Workshop Entanglement Scaling and Criticality with Tensor Networks
EPFL, December 2nd, 2022

Tensor Networks Can Resolve Fermi Surfaces

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GHENT UNIVERSITY

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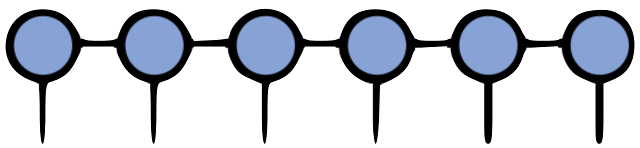
Tensor networks

In 1D: matrix product states (MPS)

⇒ area law of entanglement scaling



- gapped ground states of local hamiltonians
- many rigorous results
  (Hastings, Verstraete, Cirac, Wolf, Perez-Garcia, Schuch, Arad, Kitaev, Landau, Vazirani, Huang, …)
- numerically confirmed by 30 years of DMRG

Tensor networks

In 1D: critical point ≈ conformal field theory

$$S = \frac{c}{3}\log(L/a)$$

Finite bond dimension

$\Rightarrow$ finite entanglement

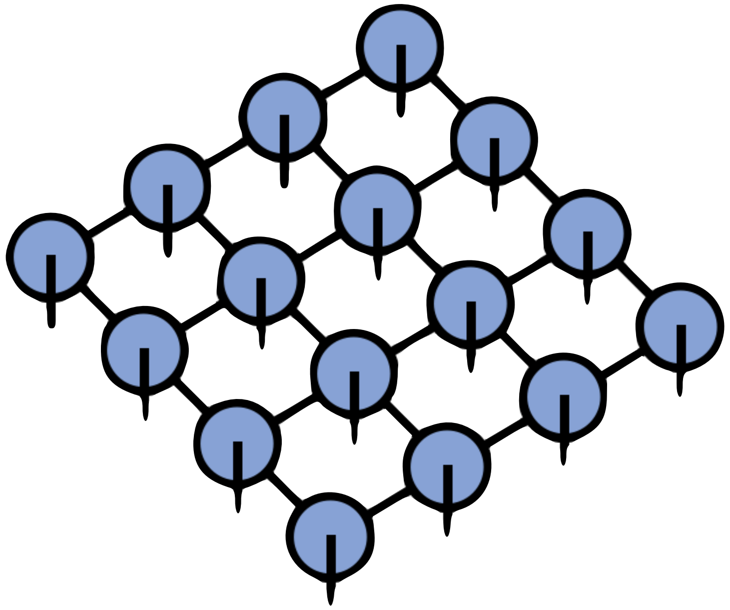
$\Rightarrow$ finite correlation length $\qquad \xi \sim D^{\kappa} \qquad \kappa = \frac{6}{c\left(\sqrt{12/c}+1\right)}$

$\Rightarrow$ finite size scaling $\qquad D \gtrsim= (L/a)^{\frac{c}{6}\left(1+\sqrt{\frac{12}{c}}\right)}$

$\Rightarrow$ finite entanglement scaling, scaling hypothesis

Nishino et al, Tagliacozzo et al, Pollmann et al,
Pirvu et al, McCulloch et al, Vanhecke et al, …

Tensor networks



In 2D: projected entangled-pair states (PEPS)

⇒ area law of entanglement scaling

- ➞ short-range entangled states
  (topologically trivial)

- ➞ PEPS representation of Levin-Wen string nets
  (non-chiral topological order)

- ➞ Certain scale-invariant states
  (Rokshar-Kivelson / quantum Lifshitz)

- ➞ Certain chiral states
  (Dubail, Read, Wahl, Tu, Schuch, Cirac, Poilblanc, …)

Tensor networks

In 2D: entanglement scaling in critical points

➞ typical critical point $S \sim L + O(\log L)$

P Corboz, P Czarnik, G Kapteijns, and L Tagliacozzo, Phys Rev X 8, 031031 (2018)
M Rader and A Laüchli, Phys Rev X 8, 031030 (2018)
P Czarnik and P Corboz, Phys Rev B 99, 245107 (2019)
B Vanhecke, J Hasik, F Verstraete, and L Vanderstraeten, Phys Rev Lett 129, 200601 (2022)

➞ Fermi surface $S \sim L \log L$

Can PEPS capture the physics of Fermi surfaces in some scaling limit?

Gaussian fermionic PEPS

Fermionic versions of PEPS can be introduced using
swap gates, Grassman numbers, graded vector spaces, …

Reproduction of Fermi surfaces can be studied with free fermions

→ Restriction to Gaussian, fermionic PEPS

C Kraus, N Schuch, F Verstraete, and I Cirac, Phys Rev A 81, 052338 (2010)

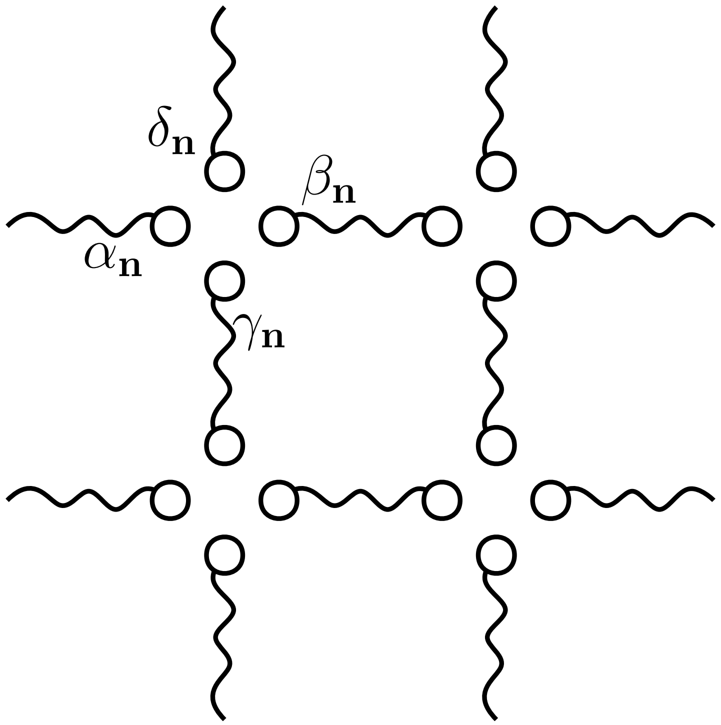
Gaussian fermionic PEPS

Create input state with virtual fermions entangled in pairs

$$|\psi_{\text{in}}\rangle = \prod_{\mathbf{n}} H_{\mathbf{n}} V_{\mathbf{n}} |0\rangle$$

$$H_{\mathbf{n}} = \frac{1}{\sqrt{2}}\left(1 + \beta_{\mathbf{n}}^{\dagger}\alpha_{\mathbf{n}_{\rightarrow}}^{\dagger}\right)$$

$$V_{\mathbf{n}} = \frac{1}{\sqrt{2}}\left(1 + \delta_{\mathbf{n}}^{\dagger}\gamma_{\mathbf{n}_{\uparrow}}^{\dagger}\right)$$



Gaussian fermionic PEPS

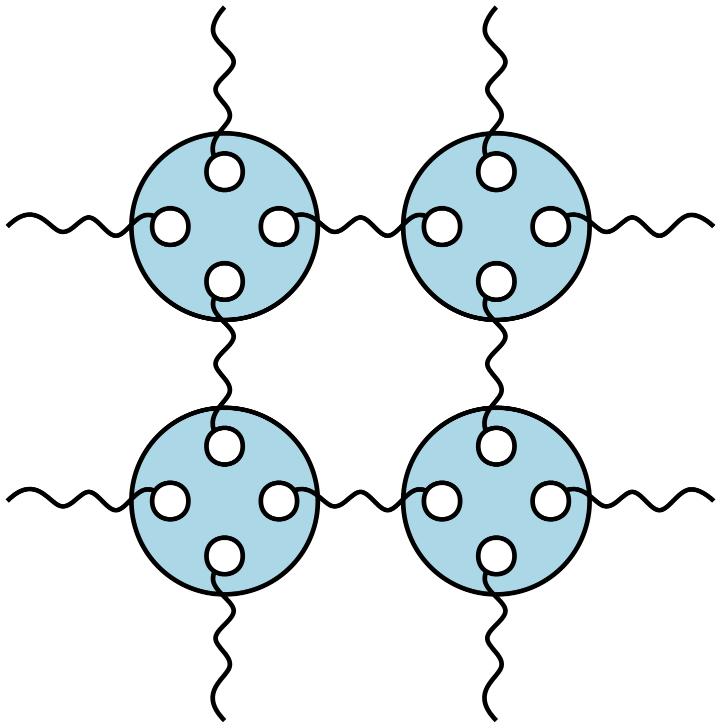
Create input state with virtual fermions entangled in pairs

$$|\psi_{\text{in}}\rangle = \prod_{\mathbf{n}} H_{\mathbf{n}} V_{\mathbf{n}} |0\rangle$$

$$H_{\mathbf{n}} = \frac{1}{\sqrt{2}} \left(1 + \beta_{\mathbf{n}}^{\dagger} \alpha_{\mathbf{n}_{\rightarrow}}^{\dagger}\right)$$

$$V_{\mathbf{n}} = \frac{1}{\sqrt{2}} \left(1 + \delta_{\mathbf{n}}^{\dagger} \gamma_{\mathbf{n}_{\uparrow}}^{\dagger}\right)$$

Project this $|\psi_{\text{in}}\rangle$ locally to the physical level with



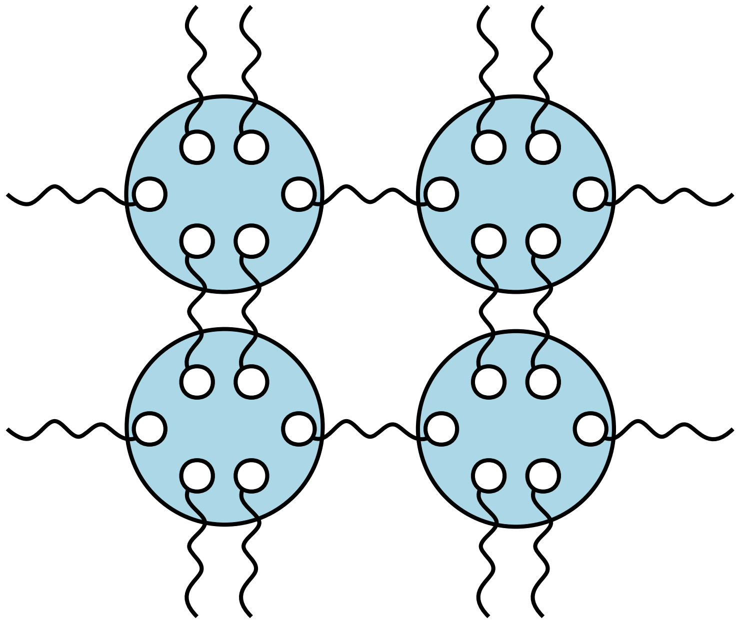
$$A_{\mathbf{n}} = [A_{\mathbf{n}}]_{lrdu}^{k} \, a_{\mathbf{n}}^{\dagger\,k} \alpha_{\mathbf{n}}^{l} \beta_{\mathbf{n}}^{r} \gamma_{\mathbf{n}}^{d} \delta_{\mathbf{n}}^{u}$$

$$\Rightarrow |\psi_{\text{out}}\rangle = \langle 0|_{\text{virt}} \prod_{\mathbf{n}} A_{\mathbf{n}} |\psi_{\text{in}}\rangle$$

