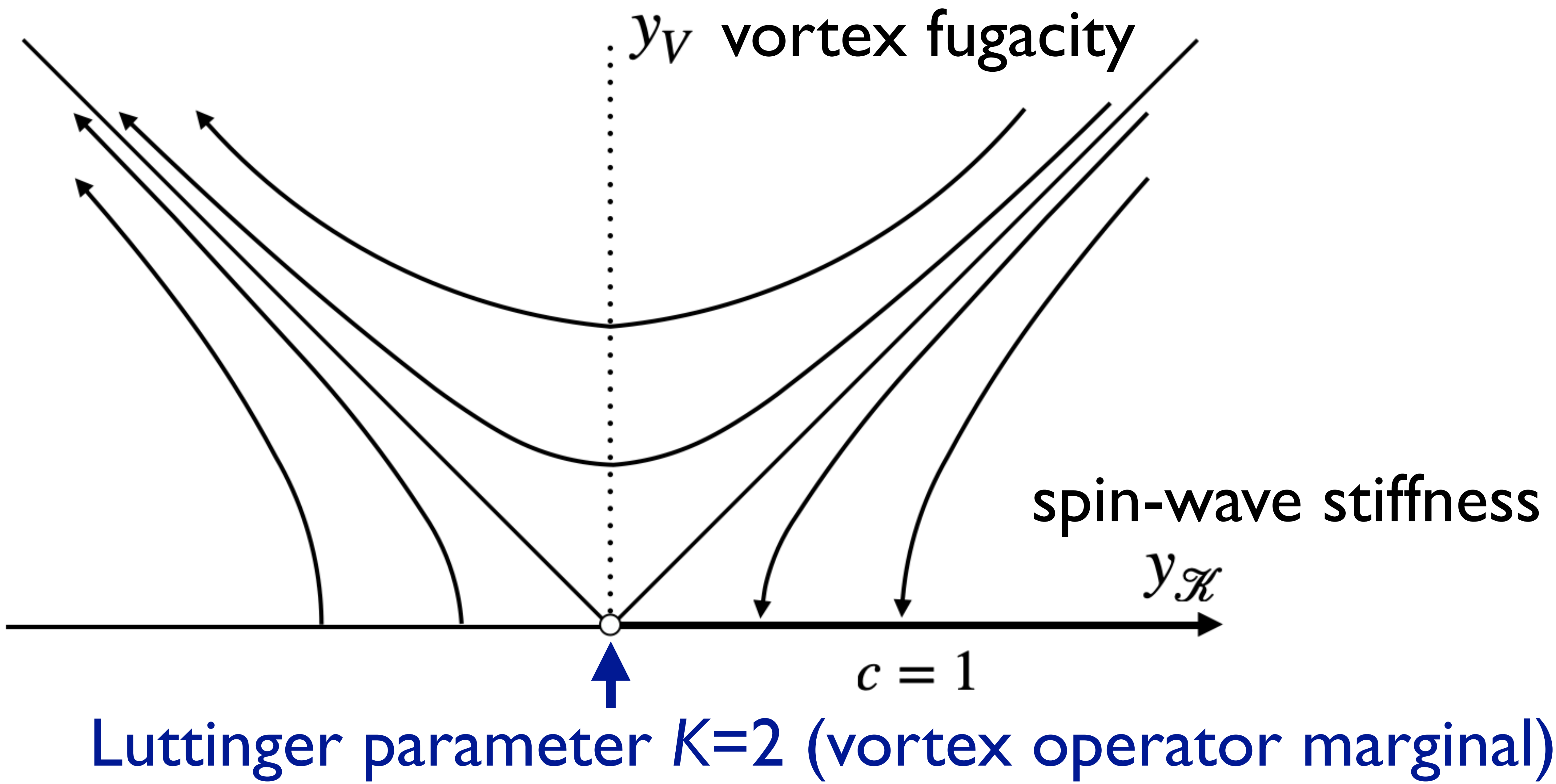
scaling dimension 2/*K*
marginal at *K*=2

Single vortex creation / annihilation operator

y_V vortex fugacity

y_K renormalization of Luttinger parameter *K*

Kosterlitz RG Flow



BKT transition: $y_V = y_{\mathcal{K}} = g$

slow decay

↓

log-corrections

$$\frac{dg}{dl} = -g^2$$

$$g \sim \frac{1}{l} \sim \frac{1}{\ln L}$$

BKT Transition in S=1/2 XXZ Chain

$$\mathcal{H} = \sum_j \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right)$$

Single vortex creation/annihilation operator

$$\cos 2\phi \sim (-1)^j \vec{S}_j \cdot \vec{S}_{j+1}$$

$$\sin 2\phi \sim (-1)^j S_j^z$$

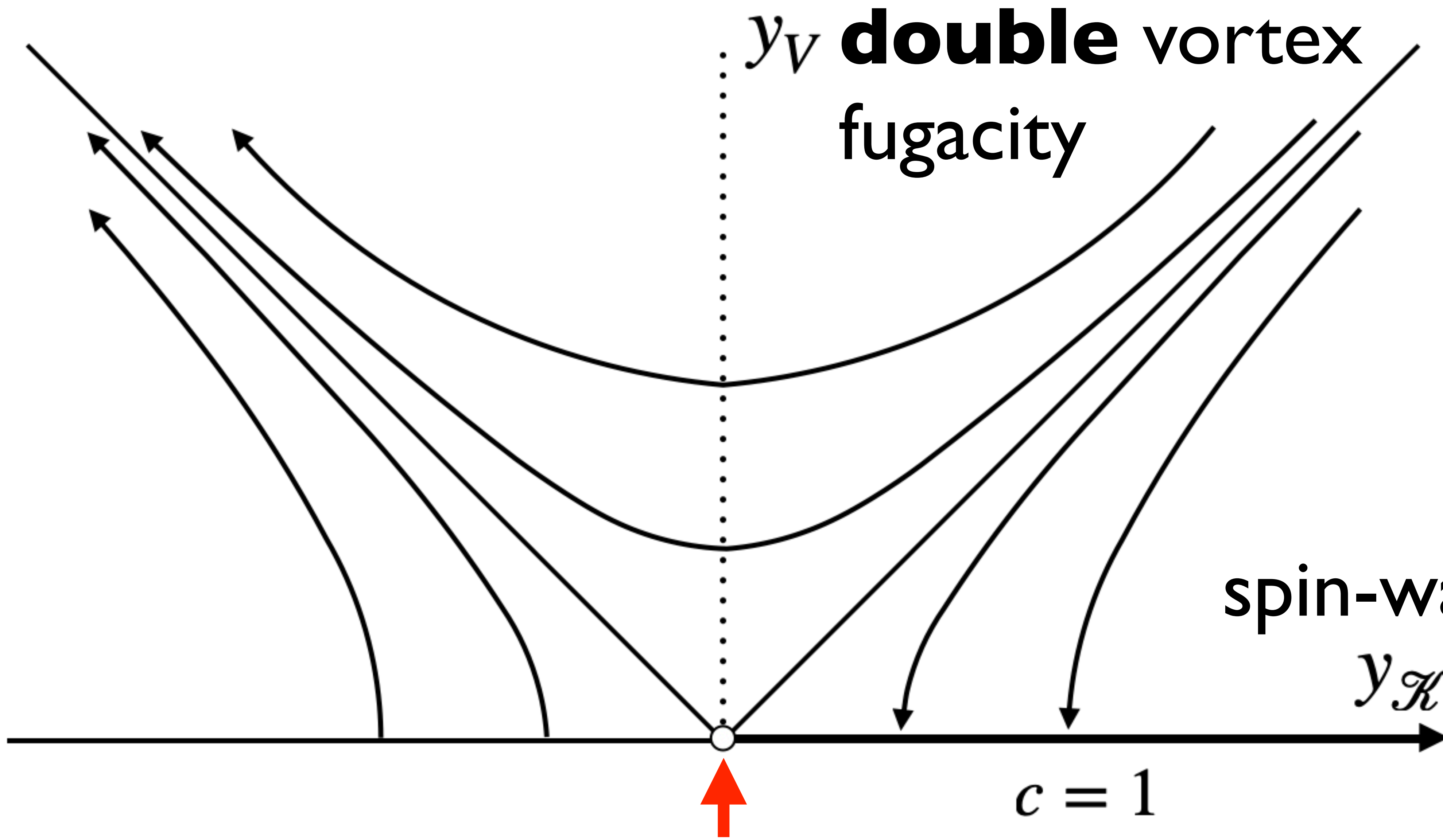
Forbidden in Hamiltonian by the translation symmetry!

(**Haldane 1980** → "Haldane conjecture"
related to Lieb-Schultz-Mattis theorem etc.)

The leading (most relevant) perturbation is thus
double vortex creation/annihilation op. $\cos 4\phi$

BTK Transition in S=1/2 XXZ Chain

$$\mathcal{L} = \frac{1}{2\pi K}(\partial_\mu \phi)^2 - y_{\mathcal{K}}(\partial_\mu \phi)^2 + y_V V \qquad V = \cos 4\phi$$



Luttinger parameter ***K*=1/2**
(**double** vortex operator marginal)

BKT Transition in S=1/2 XXZ Chain

$$\mathcal{H} = \sum_j \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right)$$

BKT transition at Δ=1 (SU(2) symmetric point)

IR fixed point at the BKT transition:

Free boson (Tomonaga-Luttinger Liquid) at *K*=1/2

equivalent to Level 1 SU(2) Wess-Zumino-Witten

Effective theory in the vicinity of the BKT transition

$$\mathcal{L} = \mathcal{L}_{k=1}^{\mathrm{WZW}} + g\mathbb{J}^L \cdot \mathbb{J}^R + t\left(-\frac{1}{2} J_+^L J_+^R - \frac{1}{2} J_-^L J_+^R + J_z^L J_z^R \right)$$

$$y_V = g + t, y_{\mathcal{K}} = g - t$$

BKT transition ⇔ *t*=0 ⇔ SU(2) symmetry

Level Spectroscopy

Determination of the critical point from the finite-size spectrum [Okamoto-Nomura 1994, …]

"Double vortex" BKT transition at K=1/2 can be identified by SU(2) symmetry of the finite-size spectrum!

State-operator correspondence and Finite-Size Scaling in CFT [Cardy 1984, 1986]

$$E_n - E_0 = \frac{2\pi}{L}\left(x_n + \sum_m c_{nnm} y_m L^{2-x_m} + \dots \right)$$

BKT transition ⇔

Energy levels form SU(2) singlet, triplet, …

1D S=1/2 XXZ vs 2D Classical XY

$$\mathcal{L} = \frac{1}{2\pi K}(\partial_\mu \phi)^2 - y_\mathcal{K}(\partial_\mu \phi)^2 + y_V V$$

S=1/2 XXZ

K=1/2 (SU(2)$_1$ WZW)

$V \sim \cos 4\phi$

double vortex op.

SU(2) triplet (degenerate at BKT)

$n^x \sim \cos\theta,\ n^y \sim \sin\theta,\ n^z \sim \sin 2\phi$

$2\phi \to \phi$

Classical XY

K=2 $\quad \phi \sim \phi + \pi$

$V \sim \cos 2\phi$

single vortex op.

$\cos 2\theta, \sin 2\theta, \boxed{\sin\phi}$

half-vortex op.
(eigenstate under antiperiodic b.c.)

Nomura-Kitazawa 1998

Level Spectroscopy for 2D Stat Mech

Level spectroscopy has been developed for quantum 1D,
but why not for 2D stat mech models (such as XY)??

