X-ray Recoil Correction

We start from energy-momentum conservation. Energy conservation implies that if the atomic recoil energy is ΔE, the X-ray only carries an energy $E - \Delta E$ ($E \approx 3.2 MeV$). For the atomic mass, take that of ^{104}Pd ($96.7866\, GeV/c^2$).

$$
\begin{aligned}
& p_{X-ray} = p_{atom} \\
\Longrightarrow\ & \frac{E_\gamma}{c} = \sqrt{2 m_{atom} E_{atom}} \\
\Longrightarrow\ & \frac{E - \Delta E}{c} = \sqrt{2 m_{atom} \Delta E} \\
\Longrightarrow\ & (E - \Delta E)^2 = 2 m_{atom} c^2 \Delta E \\
\Longrightarrow\ & \Delta E^2 - 2\Delta E (E + m_{atom} c^2) + E^2 = 0 \qquad (1) \\
\Longrightarrow\ & \Delta E = \frac{2(E + m_{atom} c^2) - \sqrt{4(E + m_{atom} c^2)^2 - 4E^2}}{2} \\
\Longrightarrow\ & \Delta E = (E + m_{atom} c^2) - \sqrt{(E + m_{atom} c^2)^2 - E^2} \\
\Longrightarrow\ & \Delta E = (E + m_{atom} c^2) \left[1 - \sqrt{1 - \left(\frac{E}{E + m_{atom} c^2} \right)^2} \right] \qquad (2) \\
\Longrightarrow\ & \Delta E \approx 53 eV \qquad (3)
\end{aligned}
$$

One can also make an approximation that ΔE is much smaller than E ($\Delta E^2 \approx 0$). If this is used in (1), we obtain

$$
\begin{aligned}
& E^2 \approx 2\Delta E (E + m_{atom} c^2) \\
\Longrightarrow\ & \Delta E \approx \frac{E^2}{2(E + m_{atom} c^2)} \\
\Longrightarrow\ & \Delta E \approx 53 eV
\end{aligned}
$$

Alternatively, an approximation can be made that E is much smaller than $m_{atom} c^2$ using the Taylor expansion $\sqrt{1-x} \approx 1 - \frac{x}{2}$. Plugging this into (2) provides us with

$$
\begin{aligned}
& \Delta E \approx (E + m_{atom} c^2) \left[1 - \left(1 - \frac{1}{2} \left[\frac{E}{E + m_{atom} c^2} \right]^2 \right) \right] \\
\Longrightarrow\ & \Delta E \approx \frac{E^2}{2(E + m_{atom} c^2)} \\
\Longrightarrow\ & \Delta E \approx 53 eV
\end{aligned}
$$