Gaussian fermionic PEPS

n_i virtual pairs in direction i

$\Rightarrow \; D_i = 2^{n_i}$



Both $|\psi_{\rm in}\rangle$ and $|\psi_{\rm out}\rangle$ are Gaussian and translation invariant

$\Rightarrow$ completely characterized by $G_{ij}(\mathbf{k}) = \frac{i}{2}\langle[d^i_{\mathbf{k}}, {d^j_{\mathbf{k}}}^\dagger]\rangle$

$$d^i_{\mathbf{k}} = \frac{1}{\sqrt{N}}\sum_{\mathbf{n}} e^{-i\mathbf{k}\cdot\mathbf{n}} c^i_{\mathbf{n}}$$

$$c^1_{\mathbf{n}} = a^\dagger_{\mathbf{n}} + a_{\mathbf{n}}$$

$$c^2_{\mathbf{n}} = -i\left(a^\dagger_{\mathbf{n}} - a_{\mathbf{n}}\right)$$

Gaussian fermionic PEPS

$$G_{\text{in}}(\mathbf{k}) = \left(\bigoplus_i \begin{pmatrix} & & e^{i\mathbf{k}\cdot\mathbf{a}_i} & \\ & & & e^{i\mathbf{k}\cdot\mathbf{a}_i} \\ -e^{-i\mathbf{k}\cdot\mathbf{a}_i} & & & \\ & -e^{-i\mathbf{k}\cdot\mathbf{a}_i} & & \end{pmatrix}^{\oplus n_i} \right)$$

$$G_{\text{out}}(\mathbf{k}) = A - B(D - G_{\text{in}}(\mathbf{k}))^{-1}C$$

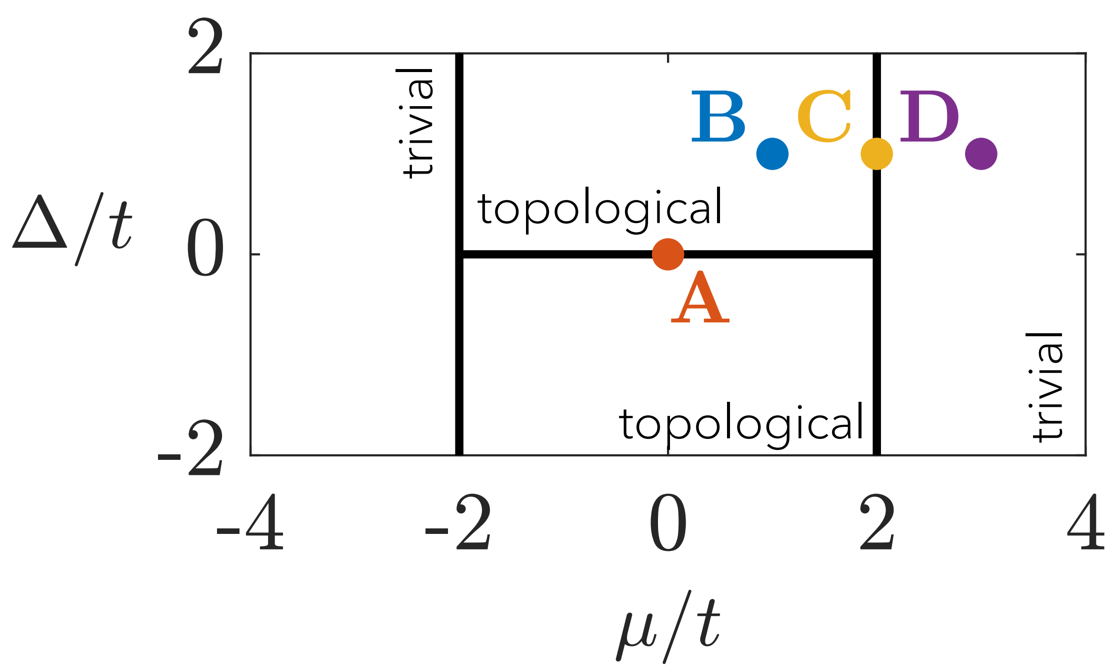
- Schur complement formula parametrized by

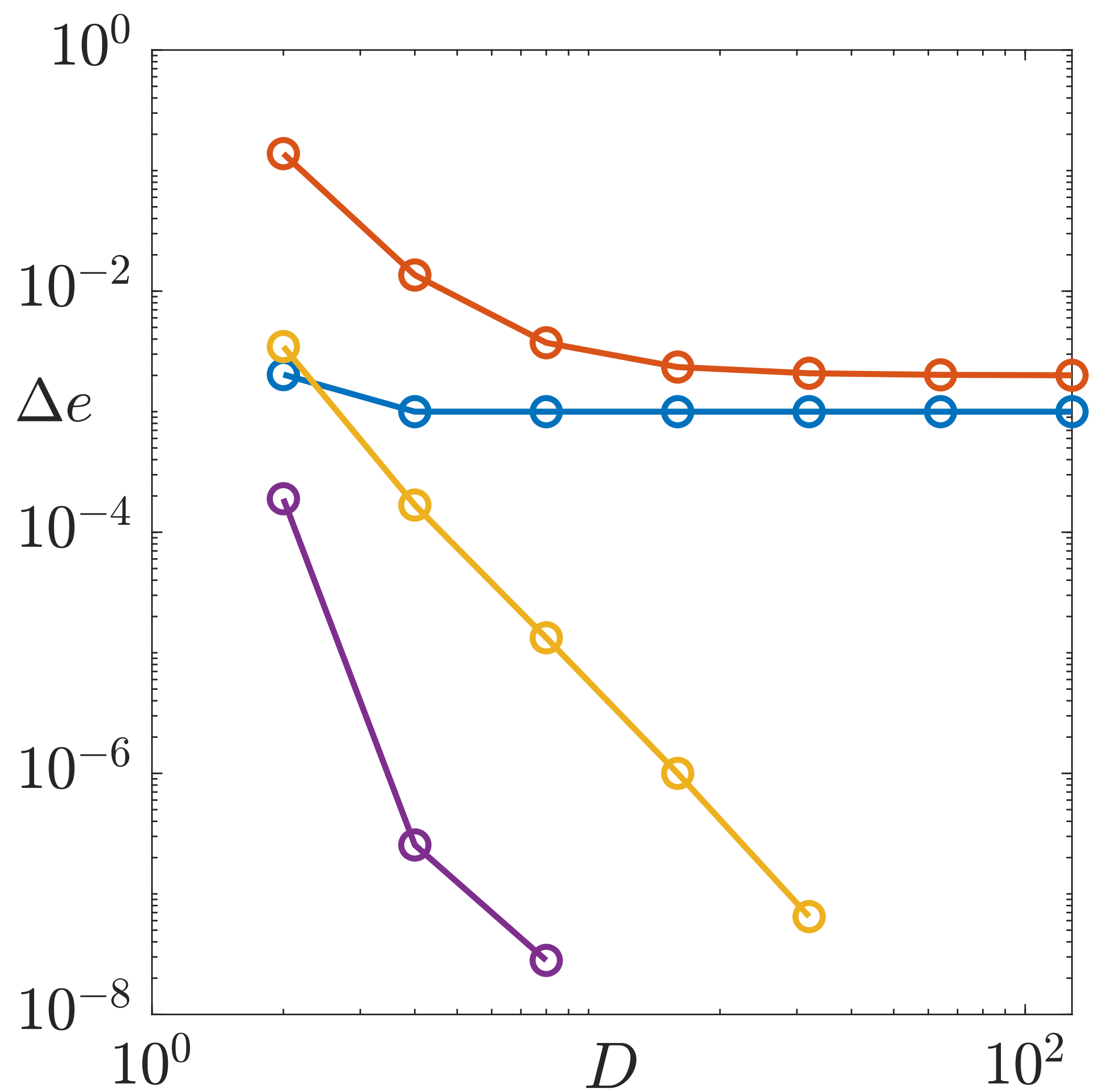
$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = -X^T$$

- Energy density is linear in $G_{\text{out}}(\mathbf{k})$ and can be minimized w.r.t. X

1d example: Kitaev chain

$$H = -\sum_n \left(t\, a_n^\dagger a_{n+1} + h.c.\right) - \mu \sum_n a_n^\dagger a_n - \sum_n \left(\Delta\, a_n^\dagger a_{n+1}^\dagger + h.c.\right)$$





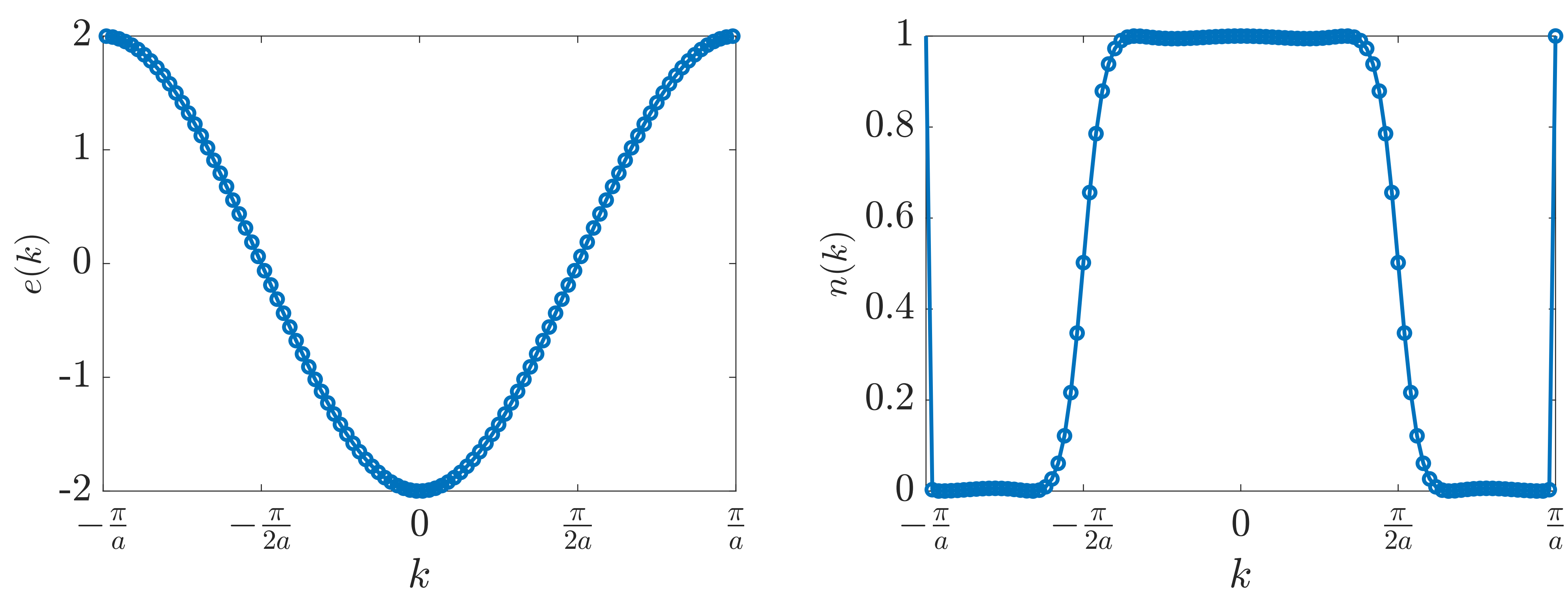
Fixed energy errors in (and between) topological phases?