1D quantum Hamiltonian ⇔

Transfer matrix for 2D stat mech

Continuous spin: series expansion of Boltzmann weight

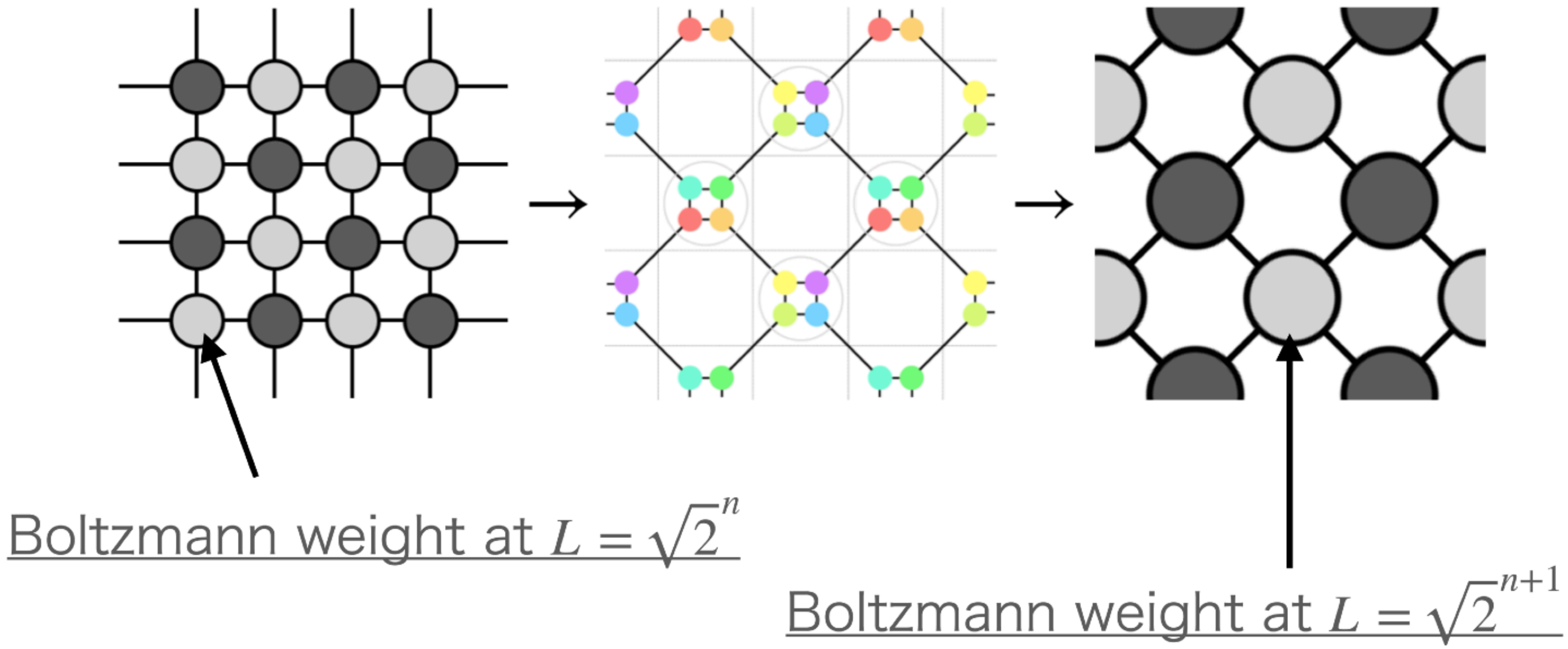
$$e^{\beta \cos(\theta_i - \theta_j)} = \sum_{n=-\infty}^{\infty} e^{in(\theta_i - \theta_j)} I_n(\beta),$$

the series may be truncated to $-15 \leqq n \leqq 15$

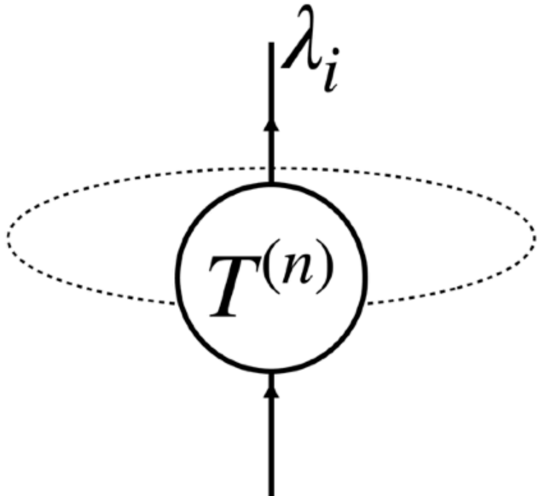
Transfer matrix still "too large" to be diagonalized
⇒ we utilize Tensor Network Renormalization

(We used Loop-TNR)

TNR Construction of Transfer Matrix



after *n* steps, a single tensor represents
a square block of linear size $L = \sqrt{2}^n$



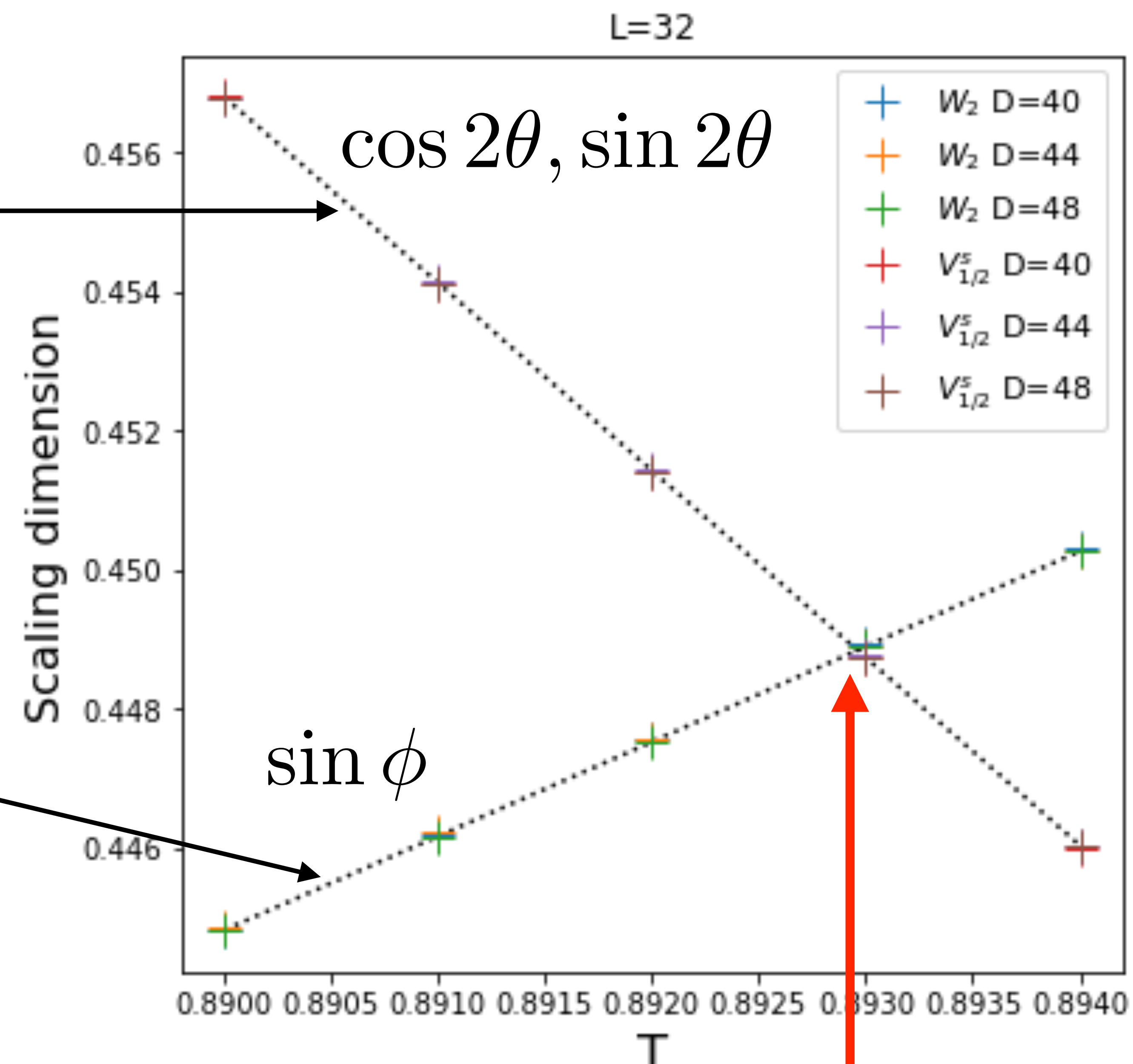
contract horizontal indices
⇒ transfer matrix in vertical direction

$$\lambda_i = e^{-LE_i(L)}$$

Identifying T_c with Level Crossing

"spin-wave" excited states under periodic b.c.

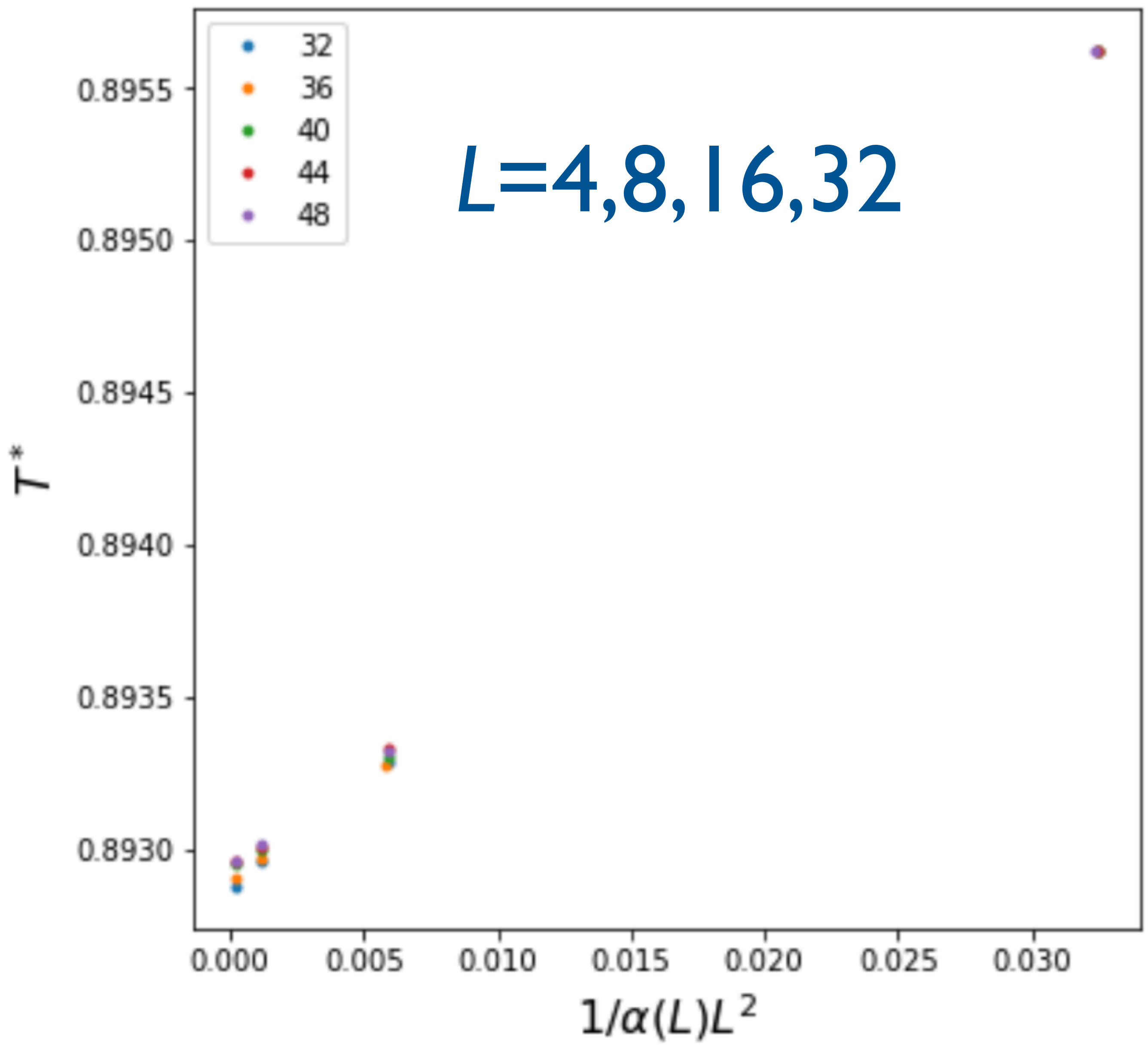
ground state under antiperiodic b.c.



