Nuclear recoil corrections for the extended nucleus

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PSI2022
October 14, 2022

Leading finite nuclear size

Atomic energy levels are shifted due to the finite nuclear size

- $E_{\rm fs}(Z\,\alpha)$ is a function of the nuclear charge Z
- expansion in α: $E_{\rm fs} = E_{\rm fs}^{(4)} + E_{\rm fs}^{(5)} + E_{\rm fs}^{(6)} + \ldots$
- $E_{\rm fs}^{(4)} = \frac{2\,\pi}{3}\,\phi^2(0)\,Z\,\alpha\,r_C^2$, where $r_C^2 = \int d^3r\; r^2\,\rho_E(r)$, it includes complete dependence on the nuclear mass through $\phi^2(0) = (\mu\,Z\,\alpha)^3/\pi$
- definition of the charge radius r_C depend on the nuclear spin
  $\delta H = -Z\,e\left(\frac{r_C^2}{6} + \frac{\delta_I}{M^2}\right)\vec{\nabla}\cdot\vec{E}$, where $\delta_0 = 0, \delta_{1/2} = 1/8, \delta_1 = 0\;\ldots$
- overlap of r_C^2 with the so-called nuclear self-energy (important for muonic atoms)
- what is the mass dependence of the higher order terms ?

Elastic two-photon exchange

- $E^{(5)}_{\rm fs}$ important for muon ic atoms
- $E^{(5)}_{\rm fs} = -\frac{\pi}{3}\,\phi^2(0)\,(Z\,\alpha)^2\,m\,r_F^3$, where $r_F^3 = \int d^3r_1\ \int d^3r_2\ \rho(r_1)\,\rho(r_2)\,|\vec{r}_1 - \vec{r}_2|$
- This result is valid in the nonrecoil limit, thus what are the nuclear recoil corrections?
- $E^{(5)}_{\rm recfs} = -\frac{m}{M}\,\phi^2(0)\,(Z\,\alpha)^2\left[\frac{7}{6} - 2\,\gamma - 2\,\ln(m\,r_L)\right]r_C^2$, where

  $\int d^3r_1 \int d^3r_2\,\rho(\vec{r}_1)\,\rho(\vec{r}_2)\,|\vec{r}_1 - \vec{r}_2|^2\,\ln(m\,|\vec{r}_1 - \vec{r}_2|) = 2\,r_C^2\,\ln(m\,r_L)$
- it is significantly enhanced r_F^3 versus r_C^2
- inclusion of the (electron-nucleus) Breit interaction leads to spurious terms that are linear in r_C

Elastic three-photon exchange

In the infinite nuclear mass limit

$$
\begin{aligned}
E^{(6)}_{\rm fns}(nS) &= -(Z\,\alpha)^6\, m^3\, r_C^2\, \frac{2}{3\,n^3}\left[\frac{9}{4n^2} - 3 - \frac{1}{n} + 2\,\gamma - \ln\frac{n}{2} + \Psi(n) + \ln(m\, r_{C2}\, Z\,\alpha)\right] \\
&\quad + (Z\,\alpha)^6\, m^5\, r_C^4\, \frac{4}{9\,n^3}\left[-\frac{1}{n} + 2 + 2\,\gamma - \ln\frac{n}{2} + \Psi(n) + \ln(m\, r_{C1}\, Z\,\alpha)\right] \\
&\quad + (Z\,\alpha)^6\, m^5\, r_{CC}^4\, \frac{1}{15\,n^5}\,, \\
E^{(6)}_{\rm fns}(nP_{1/2}) &= (Z\,\alpha)^6\, m\left(\frac{m^2\, r_C^2}{6} + \frac{m^4\, r_{CC}^4}{45}\right)\frac{1}{n^3}\left(1 - \frac{1}{n^2}\right), \\
E^{(6)}_{\rm fns}(nP_{3/2}) &= (Z\,\alpha)^6\, m^5\, r_{CC}^4\, \frac{1}{45\,n^3}\left(1 - \frac{1}{n^2}\right), \\
E^{(6)}_{\rm fns}(nL_J) &= 0 \text{ for } L > 1\,,
\end{aligned}
$$

where $r_{CC}^4 = \langle r^4\rangle$ and the effective nuclear charge radii r_{C1} and r_{C2} encode the high-momentum contributions and are expectedto be of the order of r_C.