1d example: Kitaev chain

Real-space correlation matrix is real ⇒ $G(-\mathbf{k}) = G^*(\mathbf{k})$

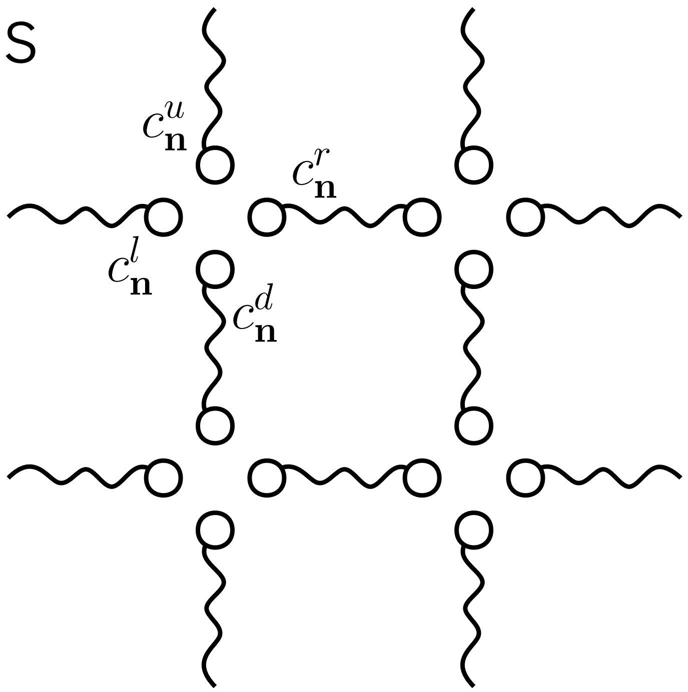
In time reversal invariant momenta ($\mathbf{k} = -\mathbf{k}$): real $G(\mathbf{k})$

- ⇒ definite parity: $P_{\mathbf{k}} = \mathrm{Pf}\left(G(\mathbf{k})\right) = \pm 1$
- ⇒ input state: $P_{\mathrm{in}\,\mathbf{k}} = \mathrm{Pf}\left(G_{\mathrm{in}}(\mathbf{k})\right) = -1$
- ⇒ Gaussian fermionic MPS: $P_{\mathrm{out}\,\mathbf{k}} = \langle (-1)^{a_{\mathbf{k}}^\dagger a_{\mathbf{k}}} \rangle = \pm P_{\mathrm{in}\,\mathbf{k}}$



Gaussian fermionic PEPS: update

More elementary ansatz based on Majorana bonds



$$G_{\text{in}}(\mathbf{k}) = \bigoplus_i \begin{pmatrix} 0 & e^{i\mathbf{k}\cdot\mathbf{a}_i} \\ -e^{-i\mathbf{k}\cdot\mathbf{a}_i} & 0 \end{pmatrix}^{\oplus\chi_i}$$

$$D_i = \sqrt{2}^{\chi_i}$$

Parity depends on bond dimension!

$$P_{\text{in}\,\mathbf{0}} = \text{Pf}\left(G_{\text{in}}(\mathbf{0})\right) = 1$$

$$P_{\text{out}\,\mathbf{k}} = \langle(-1)^{a^\dagger_{\mathbf{k}} a_{\mathbf{k}}}\rangle = \pm P_{\text{in}\,\mathbf{k}}$$

$$P_{\text{in}\,\frac{\mathbf{b}_i}{2}} = \text{Pf}\left(G_{\text{in}}(\tfrac{\mathbf{b}_i}{2})\right) = (-1)^{\chi_i}$$

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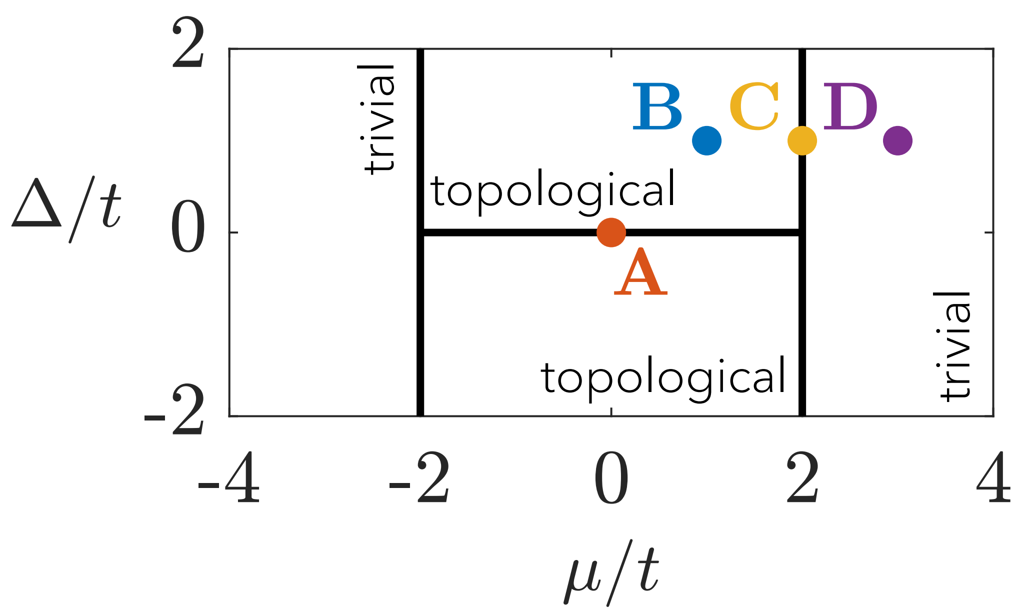
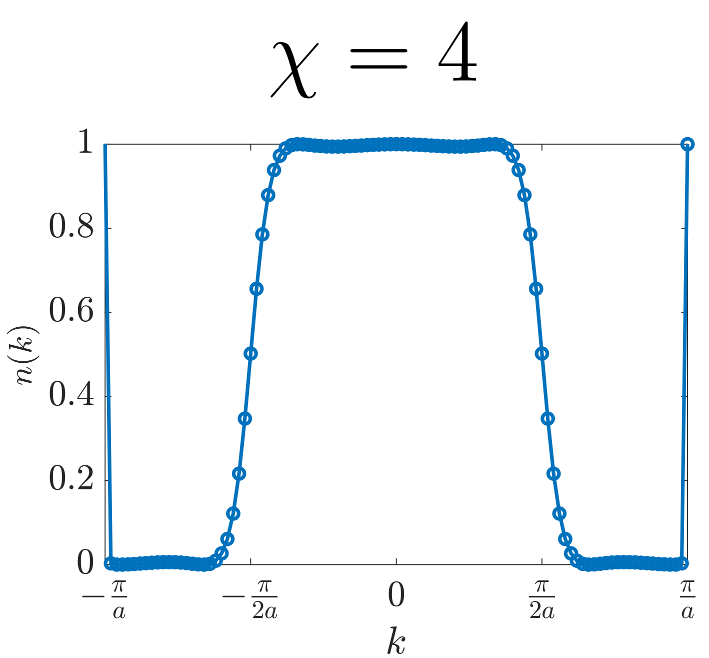
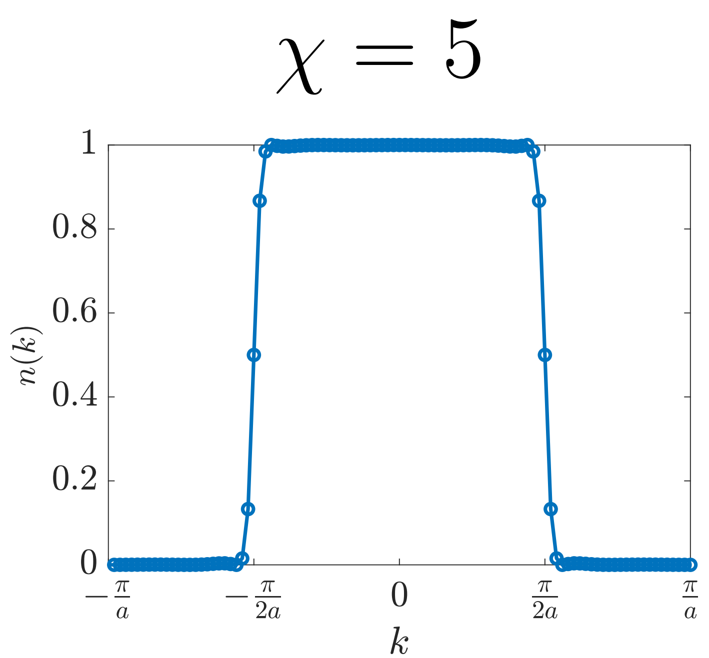
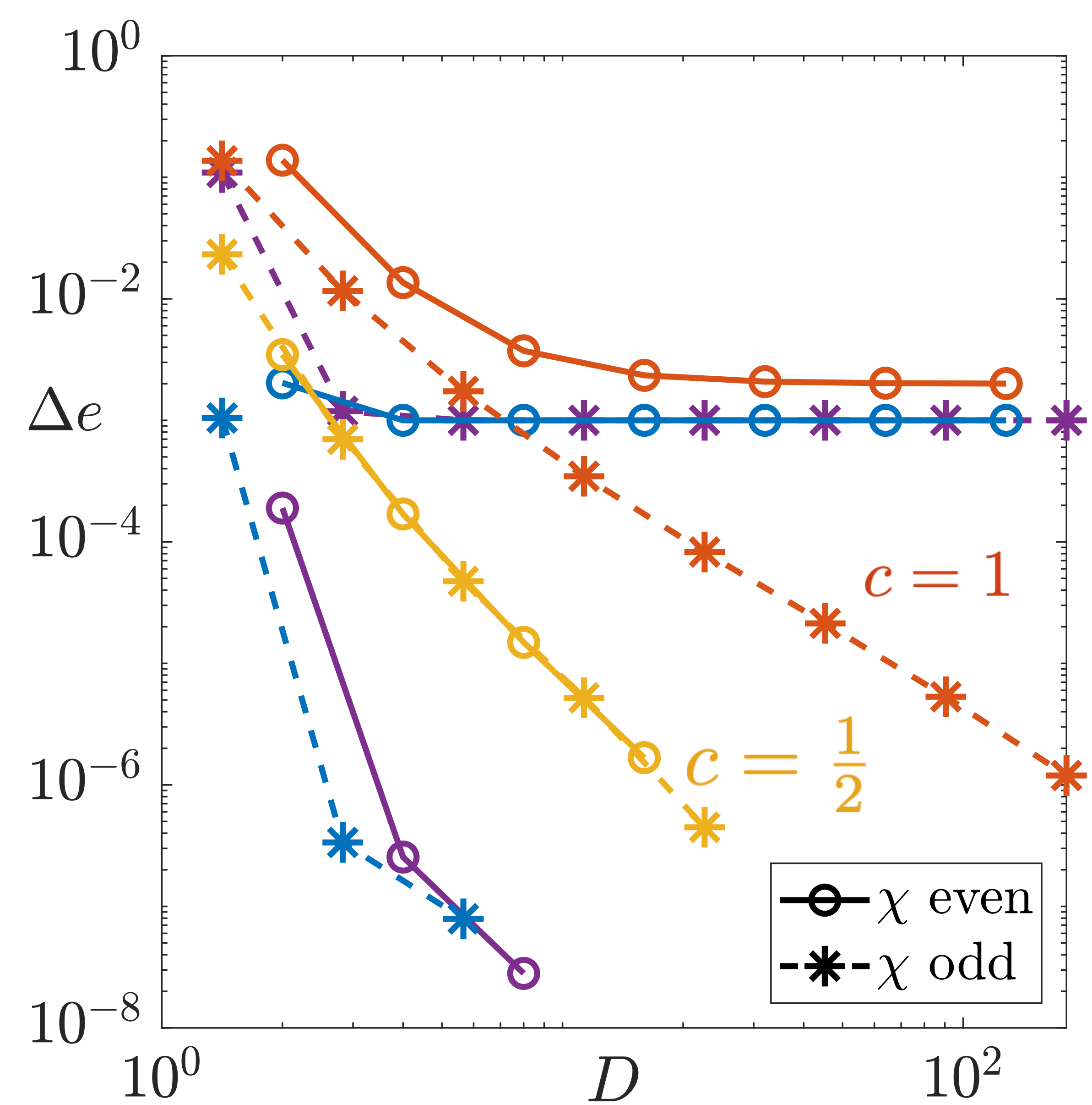