This procedure eliminates logarithmic corrections to all orders in g

extra degeneracy forming SU(2) triplet $\sim$ BKT transition

Remaining Finite-Size Effect



Extrapolate to $L=\infty$

Level crossing point weakly depends on the system size L

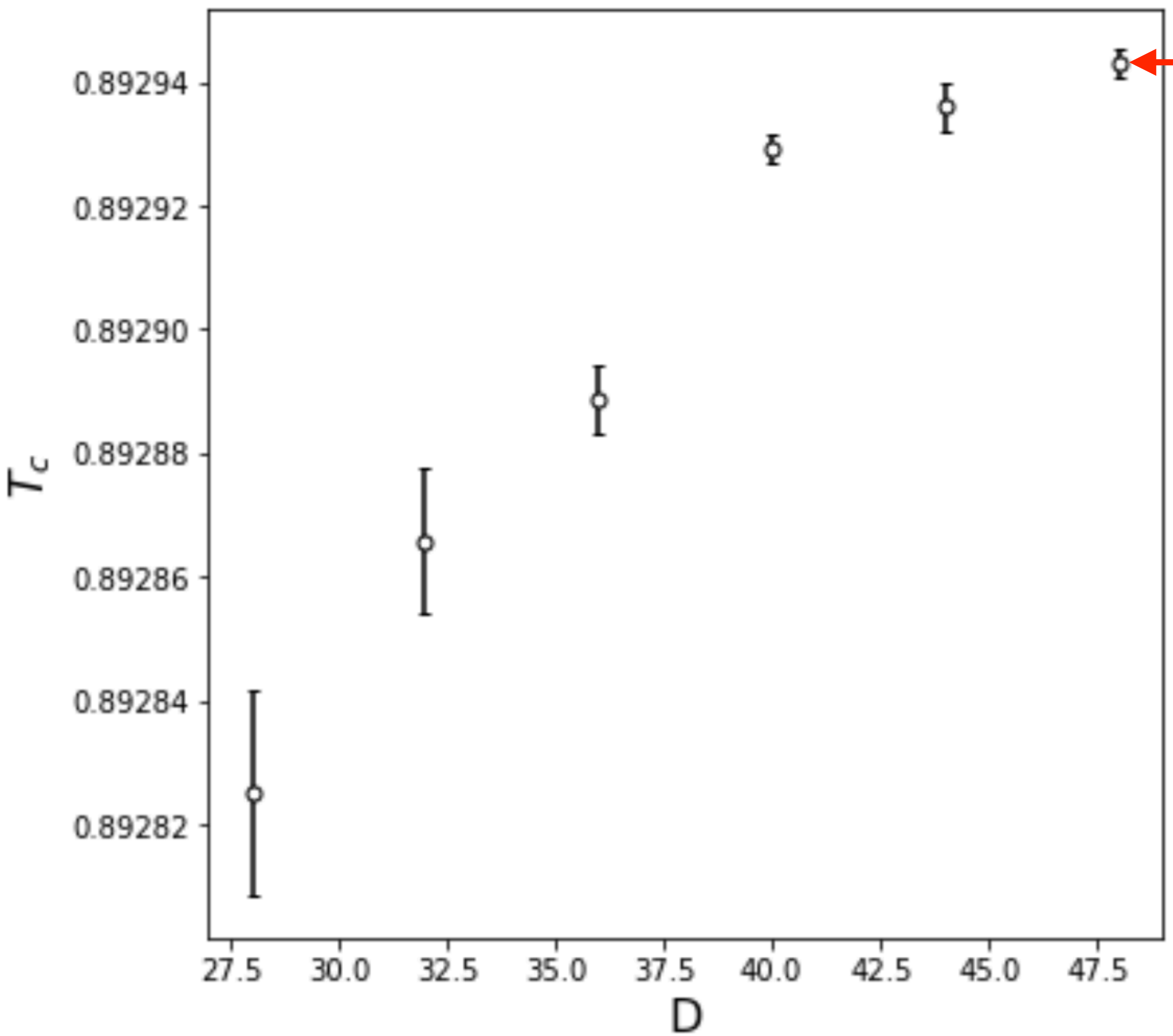
Effect of irrelevant perturbations

$$T^2, \bar{T}^2, T\bar{T}, \ldots$$

T: holomorphic part of the energy-momentum tensor

$$T^* \sim T_c + \text{const.}\frac{1}{L^2}$$

Dependence on Bond Dimension D



Our final estimate
T_c=0.892943(2)

Effect of Finite Bond-Dimension

Finite bond dimension *D* ⇔ finite "correlation length"

$$\xi_D \sim 0.3 D^{\kappa}$$

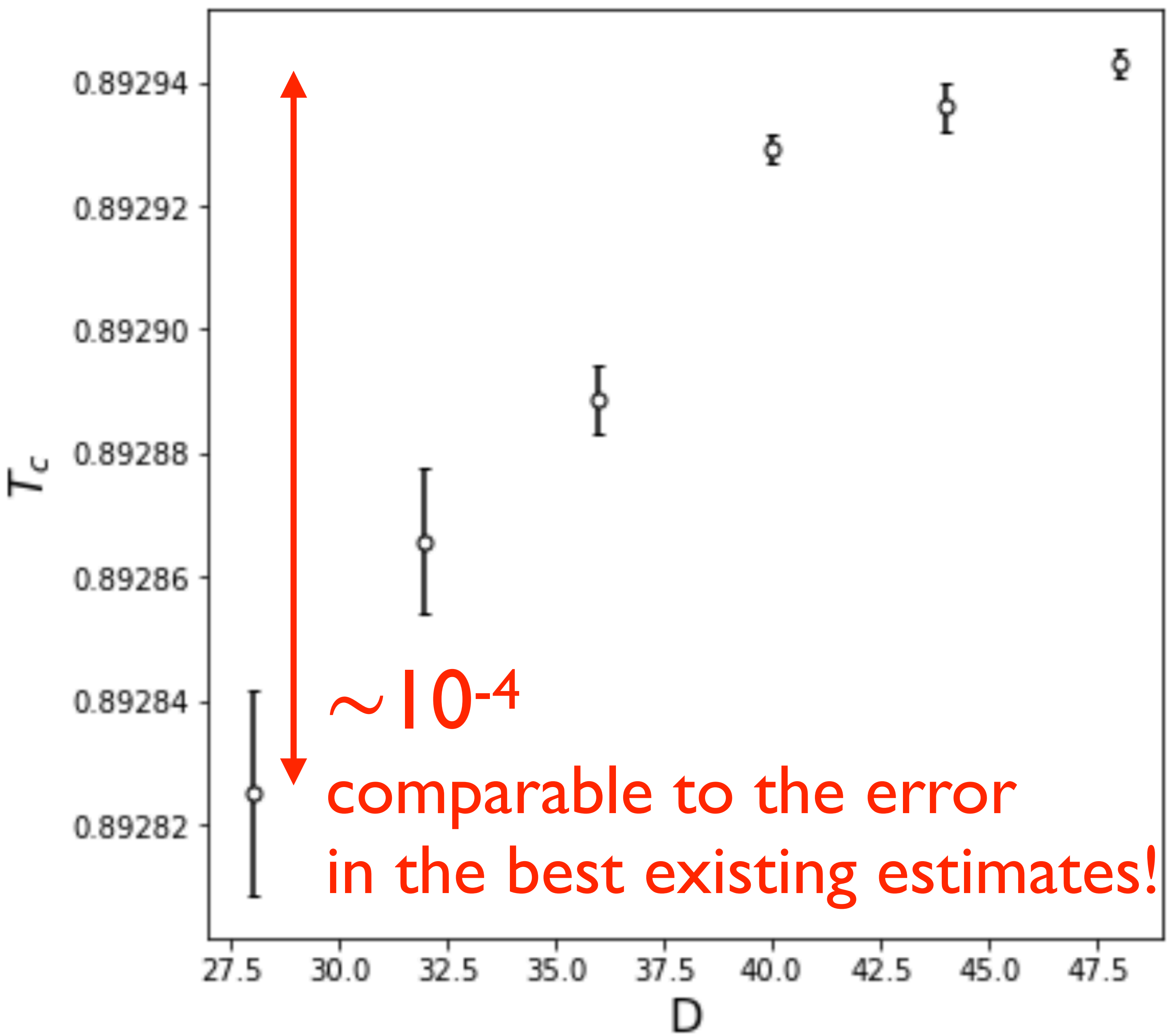
$$\kappa = \frac{6}{c\left(\sqrt{\frac{12}{c}}+1\right)}$$

[Pollmann et al. 2008]

$\xi_D > L$ low-energy finite-size spectrum almost exact!

$\xi_D < L$ low-energy spectrum still reasonably accurate, but some error due to the finite *D*

Tc dependence on *D*



D=48 gives ξ~54

enough for up to *L*=32

D=28 gives ξ~26

too small for *L*=32
BUT….

Error in (Loop-)TNR

(Loop-)TNR is often used to construct the "fixed point" tensor, which would describe the large scale behaviors, by iterating TNR many times

This approach has given accurate results for large systems, but small errors due to the finite bond dimension remain

In our approach, we study the spectrum of finite-size systems with TNR. TNR is almost exact when the system size is less than the effective correlation length.

