Dispersive Approach to Electro(Weak) Box Corrections

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Spectroscopy of μ-Atoms and Strong Interaction

Atomic spectra: solution of Schrödinger/Dirac potential problem (QM)

QED: radiative corrections — corrections to the potential $\sim \alpha_{em} \approx 1/137$

Lamb Shift is zero in the pure Coulomb problem — is entirely due to RC!

Strong Interaction is short range — enough to relegate SI effects to small corrections

Nuclear Radius — first such correction: nuclear radii from high-precision atomic spectra

Muonic atoms 200 more compact than electronic — enhanced sensitivity to nuclear radii

Mixed QCD + QED corrections are double-suppressed (α_{em} + short range)

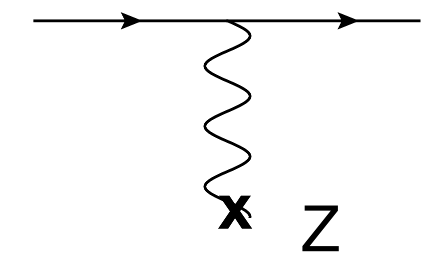
2γ-box: IR and UV finite, no enhancements — natural size; but precision goal is such that it is promoted to the main source of uncertainty; need to scan all scales from IR to UV!

Compare to EW boxes (γZ, γW) as corrections to EW precision tests:
UV sensitive (large logs) but uncertainty from intermediate scales

2γ-Potential and Lamb Shift

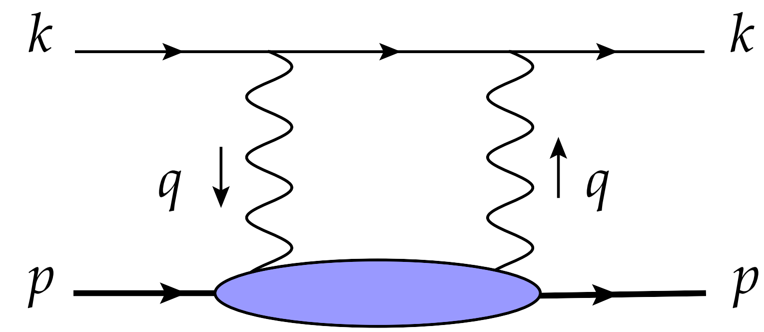
Pointlike Coulomb

$$V_C(r) = \int \frac{d^3\vec{q}}{(2\pi)^3}\frac{Ze^2}{q^2}e^{i\vec{q}\vec{r}} = \frac{Z\alpha}{r}$$



Potential from 2γ-box

$$V_{2\gamma}(r) = \int \frac{d^3\vec{\Delta}}{(2\pi)^3}T_{2\gamma}(\Delta = 0)e^{i\vec{\Delta}\vec{r}} = T_{2\gamma}(0)\delta(\vec{r})$$



Will affect S-levels, but not P-levels

Non-trivial object:
doubly-virtual Compton tensor

$$T^{\mu\nu} = \frac{i}{8\pi M}\int d^4x e^{iqx}\langle p\,|\,T j^{\nu}(x) j^{\mu}(0)\,|\,p\rangle$$

Lorentz- and Gauge-invariant decomposition

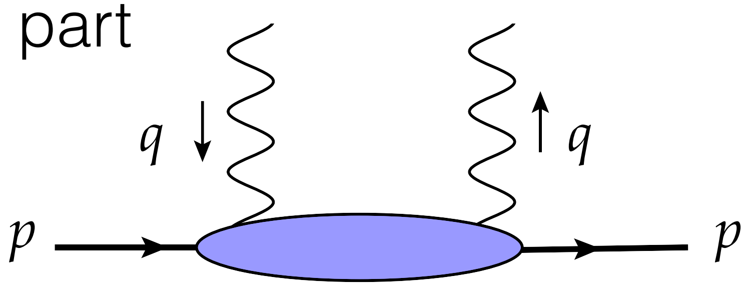
$$T^{\mu\nu} = T_1(\nu, Q^2)\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + T_2(\nu, Q^2)\frac{1}{M_T^2}\left(p - \frac{(p\cdot q)}{q^2}q\right)^{\mu}\left(p - \frac{(p\cdot q)}{q^2}q\right)^{\nu}$$