Recoil corrections ?

Nonperturbative nuclear recoil correction

Exact nonperturbative formula (a'la Shabaev):

$$E_{\rm rec} = \frac{m^2}{M}\,\frac{i}{2\pi}\int_{-\infty}^{\infty} d\omega\,\langle a|\left[p^j - D^j(\omega)\right]G(\omega+E_a)\left[p^j - D^j(\omega)\right]|a\rangle$$

where

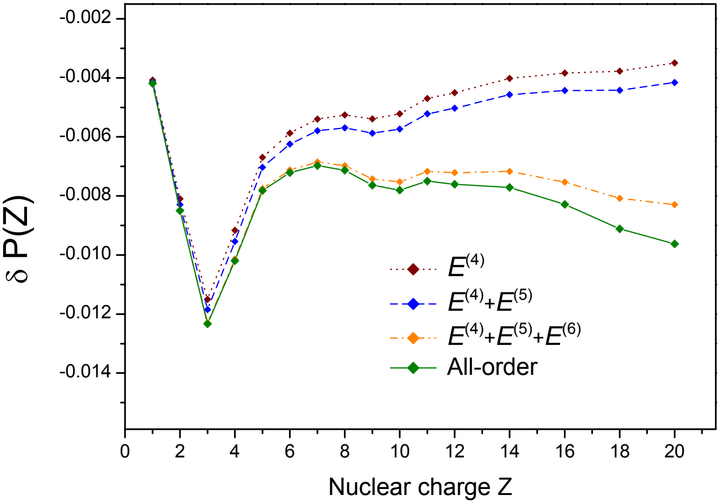
- $G(E) = [E - H_D(1-i\epsilon)]^{-1}$ is the Dirac-Coulomb Green function
- $D^j(\omega) = -4\pi Z\alpha\,\alpha^i\,G_C^{ij}(\omega,\vec{r})$, and α^i are the Dirac matrices.
- Photon propagator in the modified Coulomb gauge

$$G_C^{ij}(\omega,\vec{r}) = \delta^{ij}\,\mathcal{D}(\omega,r) + \frac{\nabla^i\nabla^j}{\omega^2}\left[\mathcal{D}(\omega,r) - \mathcal{D}(0,r)\right]$$

$$\mathcal{D}(\omega,r) = \int\frac{d^3k}{(2\pi)^3}\,e^{i\vec{k}\cdot\vec{r}}\,\frac{\rho(\vec{k}^2-\omega^2)}{\omega^2-\vec{k}^2}\,.$$

- The nuclear charge density is a function of an invariant $\rho(\vec{k}^2-\omega^2)$

Numerical results for the finite size recoil



where $\delta P = E_{\rm recfs}/[(m^2/M)(Z\,\alpha)^5/\pi]$ and $E^{(6)}_{\rm recfs} \approx -\frac{m^3}{M}\, a^{(6)}\,(Z\,\alpha)^6\, r_C$