TNR calculation of the finite-size spectrum
+ Level Spectroscopy → extremely high accuracy

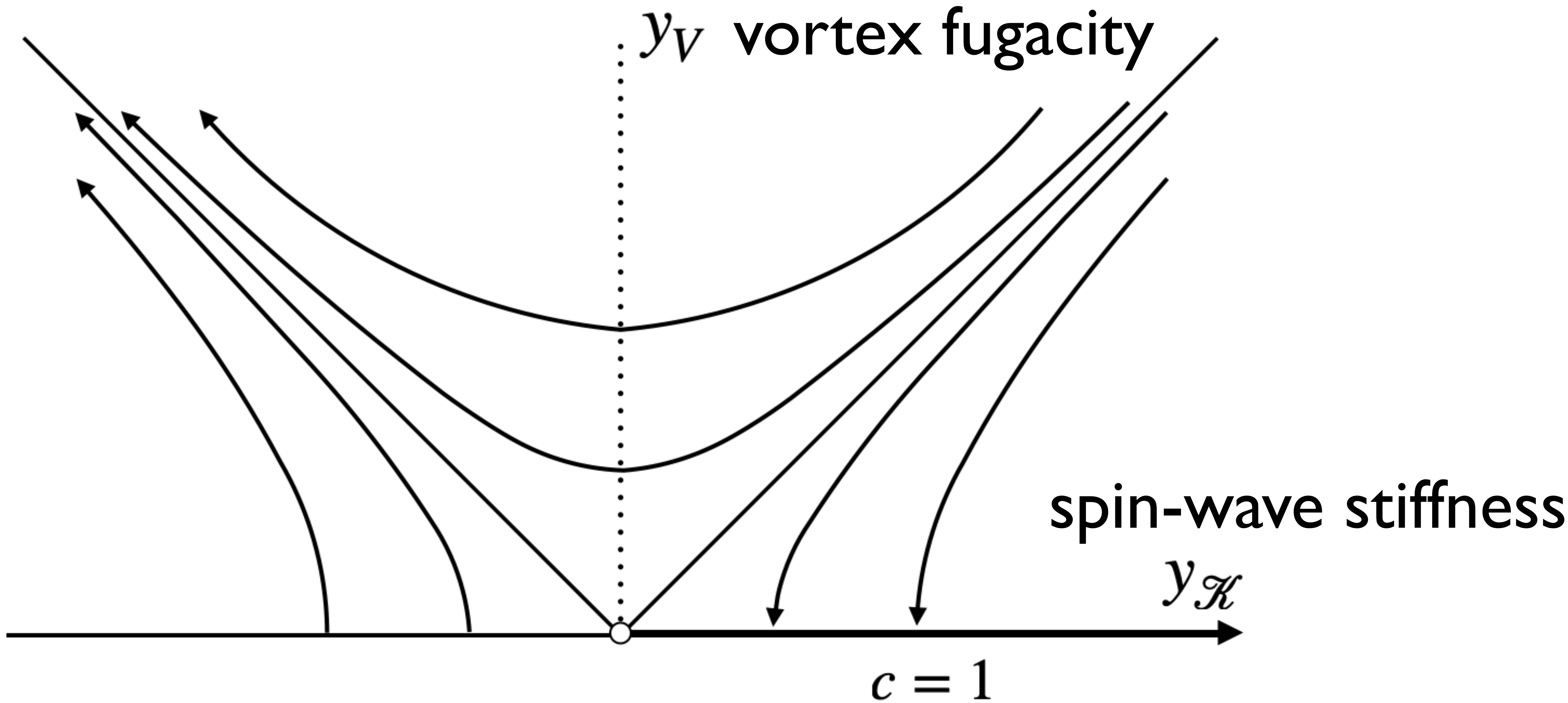
Estimates of T_c

Monte Carlo(1979)[35]	0.89
Monte Carlo(2005)[36]	0.8929
Monte Carlo(2012)[37]	0.89289
Monte Carlo(2013)[38]	0.8935
Series expansion(2009)[39]	0.89286
HOTRG(2014)[40]	0.8921
VUMPS(2019)[41]	0.8930
HOTRG(2020)[42]	0.89290(5)
present work	0.892943(2)

TABLE I. Comparison of the estimated critical temperature of the 2D classical XY model.

Kosterlitz RG Flow

You must have seen this diagram many times….



but have you ***really*** *seen the RG flow?*

ELSEVIER

Nuclear Physics B 522 [FS] (1998) 533–549

NUCLEAR PHYSICS B

Low energy effective Hamiltonian for the XXZ spin chain

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Received 19 February 1998; accepted 18 March 1998

"vacuum energy" under the twisted boundary condition with the twist angle θ [notation clash…]

$$\delta^{\mathrm{RG}} = -\frac{1}{12}\left\{1+\frac{3}{8}g_{\|}g_{\perp}^{2}\right\}+\frac{s^{2}}{2}\left\{1-\frac{g_{\|}}{2}+\frac{1}{4}g_{\perp}^{2}-\frac{7}{32}g_{\|}g_{\perp}^{2}\right\}$$
$$+\frac{|s|}{16}\left\{2g_{\perp}^{2}-g_{\|}g_{\perp}^{2}\right\}+\frac{\theta^{2}}{2}\left\{1+\frac{g_{\|}}{2}+\frac{g_{\|}^{2}}{4}-\frac{g_{\perp}^{2}}{4}+\frac{g_{\|}^{3}}{8}-\frac{g_{\|}g_{\perp}^{2}}{32}\right\}+O(g^{4}), \quad (4.6)$$

Energy Levels up to 2nd Order

- split between $x_{V^s_{1/2}}, x_{V^c_{1/2}}$ should be odd in y_V
- SU(2) triplet should be formed on the BKT transition line $y_\mathcal{K} = y_V$

⇒ uniquely determines the energy levels up to $O(y^2)$

$$x_{W_{\pm 2}} = \frac{1}{2} - \frac{y_\mathcal{K}}{4} + \frac{1}{4} y_V^2,$$
$$x_{V^s_{1/2}} = \frac{1}{2} + \frac{y_\mathcal{K}}{4} - \frac{y_V}{2} + \frac{1}{8}(y_\mathcal{K}^2 + 2y_\mathcal{K} y_V - y_V^2),$$
$$x_{V^c_{1/2}} = \frac{1}{2} + \frac{y_\mathcal{K}}{4} + \frac{y_V}{2} + \frac{1}{8}(y_\mathcal{K}^2 - 2y_\mathcal{K} y_V - y_V^2),$$

Obtaining Running Coupling Constants

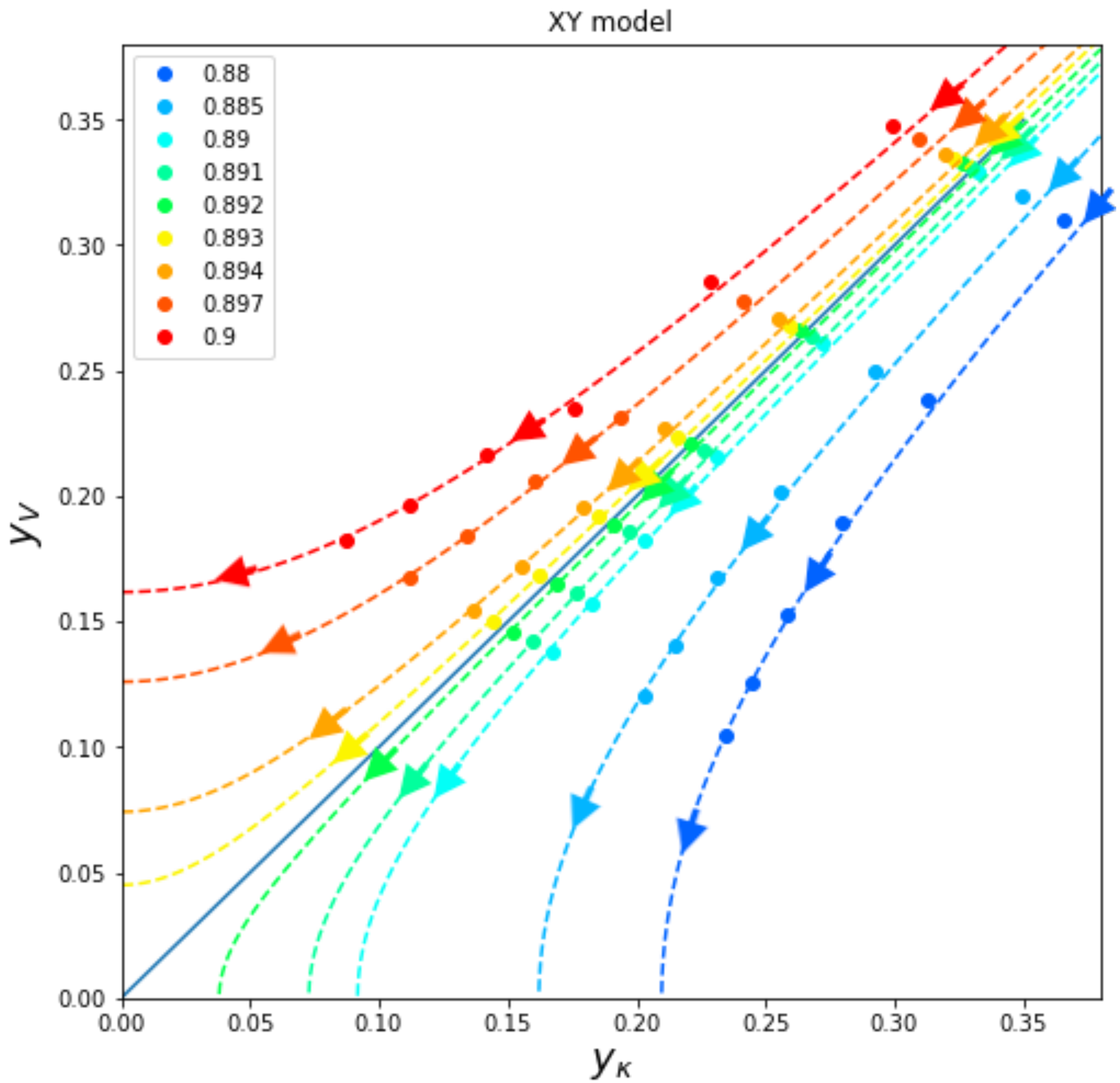
$$x_{W_{\pm 2}} = \frac{1}{2} - \frac{y_{\mathcal{K}}}{4} + \frac{1}{4} y_V^2,$$
$$x_{V^s_{1/2}} = \frac{1}{2} + \frac{y_{\mathcal{K}}}{4} - \frac{y_V}{2} + \frac{1}{8}(y_{\mathcal{K}}^2 + 2 y_{\mathcal{K}} y_V - y_V^2),$$
$$x_{V^c_{1/2}} = \frac{1}{2} + \frac{y_{\mathcal{K}}}{4} + \frac{y_V}{2} + \frac{1}{8}(y_{\mathcal{K}}^2 - 2 y_{\mathcal{K}} y_V - y_V^2),$$

$$y_{\mathcal{K}} \sim 2 - 4 x_{W_{\pm 2}} + (x_{V^c_{1/2}} - x_{V^s_{1/2}})^2,$$
$$y_V \sim (x_{V^c_{1/2}} - x_{V^s_{1/2}})/(1 - \frac{1}{2} y_{\mathcal{K}}),$$

We can estimate y_K & y_V from the finite-size energy levels

Less accuracy than T_c, but we can apply to larger systems (up to L=512)

Visualization of Kosterlitz RG Flow!