2γ-correction to the Lamb shift (nS-nP)

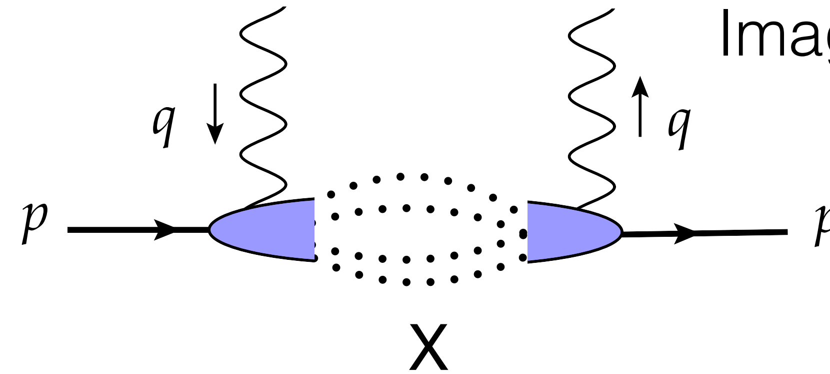
$$\Delta E = -\frac{\alpha^2}{2\pi m_l M_d}\phi_n^2(0)\int d^4q\,\frac{(q^2 + 2\nu^2)T_1(\nu, q^2) - (q^2 - \nu^2)T_2(\nu, q^2)}{q^4[(q^2/2m_l)^2 - \nu^2]}$$

Forward Compton Amplitudes from Dispersion Relation

Real part



Imaginary part



Time-ordered product

$$\int d^4x e^{iqx} \langle p \,|\, T j^\nu(x) j^\mu(0) \,|\, p \rangle$$

Commutator

$$\int d^4x e^{iqx} \langle p \,|\, [j^\nu(x), j^\mu(0)] \,|\, p \rangle$$

Insert full set of on-shell intermediate hadronic states X:

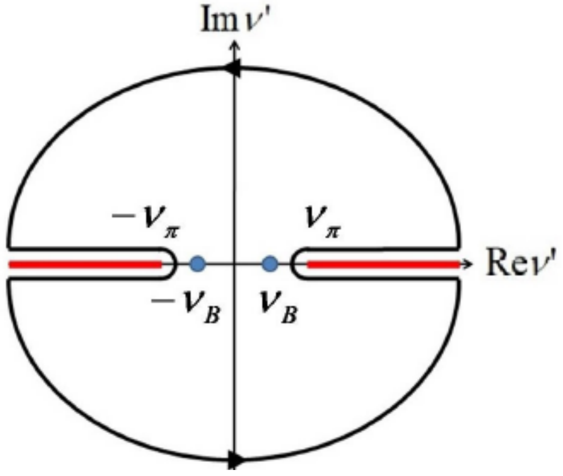
Optical Theorem relates $\mathbf{Im}T_{1,2}$ to structure functions $F_{1,2}$ = data

$$\begin{aligned} \mathrm{Im}T_1(\nu, Q^2) &= \frac{1}{4M_d} F_1(\nu, Q^2) \\ \mathrm{Im}T_2(\nu, Q^2) &= \frac{1}{4\nu} F_2(\nu, Q^2). \end{aligned}$$