Nuclear structure effects in hyperfine splitting

- $\delta E_{\rm nucl} = \delta^{(1)} E_{\rm nucl} + \delta^{(2)} E_{\rm nucl} + \ldots$ where
  $\delta^{(1)} E_{\rm nucl}$ is the two-photon exchange correction of order $(Z\,\alpha)\,E_F$,
  $\delta^{(2)} E_{\rm nucl}$ is the three-photon exchange correction of order $(Z\,\alpha)^2\,E_F$,
  $E_F = -\frac{2}{3}\,\psi^2(0)\,\vec{\mu}\cdot\vec{\mu}_e$
- $\delta^{(1)} E_{\rm nucl} = -2\,m_r Z\alpha\,r_Z E_F$ where
  r_Z is the Zemach radius defined by $r_Z = \int d^3 r_1 \int d^3 r_2\,\rho_M(r_1)\,\rho_E(r_2)\,|\vec{r}_1 - \vec{r}_2|$
- nuclear recoil correction

$$\delta^{(1)} E_{\rm fns,rec} = -\,E_F\,\frac{Z\,\alpha}{\pi}\,\frac{m}{M}\,\frac{3}{8}\left\{g\left[\gamma - \frac{7}{4} + \ln(m\,r_{M^2})\right] - 4\left[\gamma + \frac{9}{4} + \ln(m\,r_{EM})\right] - \frac{12}{g}\left[\gamma - \frac{17}{12} + \ln(m\,r_{E^2})\right]\right\}$$

$O(\alpha^2)$ corrections to hfs

- $\delta^{(2)} E_{\rm fns} = \frac{4}{3} E_F (m r_p Z\alpha)^2 \left[-\frac{1}{n} + 2\gamma - \ln\frac{n}{2} + \Psi(n) + \ln(m r_{pp} Z\alpha) + \frac{r_m^2}{4 r_p^2 n^2} \right]$
- $O(\alpha^2)$ recoil corrections are unknown
- nonperturbative formula for the recoil correction to hfs has not yet been derived (ongoing project)
- the use elastic formfactors in description of hfs is very much approximate

More accurate picture

$$\delta^{(1)} E_{\rm hfs} = E_{\rm Low} + E_{\rm 1nuc} + E_{\rm pol}$$

$$E_{\rm 1nuc} = -\frac{8\,\pi}{3}\,\alpha^2\,\frac{\psi^2(0)}{m_p + m}\,\vec{s}\cdot\langle\sum_a g_a\,\vec{s}_a\,r_{a{\rm Z}}\rangle$$

$$E_{\rm Low} = \frac{\alpha}{16}\,\psi^2(0)\,\vec{\sigma}\sum_{a\neq b}\frac{e_a\,e_b}{m_b}\left\langle 4\,r_{ab}\,\vec{r}_{ab}\times\vec{p}_b + \frac{g_b}{r_{ab}}\left[\vec{r}_{ab}\,(\vec{r}_{ab}\cdot\vec{\sigma}_b) - 3\,\vec{\sigma}_b\,r_{ab}^2\right]\right\rangle$$

Let us consider the special case of a spherically symmetric nucleus and neglect the proton-neutron mass difference.

$$E_{\rm Low} = -\frac{8\,\pi}{3}\,\alpha^2\,\frac{\psi^2(0)}{m_n}\sum_{a-{\rm protons}}\sum_b\langle r_{ab}\,g_b\,\vec{s}_b\rangle\,\vec{s}$$

Much better description for hfs in μD

Discrepancies in μD hfs

- the “experimental value” of the nuclear-structure correction in μD(2S) hfs

  $\delta E_{\mathrm{nucl,exp}} = E_{\mathrm{hfs}}(\mathrm{exp}) - E_{\mathrm{hfs}}(\mathrm{point}) = 0.0966(73)$ meV

- the numerical value of the Zemach correction with $r_Z = 2.593(16)$ fm is

  $\delta E_{Zem} = -0.1177(33)$ meV, opposite sign !

- including the nuclear vector polarizability and
  the inelastic three-photon exchange (10% effect)

  $\delta E_{\mathrm{nucl,theo}} = 0.028\,3(86)$ meV

- the difference

  $\delta E_{\mathrm{nucl,theo}} - \delta E_{\mathrm{nucl,exp}} = 0.058\,3(113)$

- Nuclear structure effects in hfs are not yet well known