L=16
32
64
128
256
512

Revisiting Ising Model

— to clarify the "origin" of the finite-*D* effect

$$\mathcal{H} \sim \mathcal{H}_{\mathrm{CFT}} + g_\sigma \int dx\ \sigma(x) + g_\epsilon \int dx\ \epsilon(x) + g \int dx\ \left(T^2(x) + \bar{T}^2(x)\right)$$

$T, \bar{T}$: energy-momentum tensor

$$x_\sigma(L) \sim \frac{1}{8} + \alpha^\sigma_\sigma {g_\sigma}^2 + \pi g_\epsilon + \alpha^\epsilon_\sigma {g_\epsilon}^2 - \frac{7\pi}{768} g + \dots$$

$$x_\epsilon(L) \sim 1 + \alpha^\sigma_\epsilon {g_\sigma}^2 + \alpha^\epsilon_\epsilon {g_\epsilon}^2 + \frac{7\pi}{48} g + \dots$$

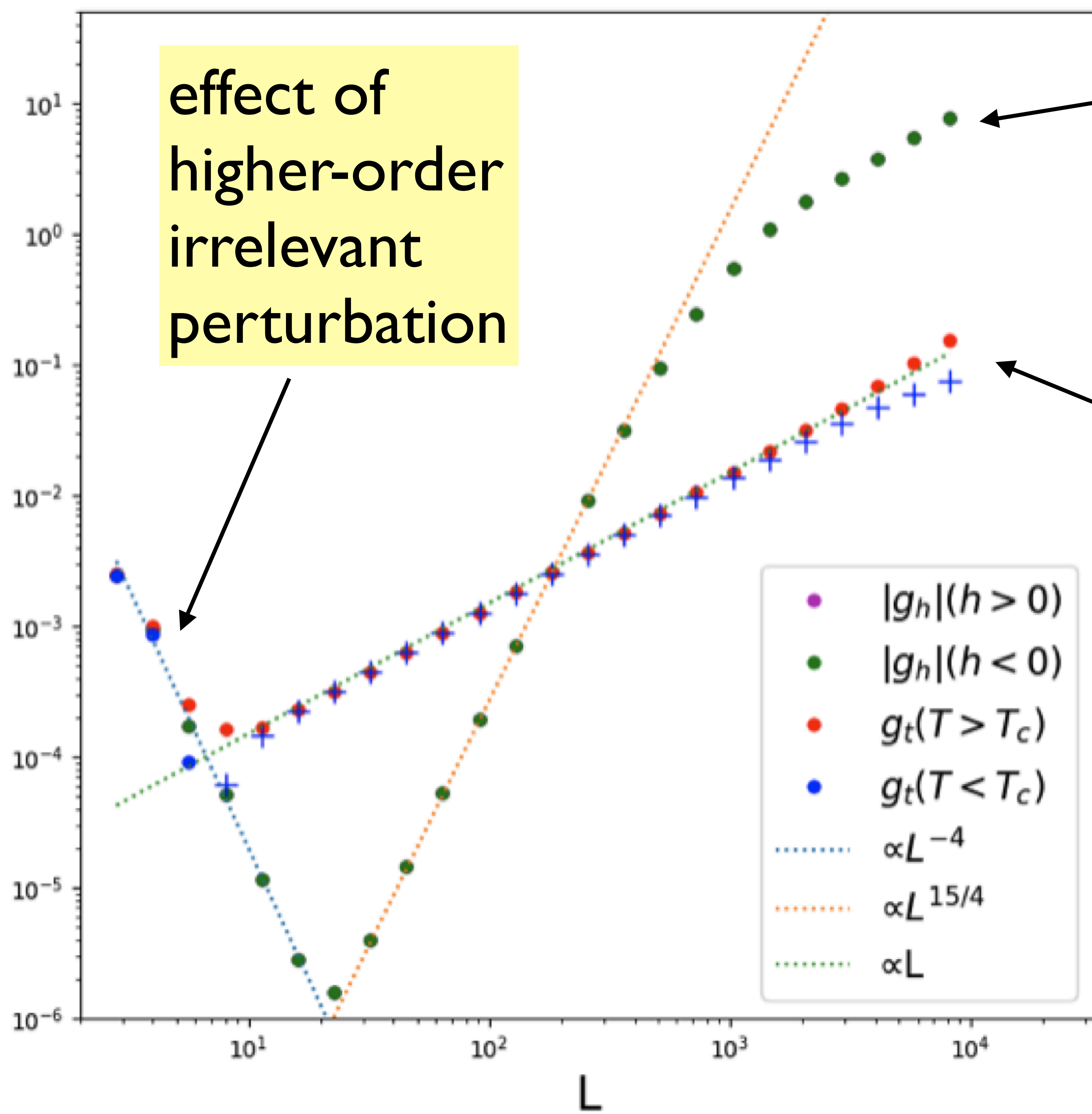
We can eliminate the effects of leading irrelevant *g* by

$$x_\sigma(L) - \frac{1}{8} + \frac{1}{16}\left(x_\epsilon(L) - 1\right) \sim \pi g_\epsilon + \left(\alpha^\epsilon_\sigma + \frac{1}{16}\alpha^\epsilon_\epsilon\right) {g_\epsilon}^2 + \left(\alpha^\sigma_\sigma + \frac{1}{16}\alpha^\sigma_\epsilon\right) {g_\sigma}^2 + \dots$$

1

$$x_\sigma(L) - \frac{1}{8} + \frac{1}{16}\left(x_\epsilon(L) - 1\right) \;\sim {g_h}^2 + g_t$$

$$g_h \propto g_\sigma, g_t \propto g_\epsilon$$



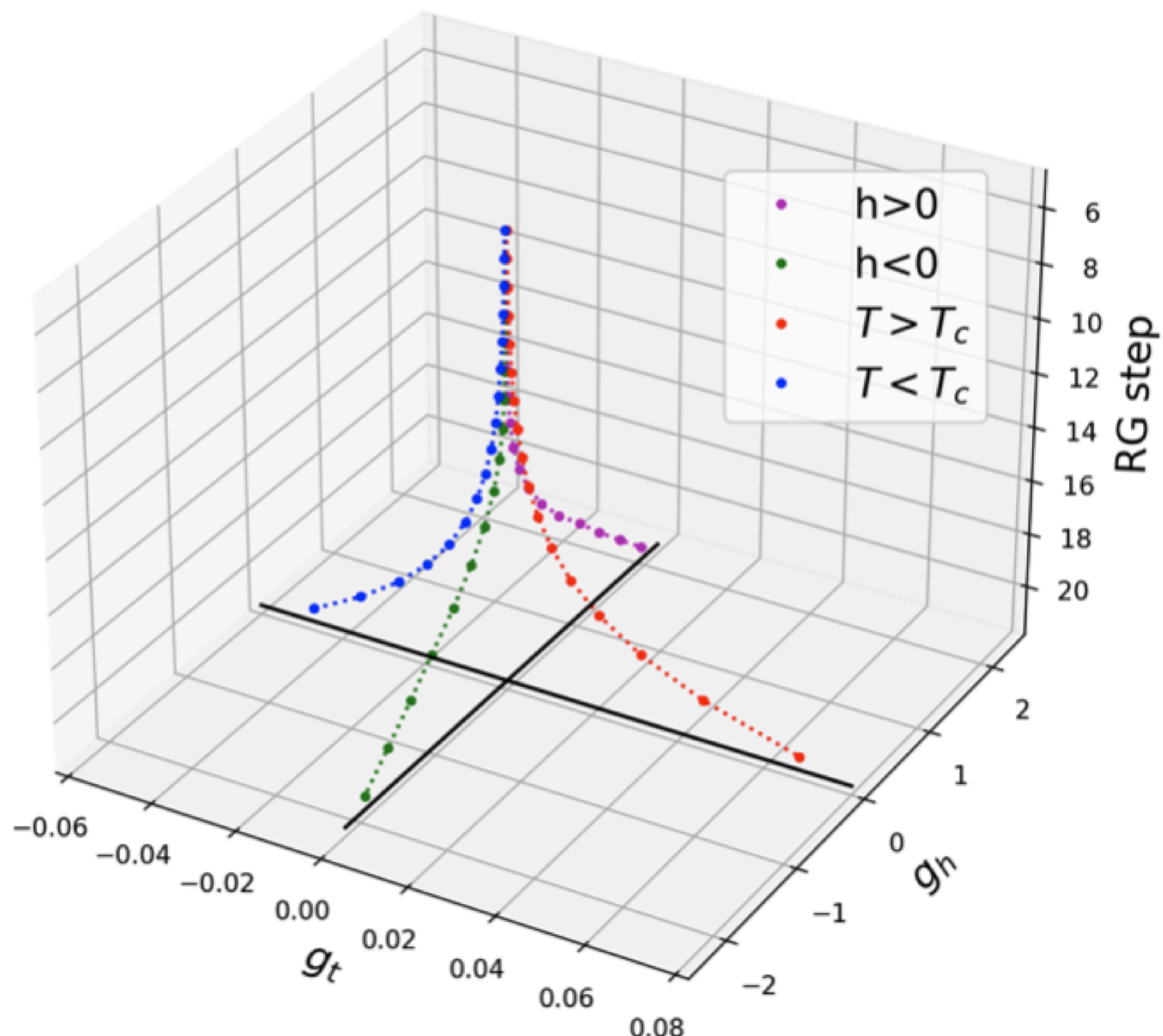
h=10^{-5}
at T=T_c

$${g_h}^2 \propto \left(L^{2-\frac{1}{8}}\right)^2 = L^{15/4}$$

T=1.0001T_c
T=0.9999T_c
with h=0

$$g_t \propto L^{2-1} = L$$

Visualizing RG Flow for Ising



What is Finite-*D* Effect?

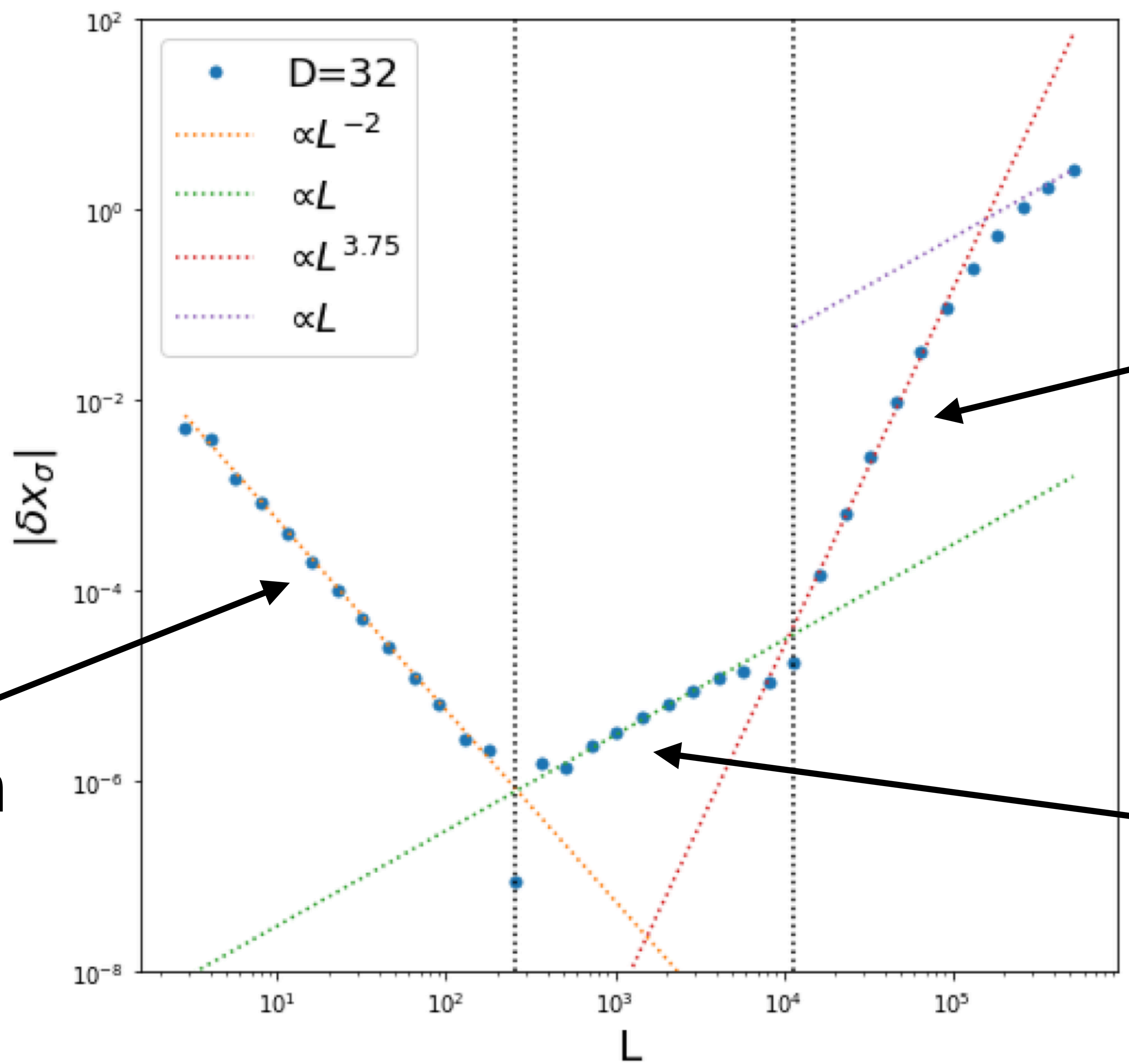
Finite-*D* ⇔ Finite correlation length

⇔ "Massive" field theory due to relevant perturbation

Natural idea, but is this true?
(Are relevant perturbations somehow automatically induced for finite-*D*??)