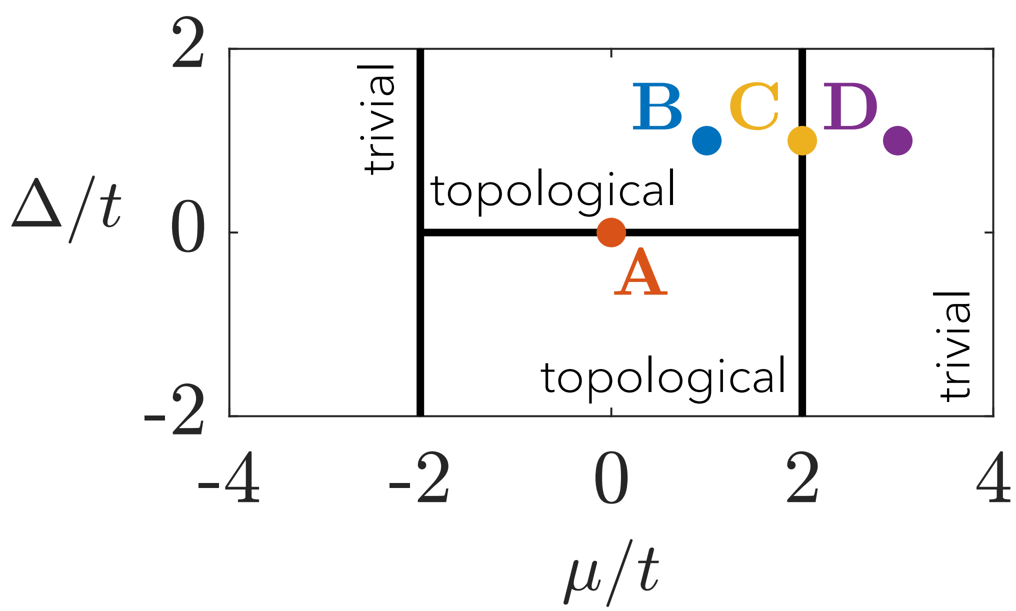


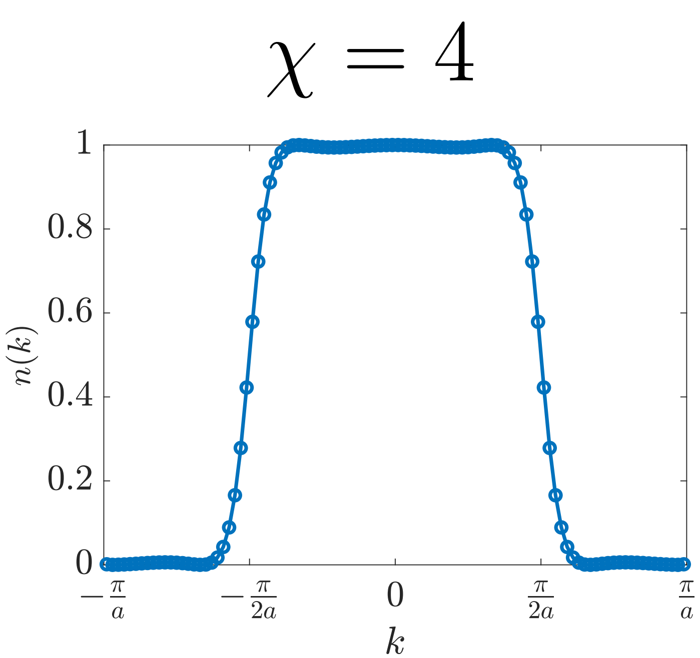


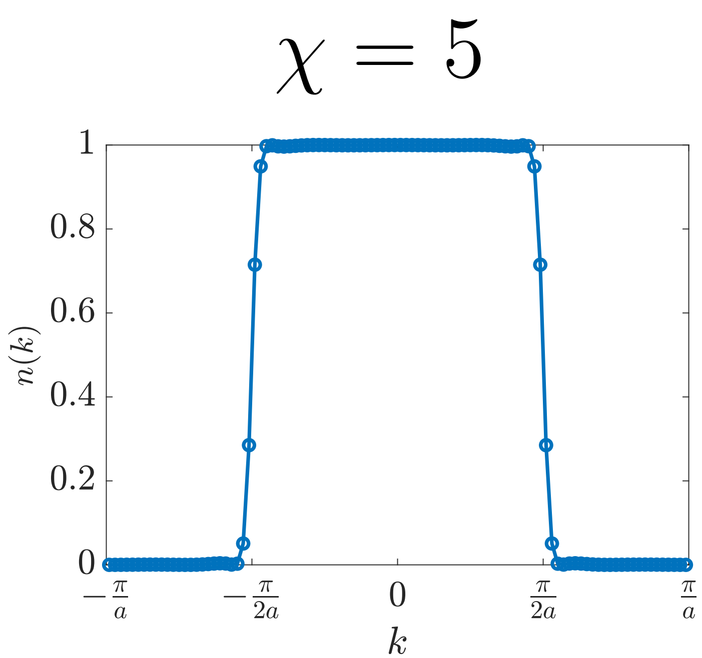


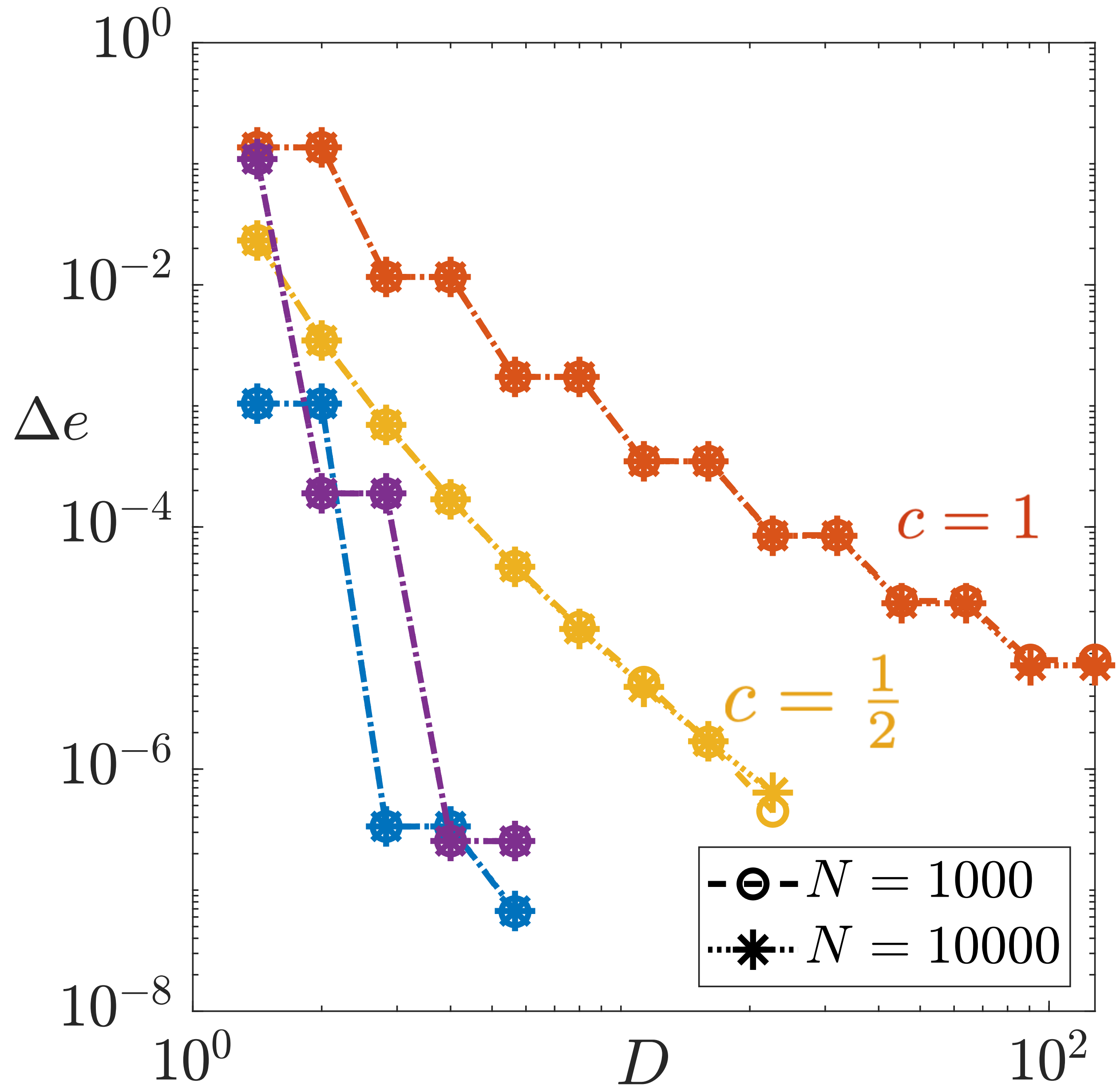
1d example: Kitaev chain

Anti-periodic boundary conditions



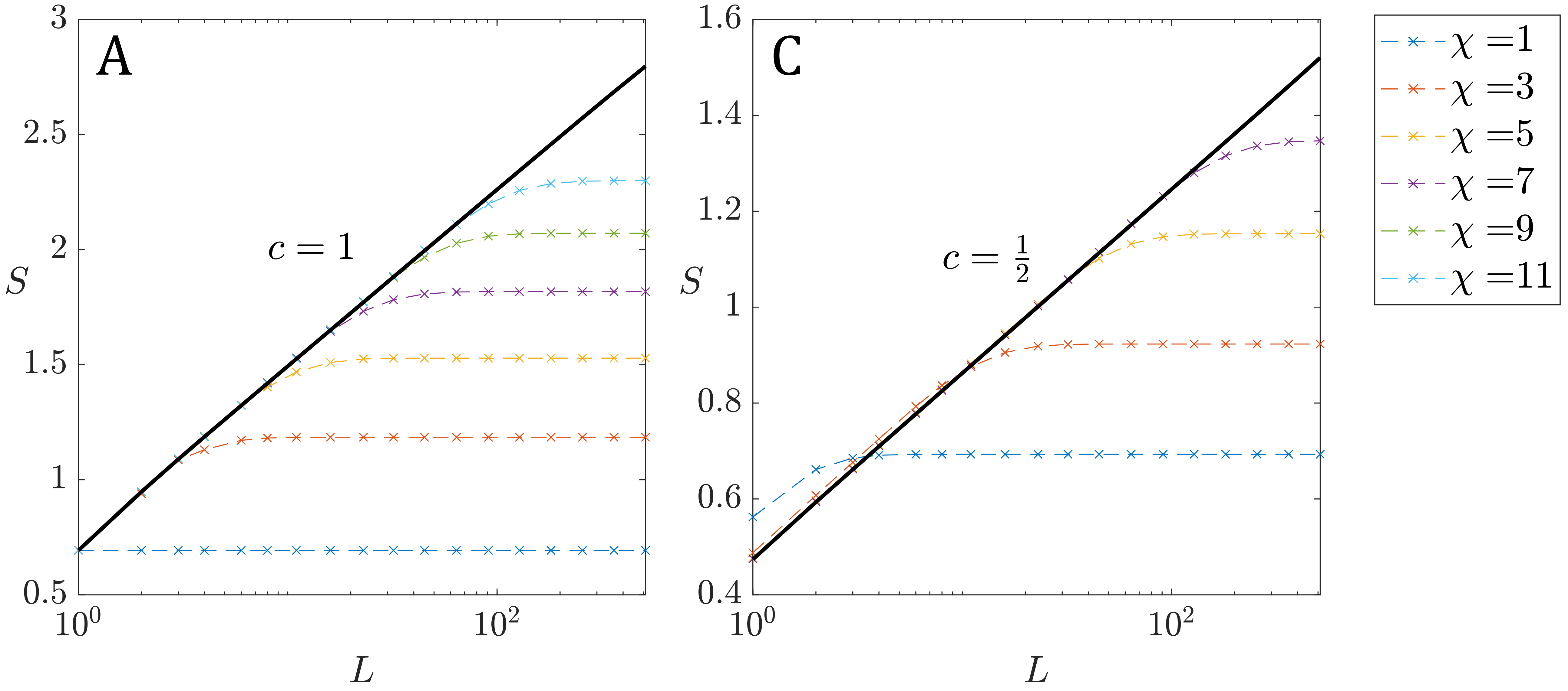






1d example: Kitaev chain

Entanglement scaling in critical points



Fermi surface in 2d

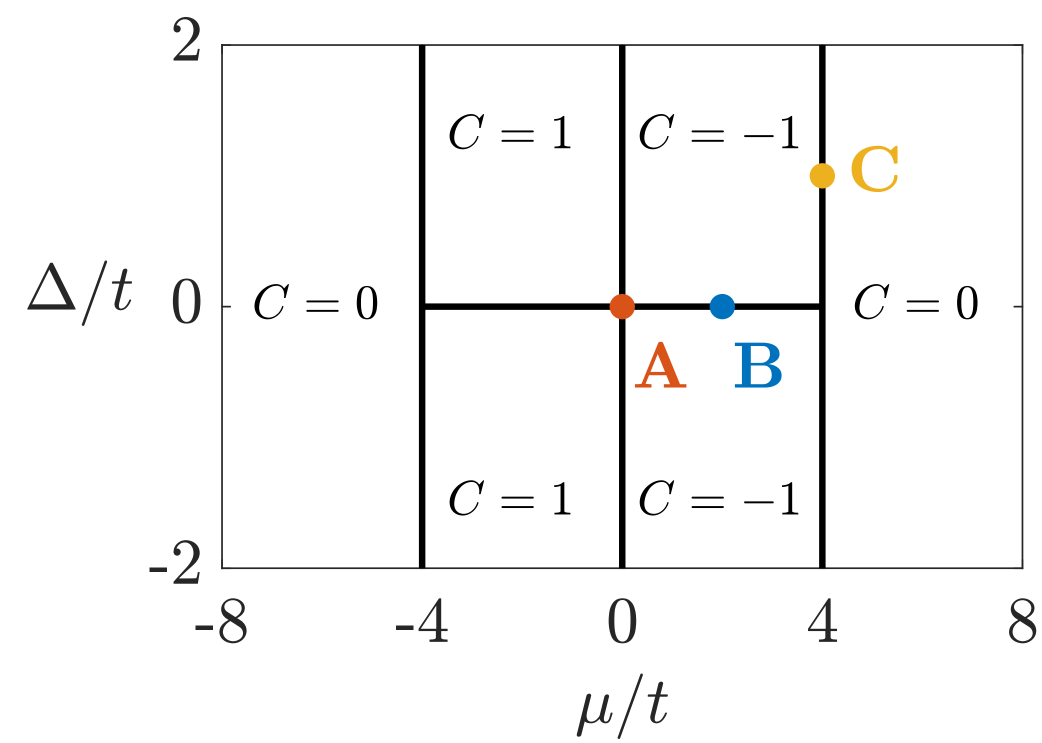
p-wave superconductor

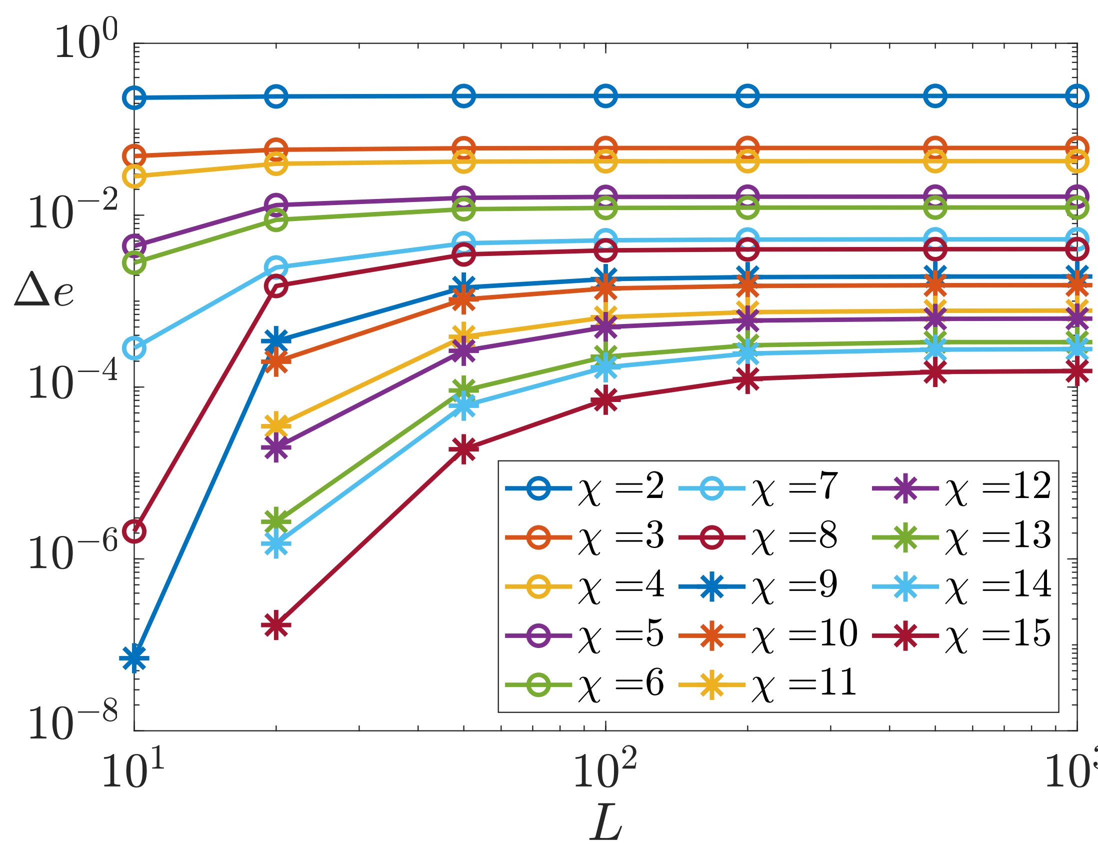