Forward Compton Amplitudes from Dispersion Relation

Reconstruct full amplitudes from Cauchy's theorem:
contour in the complex ν-plane for fixed Q^2

$$T_i(\nu, Q^2) = \frac{1}{2\pi i} \oint_C \frac{dz T_i(z, Q^2)}{z - \nu}$$



Symmetries + Analytical Structure + High-Energy Behavior:
T_1 only known up to a function of Q^2

$$\mathrm{Re}T_1(\nu, Q^2) = \bar{T}_1(0,Q^2) + \frac{\nu^2}{2\pi M} \int_0^\infty \frac{d\nu' F_1(\nu', Q^2)}{\nu'(\nu'^2 - \nu^2)} \qquad \mathrm{Re}T_2(\nu, Q^2) = \frac{1}{2\pi} \int_0^\infty \frac{d\nu' F_2(\nu', Q^2)}{(\nu'^2 - \nu^2)}$$

2γ-correction to the Lamb shift: sum rule + subtraction contribution

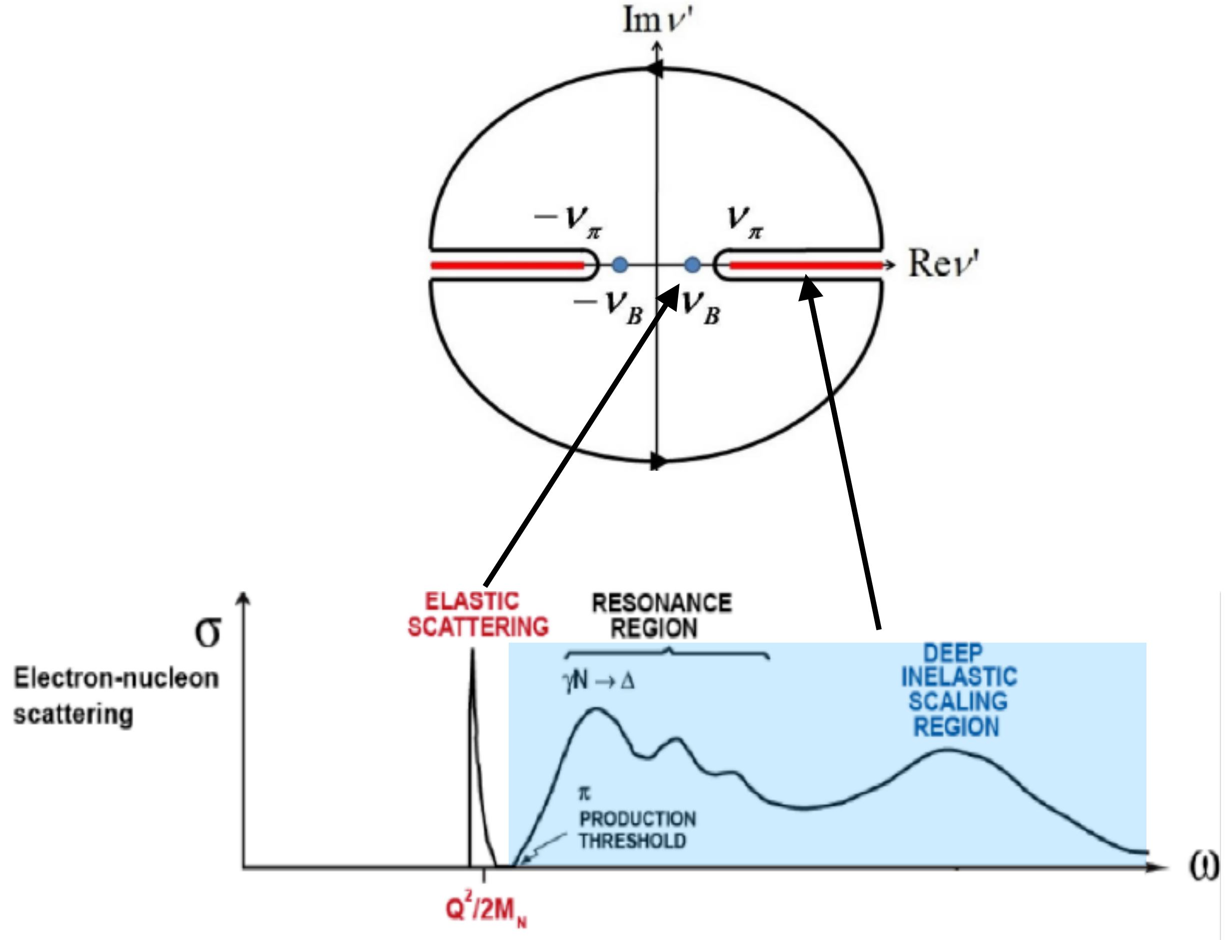
$$\Delta E \propto \int_0^\infty dQ^2 \int_0^\infty d\nu \left[A(\nu, Q^2)F_1 + B(\nu, Q^2)F_2 + C(Q^2)\bar{T}_1(0,Q^2)\right]$$