Finite-*D* TNR at the Critical Point

$$x_\sigma(L) - \frac{1}{8} \sim -\frac{7\pi}{768} g + g_t + {g_h}^2 + \dots$$



leading irrelevant perturbation

$$g \propto \frac{1}{L^2}$$

"generated"

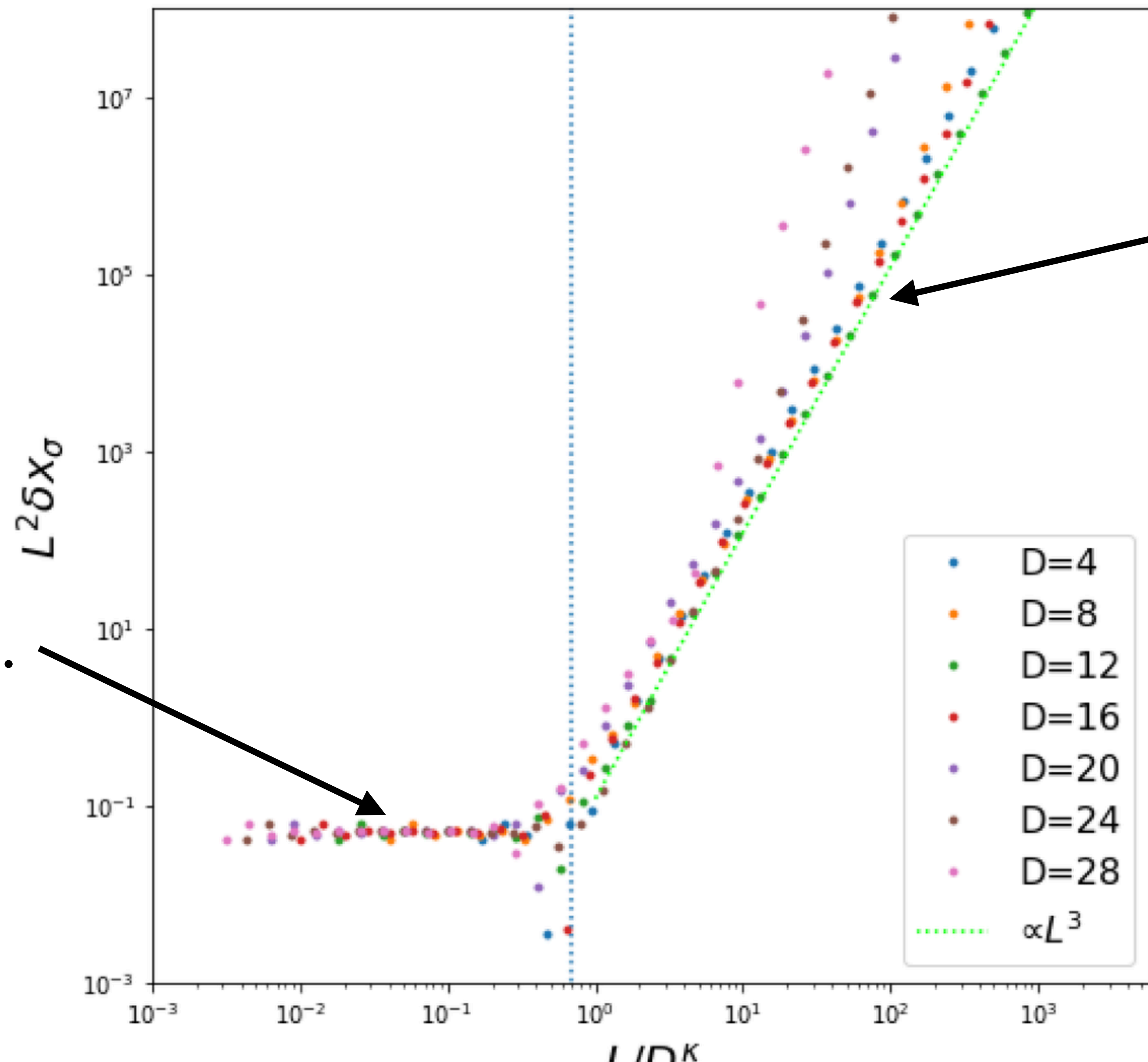
$${g_h}^2 \propto L^{15/4}$$

"generated"

$$g_t \propto L$$

Scaling in L/ξ_D

$$L^2\left(x_\sigma(L) - \frac{1}{8}\right) \sim -\frac{7\pi}{768}L^2 g + L^2 g_t + \ldots$$



$$L^2 g_t \propto L^3$$

$$L^2 g \sim \text{const.}$$

$$\kappa = \frac{6}{c\left(1+\sqrt{\frac{12}{c}}\right)}$$

[Pollmann et al. 2009]

$$\frac{L}{\xi_D} = \frac{L}{D^\kappa}$$

Conclusions (so far)

TNR + Level Spectroscopy (finite size scaling of the transfer-matrix spectra based on CFT)

allows

- visualization of RG flow by extraction of running coupling constants from the spectrum
- extremely accurate determination of the critical point
- also applicable to continuous valued 2D classical spin systems such as XY model
- finite bond dimension *D* corresponds to "automatically" induced (relevant) perturbations

Classical Heisenberg Model

$$H = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j \,,$$

continuum limit

$$\mathcal{L} = \frac{1}{2g} (\partial_\mu \vec{n})^2 \qquad \vec{n}^2 = 1$$

O(3) Nonlinear Sigma Model

coupling *g* corresponds to temperature

Asymptotic freedom ⇒ disordered at any $T>0$

$$\frac{dg}{d \log L} = \frac{g^2}{2\pi} + \ldots$$

Supported also by factorizable *S*-matrix (Zamolodchikov²)

BUT…

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 3741–3748 doi:10.1088/1751-8113/40/14/001

Quasi-long-range ordering in a finite-size 2D classical Heisenberg model

Really??

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But not easy to prove/disprove

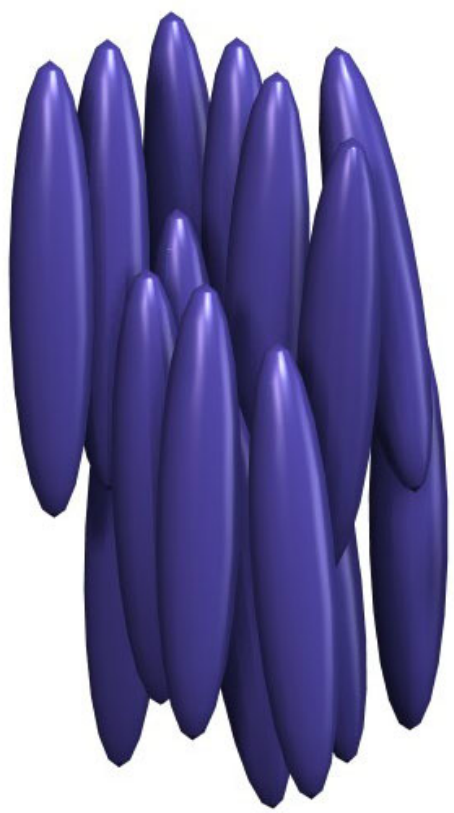
Received 24 November 2006, in final form 13 February 2007
Published 20 March 2007
Online at stacks.iop.org/JPhysA/40/3741

Abstract
We analyse the low-temperature behaviour of the classical isotropic ferromagnetic Heisenberg model on a two-dimensional square lattice of finite size. Presence of a residual magnetization in a finite-size system enables us to use a low-temperature approximation, which is however more restricting than the usual spin-wave approximation known to give reliable results for the XY model at low temperatures T. For the system considered, we find that the spin–spin correlation function decays as $1/r^{\eta(T)}$ for large separations r bringing about the presence of a quasi-long-range ordering. We give analytic estimates for the exponent $\eta(T)$ in different regimes and support our findings by Monte Carlo simulations of the model on lattices of different sizes at different temperatures.

PACS numbers: 05.50.+q, 75.10

difficult to distinguish massive with very large correlation length from massless…

Z_2 vortex-driven transition?



Nematic liquid crystal:
symmetric rod-like molecules
(no distinction between head & tail)

$$H = -J \sum_{\langle i,j \rangle} (\vec{S}_i \cdot \vec{S}_j)^2,$$

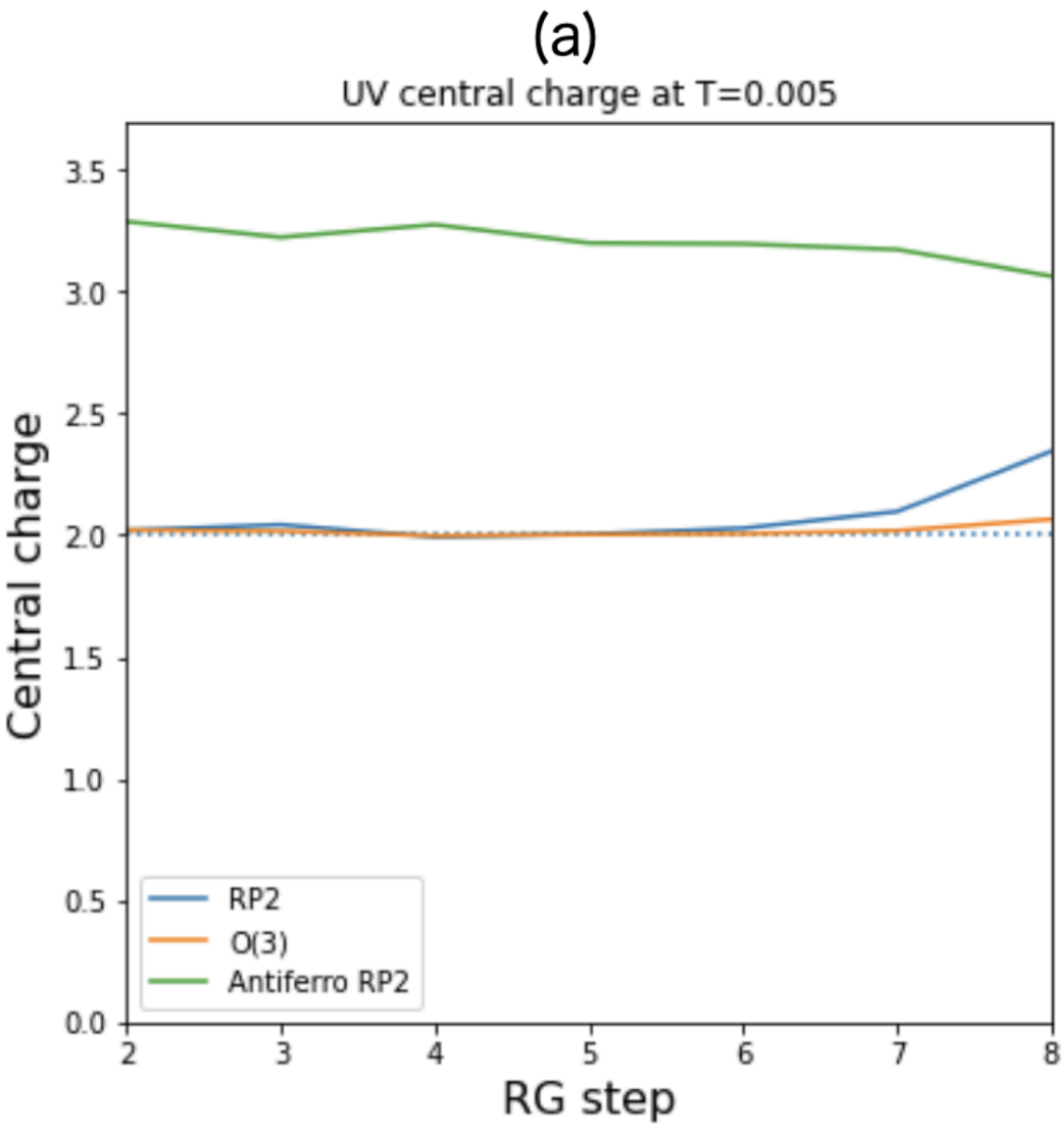
Lebwohl-Lascher 1972
Kawamura-Miyashita 1984

Target space = RP^2

$\pi_1(\mathrm{RP}^2) = \mathbb{Z}_2$ Z_2 vortex!