$$H = H_t + H_\mu + H_\Delta$$

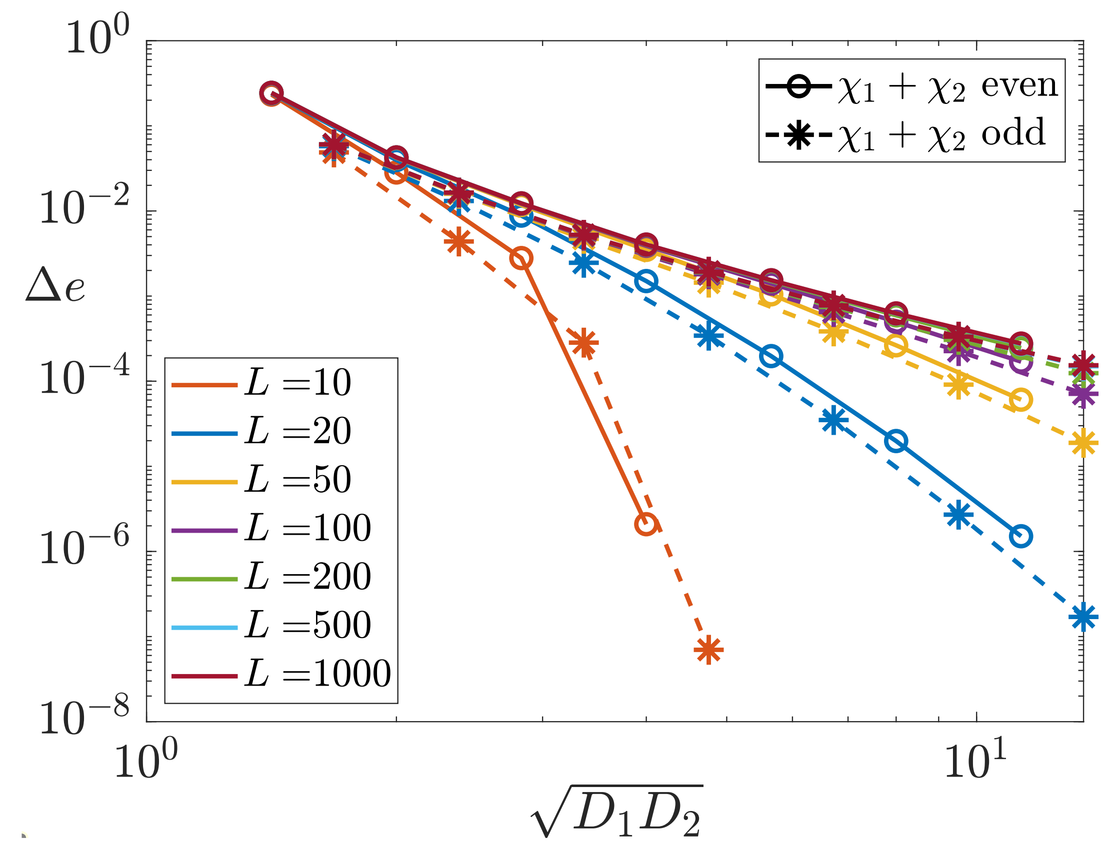
$$H_t = -t \sum_{\mathbf{n}} \left(a^\dagger_{\mathbf{n}} a_{\mathbf{n}_\rightarrow} + a^\dagger_{\mathbf{n}} a_{\mathbf{n}_\uparrow} + h.c. \right)$$

$$H_\mu = -\mu \sum_{\mathbf{n}} a^\dagger_{\mathbf{n}} a_{\mathbf{n}}$$

$$H_\Delta = -\Delta \sum_{\mathbf{n}} \left(a^\dagger_{\mathbf{n}} a^\dagger_{\mathbf{n}_\rightarrow} + i a^\dagger_{\mathbf{n}} a^\dagger_{\mathbf{n}_\uparrow} + h.c. \right)$$