Carlson, Vanderhaeghen, Phys Rev A84 (2011)

Subtraction — the only problematic term

Input into dispersion integral

Dispersion in energy: $W^2 = M^2 + 2M\nu - Q^2$
scanning hadronic intermediate states



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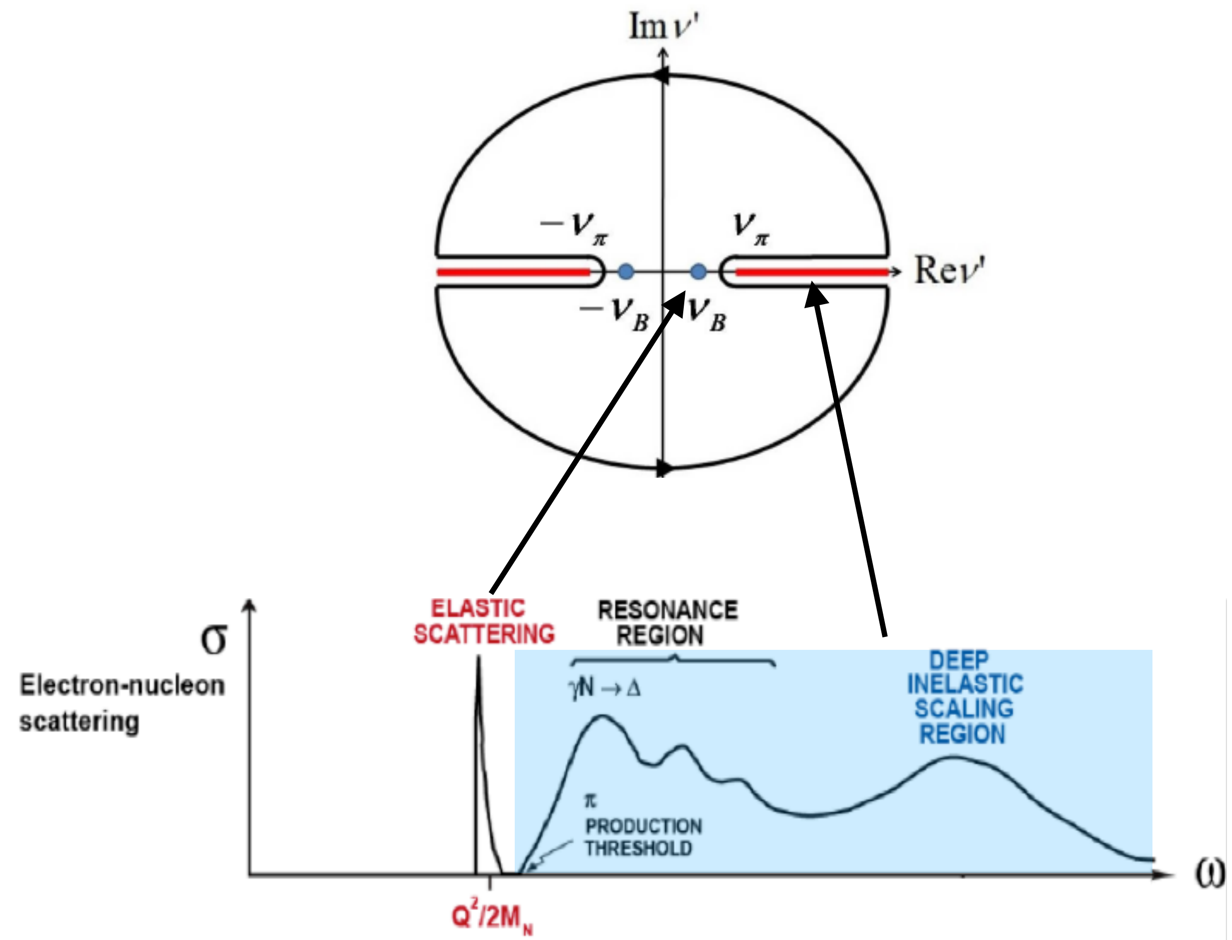
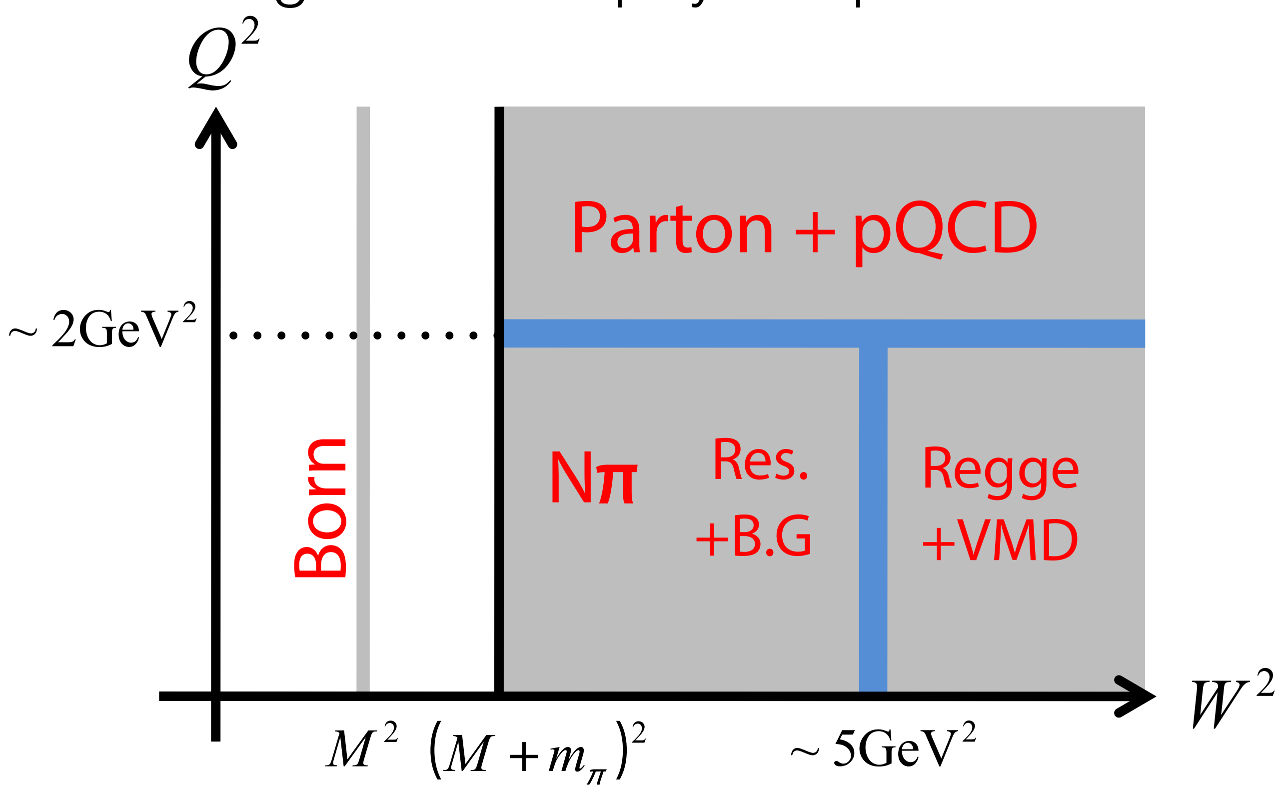