BKT-like transition driven by the Z_2 vortices?
But the RG equation also implies asymptotic freedom..

UV Fixed Point



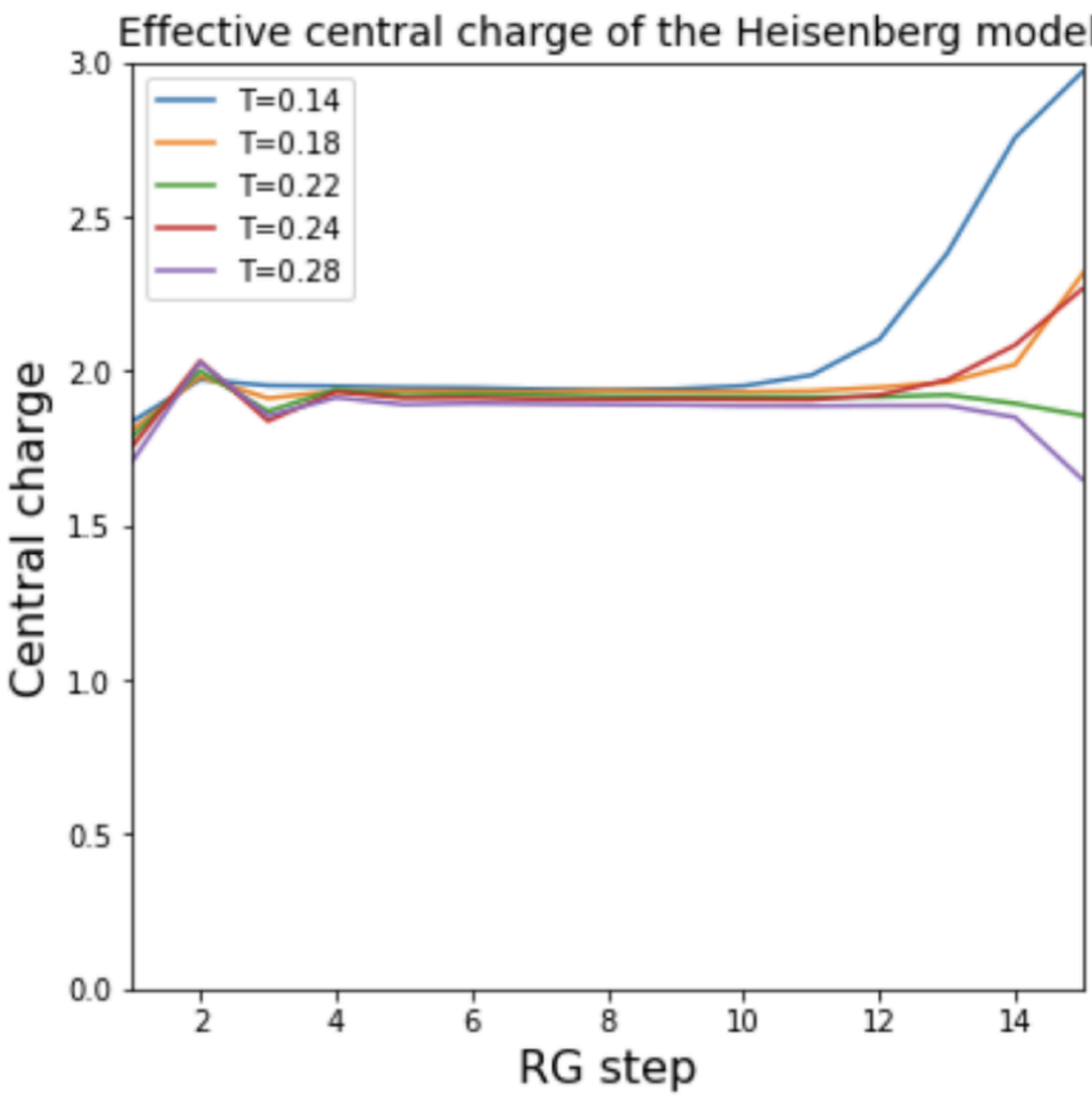
$T \to 0$
Spontaneous Symmetry Breaking of SO(3)

2 independent Nambu-Goldstone modes
$\Rightarrow c=2$

Confirmed by TNR

$$E_0(L) \sim \epsilon_0 L - \frac{\pi c}{6L}.$$

Heisenberg Model at Intermediate T

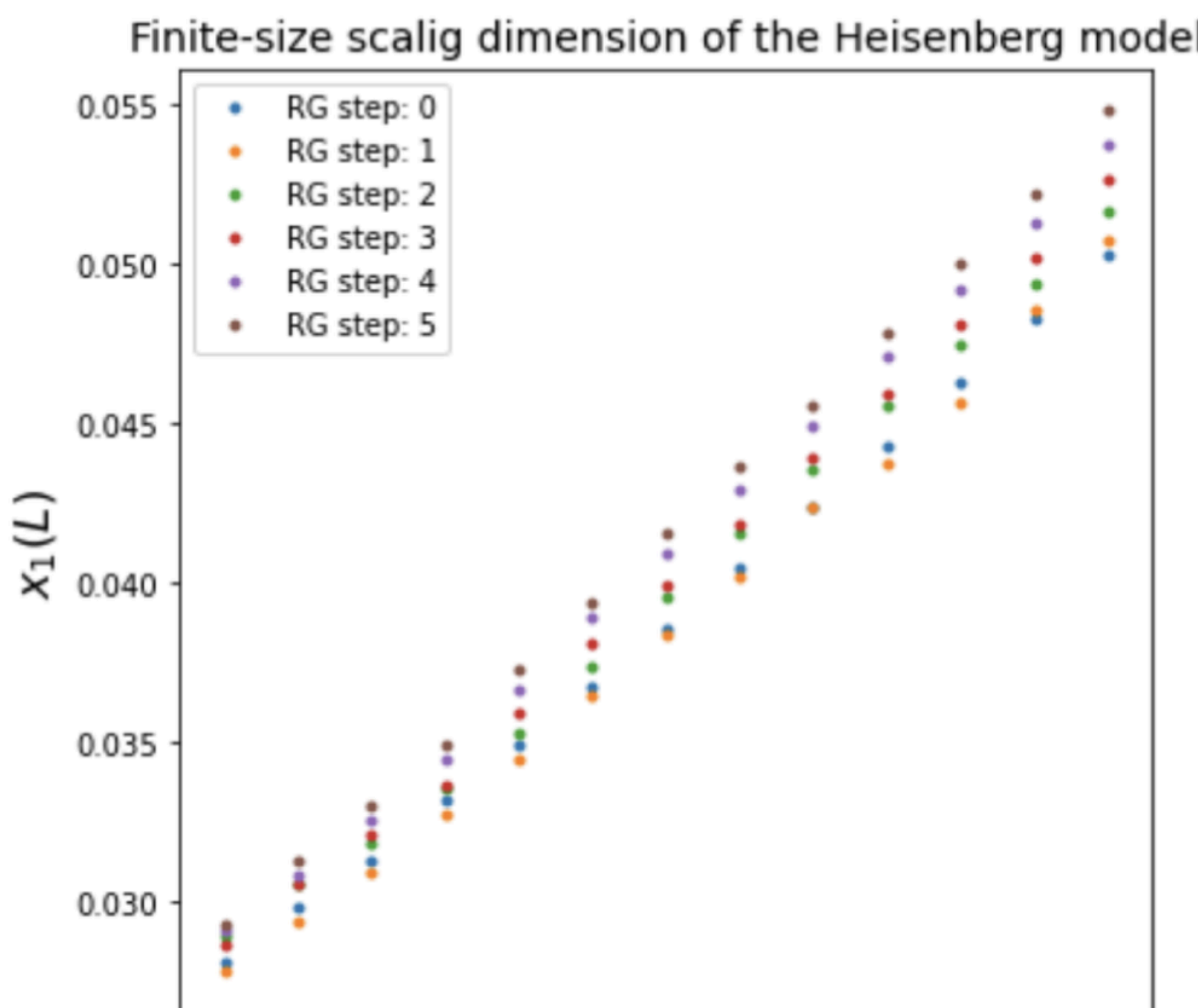


$$E_0(L) \sim \epsilon_0 L - \frac{\pi c}{6L}.$$

$c \sim 1.94 < 2$ no candidate CFT

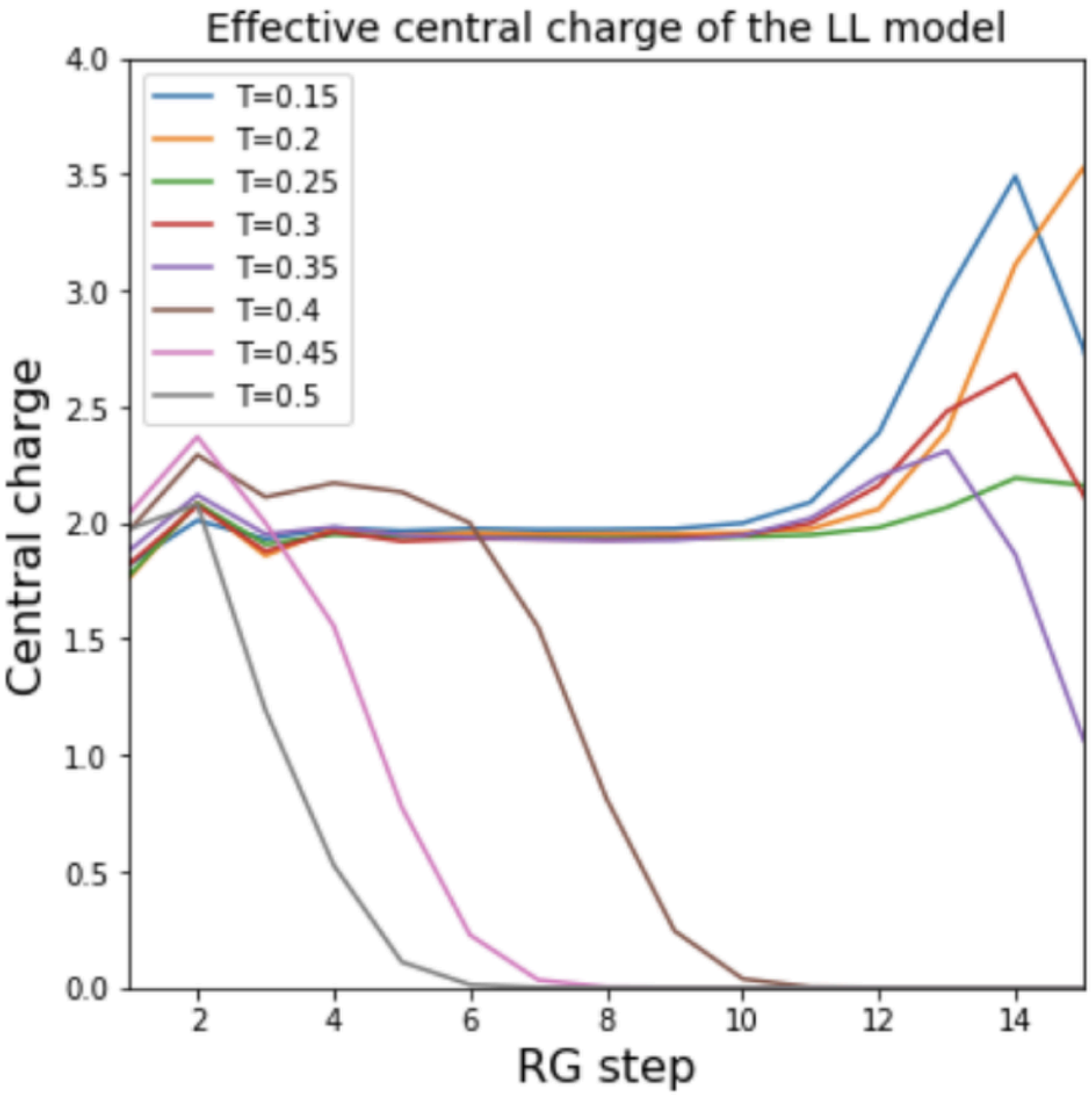
$$\frac{2\pi x_n}{L} = E_n(L) - E_0(L).$$