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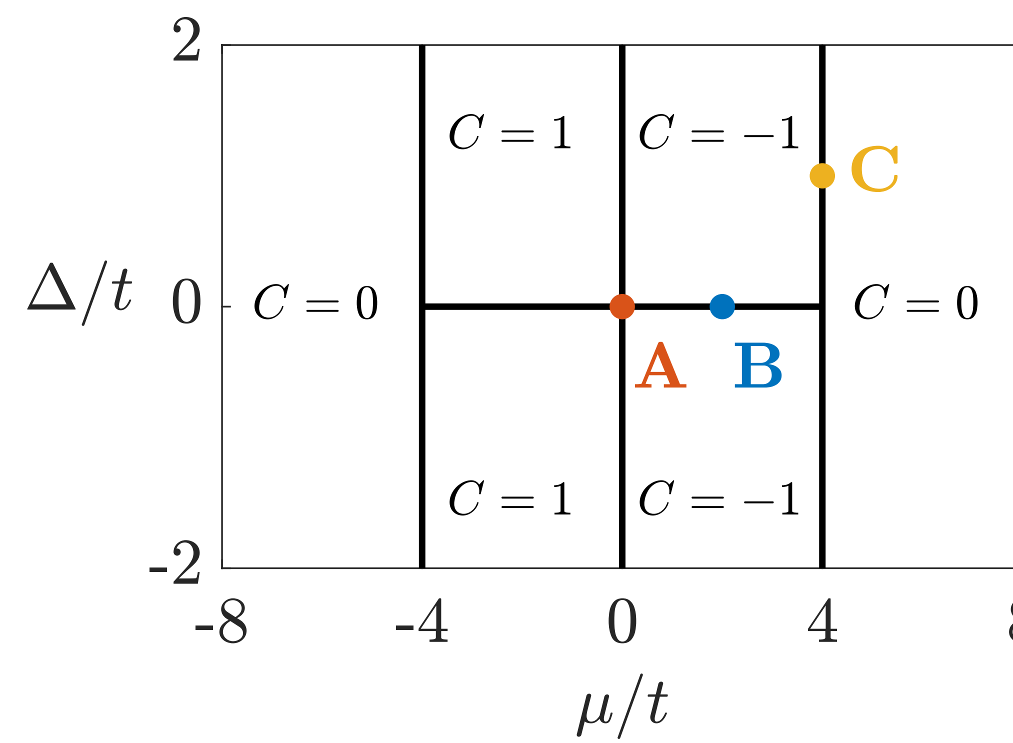