Dispersion in Q^2:
scanning dominant physics pictures

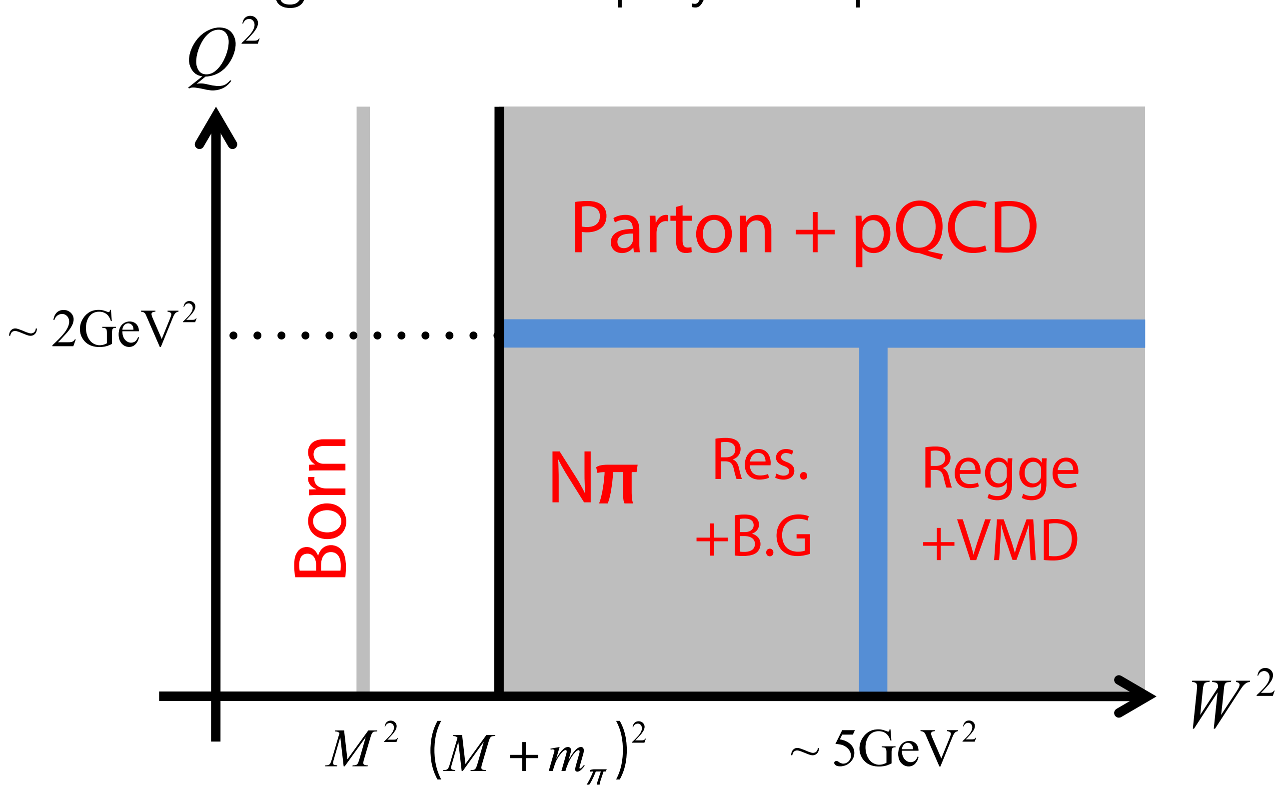


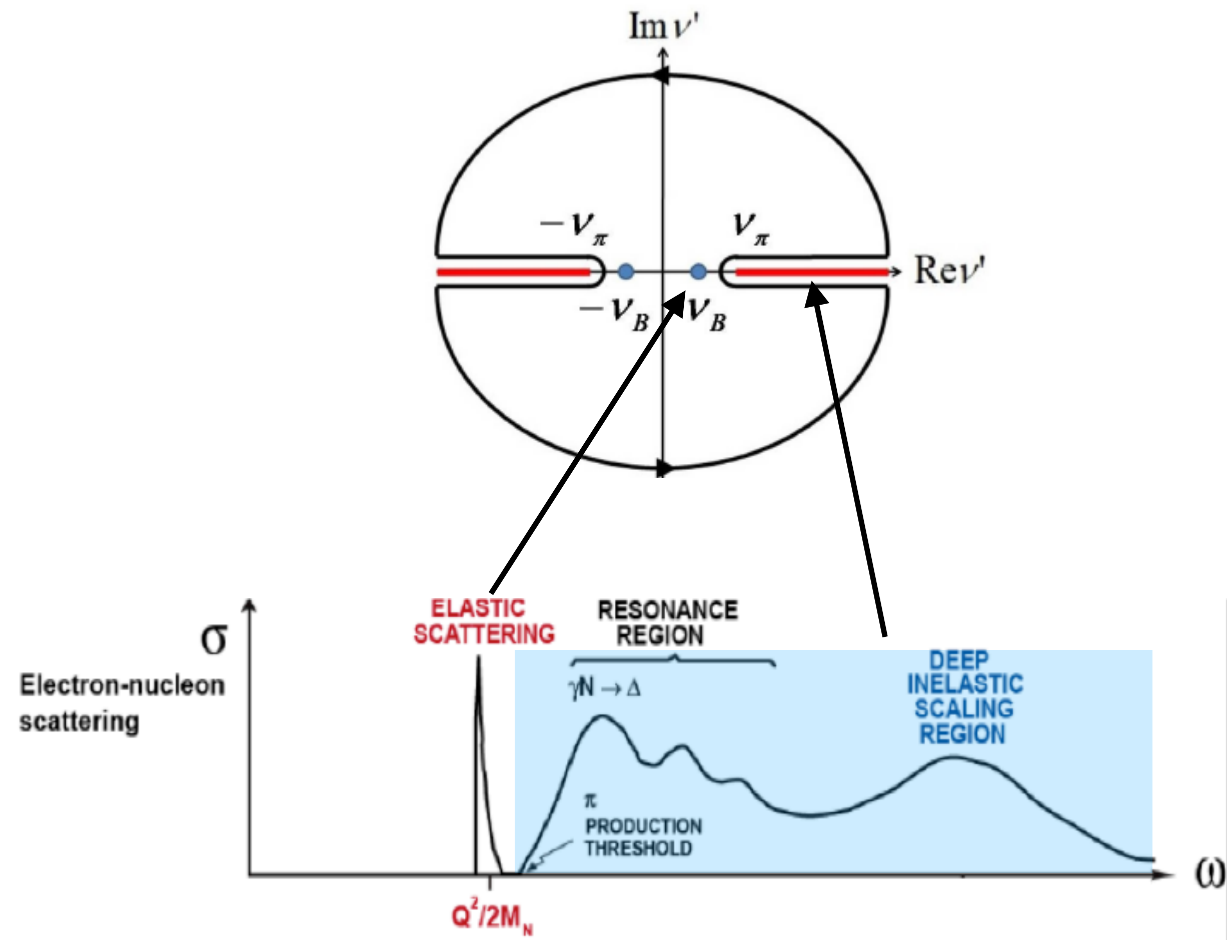
Boundaries between regions - approximate

Input into dispersion integral

Dispersion in energy: $W^2 = M^2 + 2M\nu - Q^2$
scanning hadronic intermediate states

Dispersion in Q^2:
scanning dominant physics pictures





Boundaries between regions - approximate

All regions contribute but weigh differently: 2γ-box for atoms vs scattering; $\gamma Z/\gamma W$ boxes

E-M structure functions measured in a wide kinematical range
— can evaluate dispersion integral directly without bothering about physics!