does not converge



likely to support "asymptotic freedom" RG flow c=2 → c=0

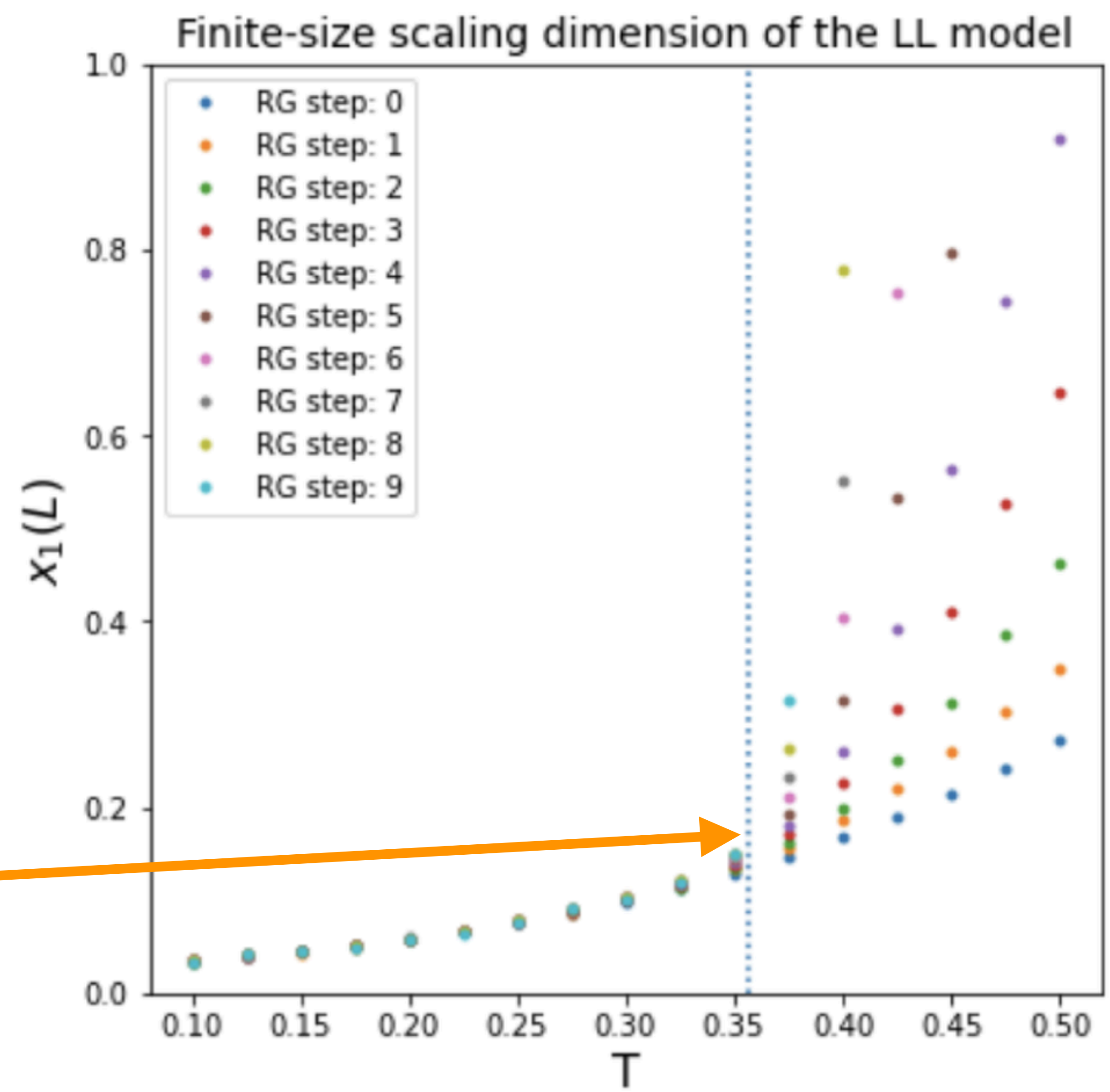
RP^2 Model at Intermediate T



$$E_0(L) \sim \epsilon_0 L - \frac{\pi c}{6L}.$$

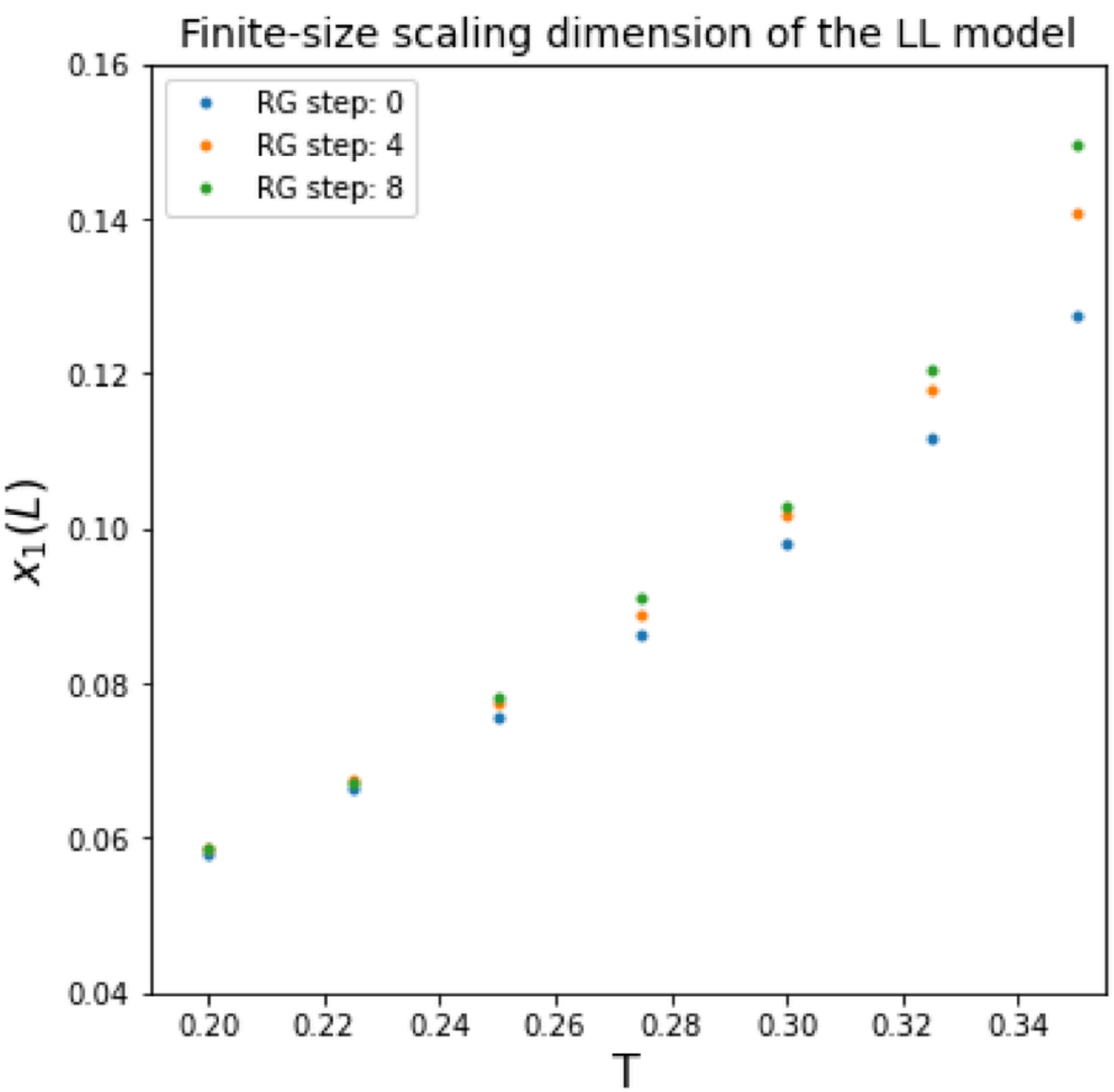
$c \sim 1.92 < 2$ no candidate CFT

$$\frac{2\pi x_n}{L} = E_n(L) - E_0(L).$$



Z2 vortex dissociation transition(?)

RP^2 Model at $T \leqq T_*$



Spectrum does not converge well even at $T \leqq T_*$

$c \sim 1.92 < 2$

SU(2) level 4

c=2 but smallest scaling dim

$$x_{j=1} = \frac{2}{3} = 0.666\ldots \gg 0.2$$

likely to support "asymptotic freedom"

RG flow $c=2 \to c=0$, but with a strong crossover at T_*

Conclusions (Heisenberg & RP^2)

Both Heisenberg & RP^2 Models
show c=2 at the UV fixed point for $T \to 0$
consistently with 2 (would-be) Nambu-Goldstone modes

At intermediate temperatures, the effective central charge
exhibit a "plateau" at c～1.9

but there is no appropriate candidate CFT
and scaling dimensions do not converge

Our results support the "asymptotic freedom"
scenario with the single disordered phase in $T>0$

Discussion (Heisenberg & RP^2)

Difficulty in distinguishing very large but finite correlation length from criticality

Even when (system size) < (correlation length) the Hamiltonian/transfer matrix spectrum gives useful information!

We do not completely rule out the possibility of the critical point at $T \sim T_*$ or the critical phase in $T \lesssim T_*$ suggested in several papers. However, to pursue this viewpoint, one would need to explain the effective central charge and the spectra obtained in the present TNR study, which is perhaps more difficult than simply discussing whether the correlation length diverges or not.

Conclusions (Overall)

TNR + Level Spectroscopy (finite size scaling of CFT) allows

- super accurate determination of BKT critical point
- visualization of Kosterlitz RG flow by extraction of running coupling constants from the spectrum
- identify the UV fixed point for the Heisenberg & RP^2 models
- clarify(?) the crossover in the Heisenberg & RP^2 models

Future: extension/application to more nontrivial systems & unknown physics