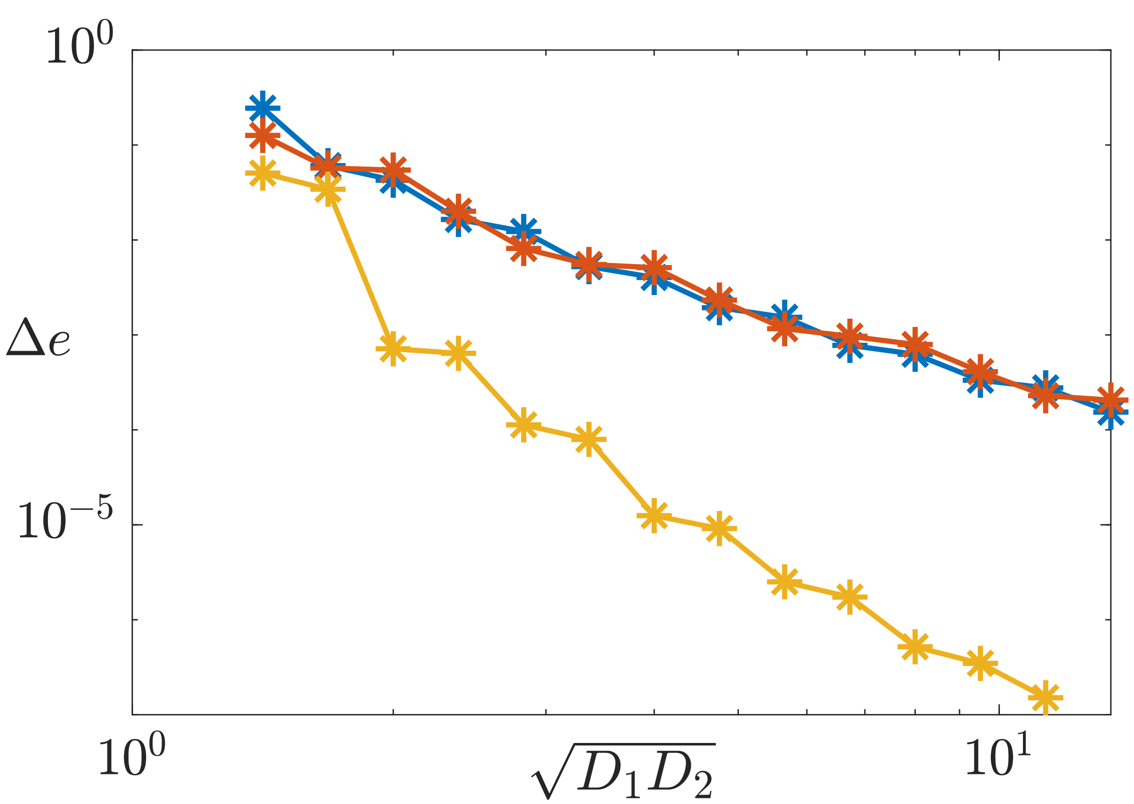

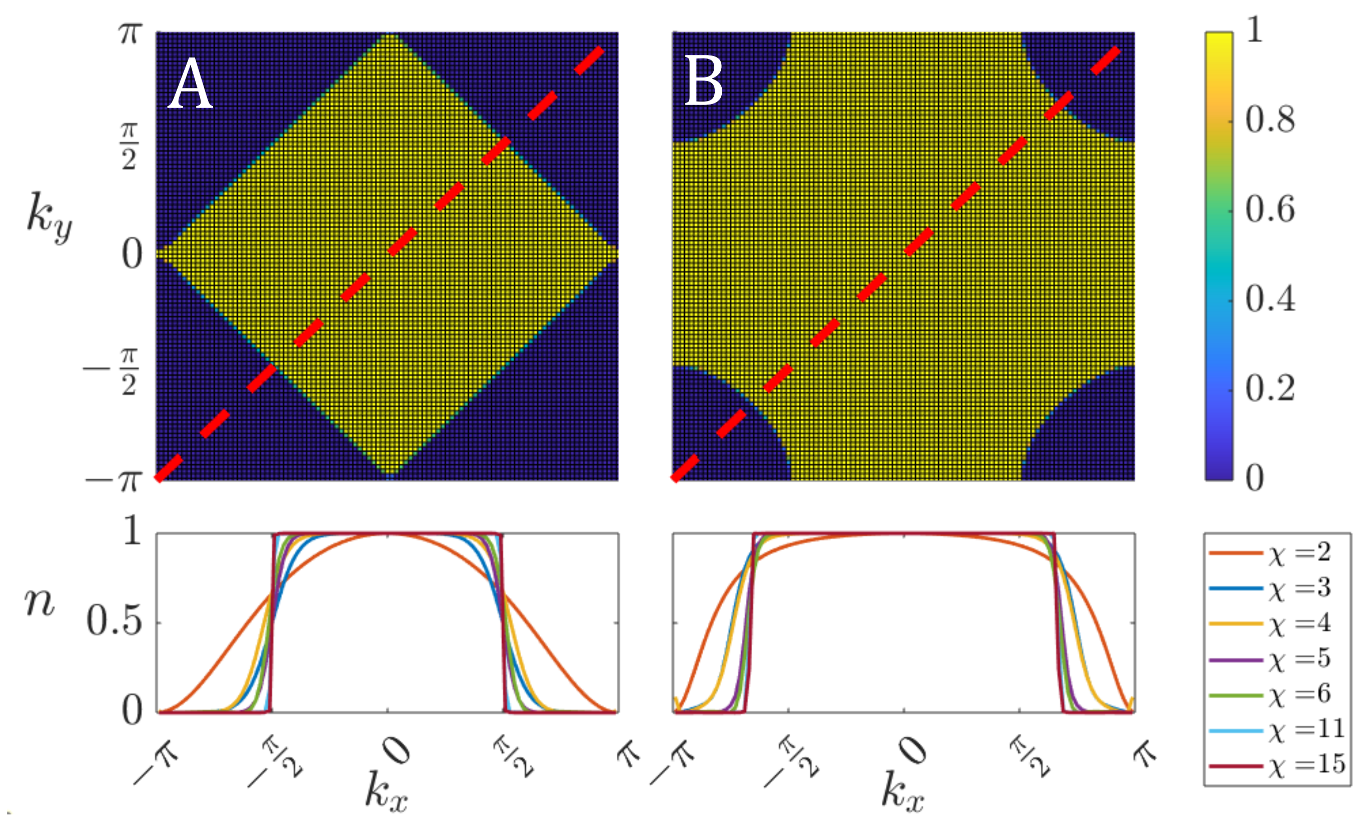

Fermi surface in 2d

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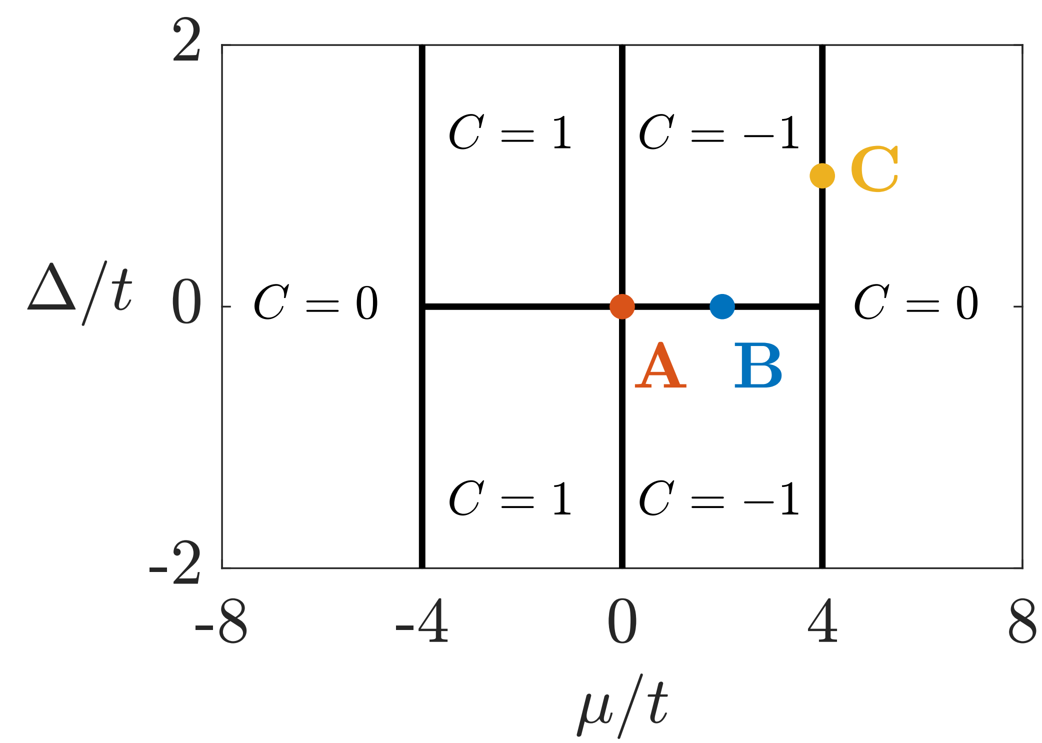
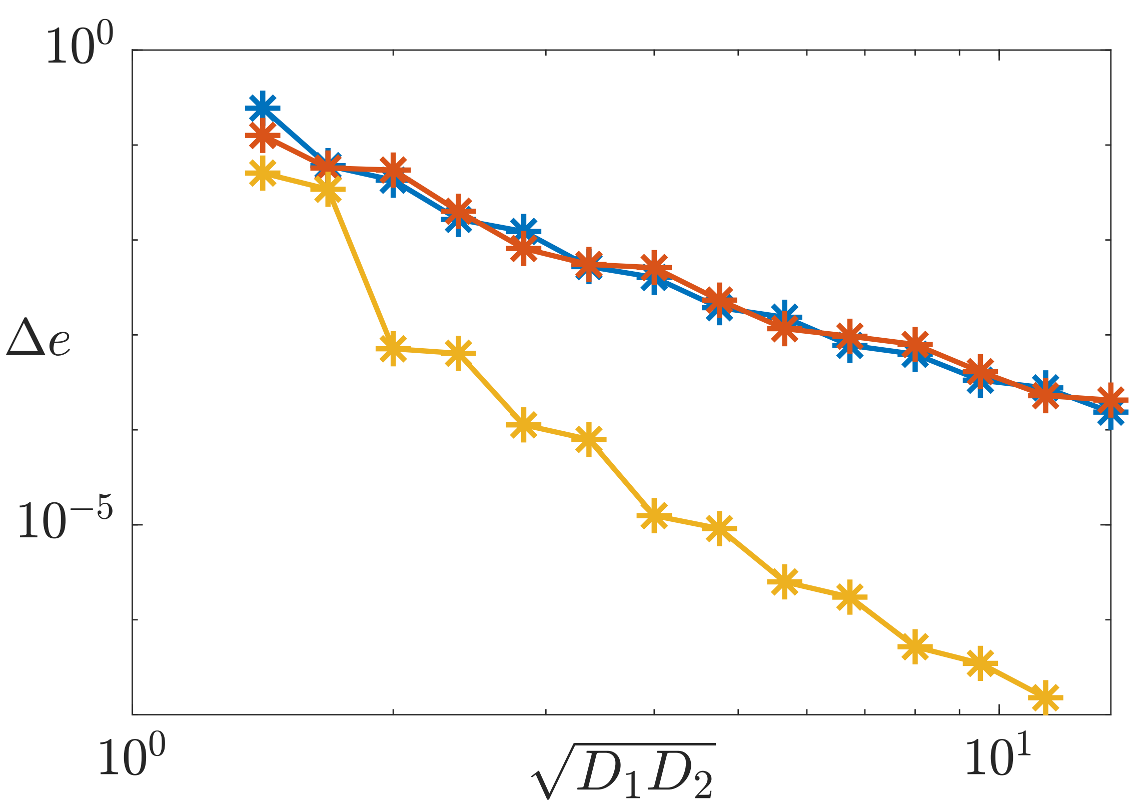
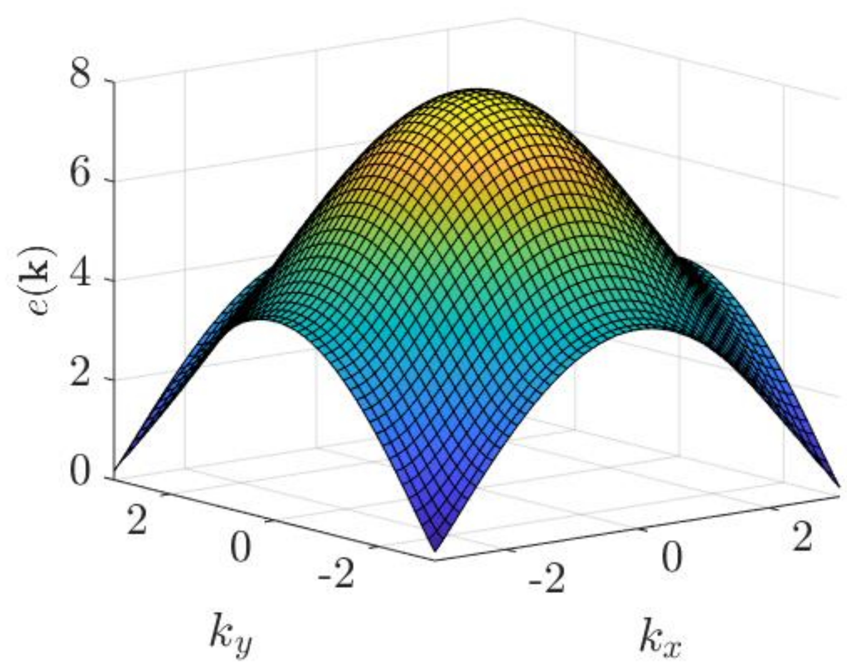


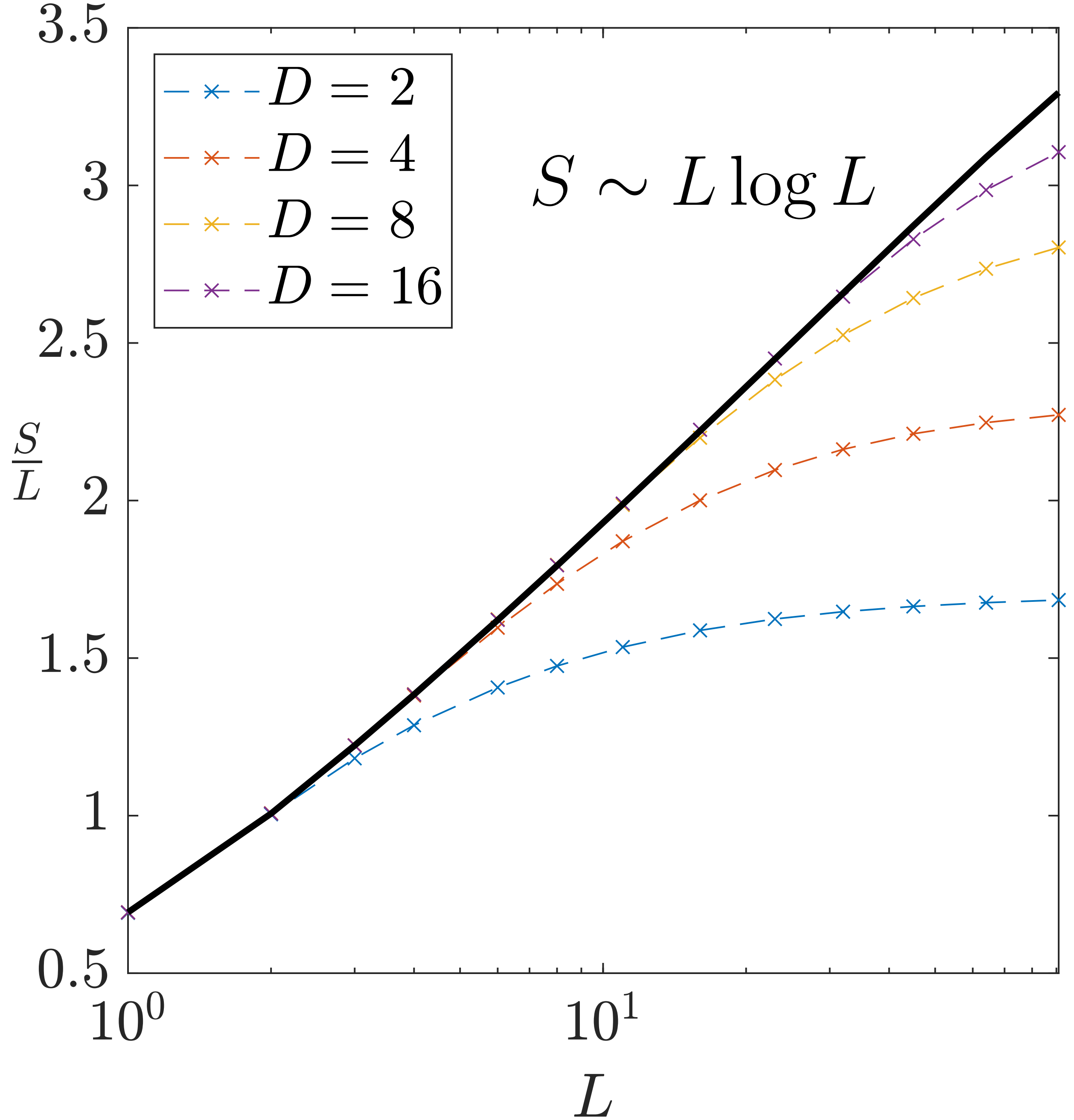


C: Dirac point



Fermi surface in 2d

Entanglement scaling



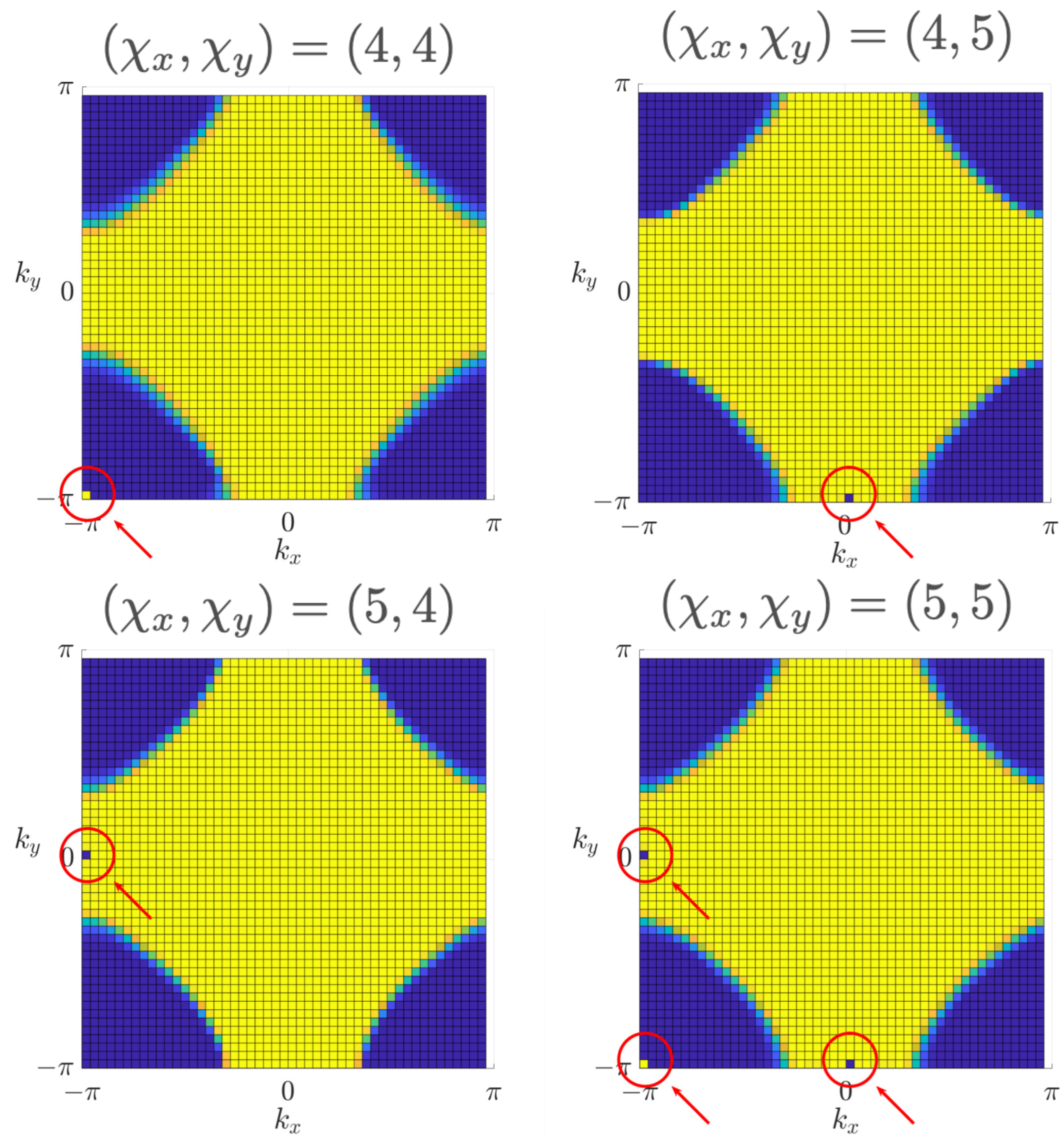
Topological considerations

Parity restrictions yield obstruction

$$P_{\mathrm{in}\,\mathbf{0}} = \mathrm{Pf}\left(G_{\mathrm{in}}(\mathbf{0})\right) = 1$$

$$P_{\mathrm{in}\,\frac{\mathbf{b}_i}{2}} = \mathrm{Pf}\left(G_{\mathrm{in}}(\tfrac{\mathbf{b}_i}{2})\right) = (-1)^{\chi_i}$$

$$P_{\mathrm{out}\,\mathbf{k}} = \langle(-1)^{a^\dagger_{\mathbf{k}}a_{\mathbf{k}}}\rangle = \pm P_{\mathrm{in}\,\mathbf{k}}$$



Gaussian fPEPS: 2^{2^d}

General Gaussian: 2^{d+1}

→ Physical significance?

Topological considerations

General quadratic model

$$H = \frac{1}{2}\sum_{\mathbf{k}} \Upsilon^\dagger_{\mathbf{k}} H_{\mathrm{BdG}}(\mathbf{k})\Upsilon_{\mathbf{k}} + E$$

$$\downarrow \quad \tilde{\Upsilon}_{\mathbf{k}} = U^\dagger(\mathbf{k})\Upsilon_{\mathbf{k}}$$

$$H = \frac{1}{2}\sum_{\mathbf{k}} \tilde{\Upsilon}^\dagger_{\mathbf{k}} \begin{pmatrix} e(\mathbf{k}) & \\ & -e(-\mathbf{k}) \end{pmatrix} \tilde{\Upsilon}_{\mathbf{k}} + E$$

$$H = \sum_{\mathbf{k}} \tilde{a}^\dagger_{\mathbf{k}} e(\mathbf{k}) \tilde{a}_{\mathbf{k}} - \frac{1}{2}\sum_{\mathbf{k}} \mathrm{tr}(e(\mathbf{k})) + E$$

$$H_{\mathrm{BdG}}(\mathbf{k}) = \begin{pmatrix} \Xi(\mathbf{k}) & \Delta(\mathbf{k}) \\ -\Delta^*(-\mathbf{k}) & -\Xi^T(-\mathbf{k}) \end{pmatrix}$$

$$\Xi^\dagger(\mathbf{k}) = \Xi(\mathbf{k}) \qquad \Delta(\mathbf{k}) = -\Delta^T(-\mathbf{k})$$

$$U(\mathbf{k}) = \begin{pmatrix} U_+(\mathbf{k}) & U_-(\mathbf{k}) \end{pmatrix}$$

(Majorana) Chern number

$$C = \frac{i}{2\pi}\int_{\mathrm{BZ}} \left(\frac{\partial}{\partial k_x}\mathrm{tr}\left(A_y(\mathbf{k})\right) - \frac{\partial}{\partial k_y}\mathrm{tr}\left(A_x(\mathbf{k})\right) \right) \mathrm{d}k_x \mathrm{d}k_y$$

$$A_i^{\alpha\beta}(\mathbf{k}) = \langle u_-^\alpha(\mathbf{k})|\frac{\partial}{\partial k_i}|u_-^\beta(\mathbf{k})\rangle$$

Topological considerations

Dimensional reduction

$$C = \frac{i}{2\pi}\int_{\mathrm{BZ}}\left(\frac{\partial}{\partial k_x}\mathrm{tr}\left(A_y(\mathbf{k})\right) - \frac{\partial}{\partial k_y}\mathrm{tr}\left(A_x(\mathbf{k})\right)\right)\mathrm{d}k_x\mathrm{d}k_y$$

appropriate gauge: $\frac{\partial}{\partial k_y}\mathrm{tr}\left(A_x(\mathbf{k})\right) = 0$

$$C = \int_{-\pi}^{\pi}\frac{\partial}{\partial k_x}\mathrm{CS}_1(k_x)\mathrm{d}k_x$$

$$\mathrm{CS}_1(k_x) = \int_{-\pi}^{\pi}\frac{i}{2\pi}\mathrm{tr}\left(A_y(\mathbf{k})\right)\mathrm{d}k_y$$

for $k_x = 0, \pi$:

$$\exp\left(-2\pi i\,\mathrm{CS}_1(k_x)\right) = P_{(k_x,0)}P_{(k_x,\pi)}$$

$$C = 2\left[\mathrm{CS}_1(k_x = \pi) - \mathrm{CS}_1(k_x = 0)\right] \quad \mathrm{mod}\ 2$$

$$C = 2\left[\mathrm{CS}_1(k_y = \pi) - \mathrm{CS}_1(k_y = 0)\right] \quad \mathrm{mod}\ 2$$

$$\Rightarrow e^{i\pi C} = P_{(0,0)}P_{(\pi,0)}P_{(0,\pi)}P_{(\pi,\pi)}$$

Topological considerations

Odd Majorana Chern numbers require obstructed parities

$$e^{i\pi C} = P_{(0,0)} P_{(\pi,0)} P_{(0,\pi)} P_{(\pi,\pi)}$$

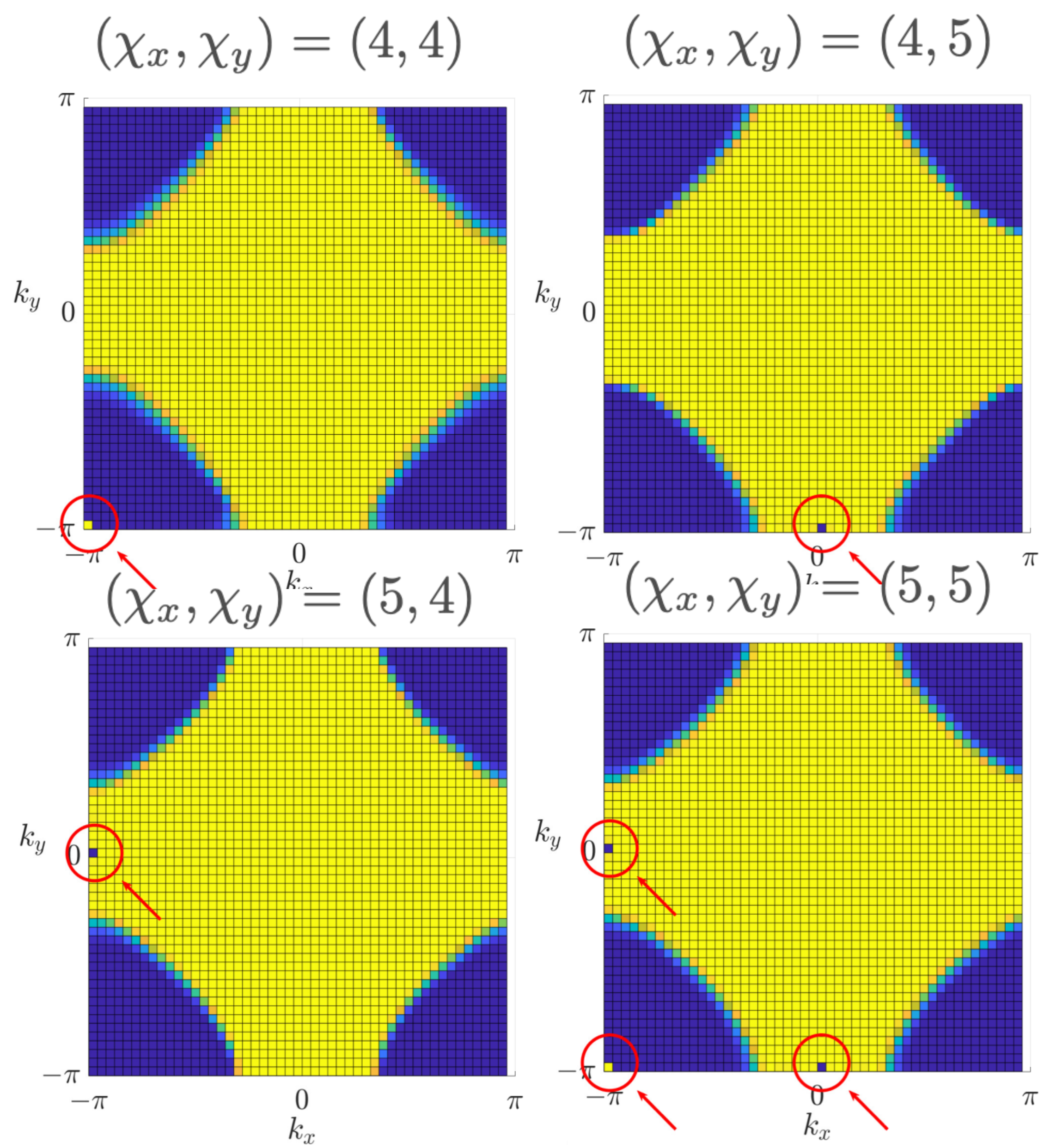
PEPS becomes singular with critical correlations to realize chirality

→ No-go theorem

N Read and J Dubail, PRB (2015)

→ Exact criticality at low D

T B Wahl, H Tu, N Schuch and J I Cirac, PRL (2013)



Outlook and Conclusions

- PEPS can capture 2D quantum criticality in an efficient scaling limit
- Fermi surfaces do not pose intrinsic difficulties
- Topology of nearby gapped phases can lead to obstructions

Outlook and Conclusions

Open questions and further research:

- → Characterize the nature of the entanglement scaling/ come up with scaling hypothesis
- → Extract generic tensor from the Gaussian ansatz
  - → Use this as initial guess → interactions
  - → Apply Gutzwiller projection → relevant interacting states
- → Obstructions and interplay with symmetries, characterization, …