Subtraction Function:
Low-Energy Expansion vs Finite Energy Sum Rule

Subtraction — the only problematic term

To reconstruct it — need additional information

Low-Energy Expansion:

General properties of Compton amplitude with low-energy photons
+ low-energy dynamics (e.g. pions or nucleons)

$$\bar{T}_1(0,Q^2) = T_1^B(0,Q^2) - T_1^{pole}(0,Q^2) + \frac{Q^2}{e^2}\beta_M(Q^2)$$

Current state-of-the-art for muonic atoms - chiral effective framework
Accounts for lowest relevant d.o.f., predicts low-Q^2 behavior
Many people in this room contributed to LE approach

Finite Energy Sum Rules:

Use data at low energy + information from high energies

Guess the correct d.o.f. (dynamic at HE, static at LE) — duality

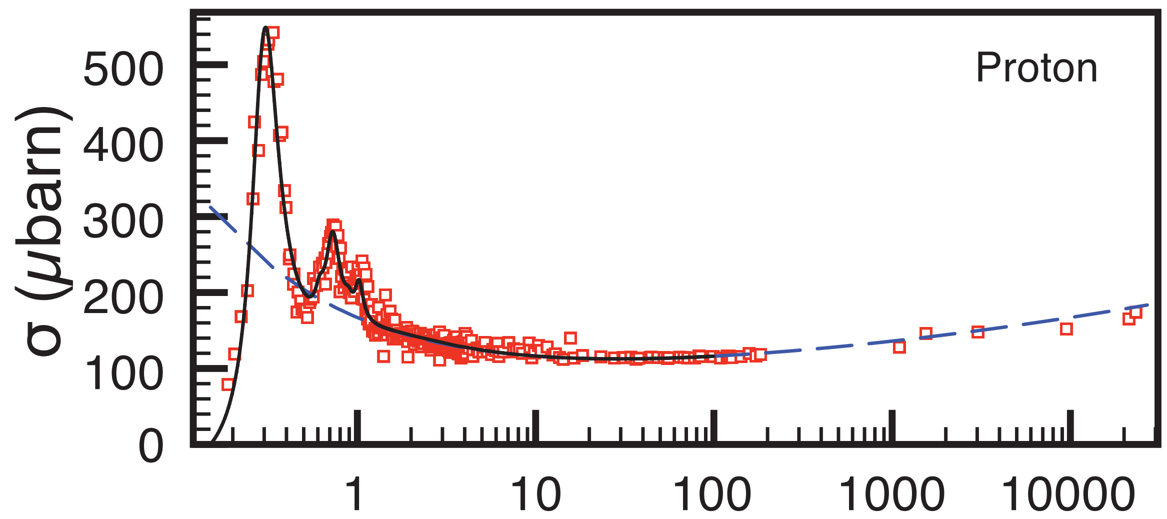
Analyticity then constrains the LE subtractions!

Subtraction Function from FESR

HE photoabsorption:
Regge behavior for $\nu \geq$ few GeV

$$F_1^R = c_P \nu^{\alpha_P} + c_f \nu^{\alpha_f}, \quad \alpha_P \approx 1.09,\ \alpha_f \approx 0.5$$

MG, Hobbs, Londergan, Szczepaniak, Phys Rev C84 (2011)



HE behavior explains the need for subtraction (but does not determine the subtraction fn.)

Important: no constant trajectories (data and theory)

Define two analytical functions that possess the "same" HE asymptotics

$$\mathrm{Re}T_1(\nu, Q^2) = \bar{T}_1(0,Q^2) + \frac{\nu^2}{2\pi M}\int_{thr}^{\infty} \frac{d\nu' F_1(\nu', Q^2)}{\nu'(\nu'^2 - \nu^2)} \qquad \mathrm{Re}T_1^R(\nu, Q^2) = 0 + \frac{\nu^2}{2\pi M}\int_{0}^{\infty} \frac{d\nu' F_1^R(\nu', Q^2)}{\nu'(\nu'^2 - \nu^2)}$$

At asymptotic energy $\nu \to \infty$ the two functions can at most differ by a constant

$$C_\infty(Q^2) \equiv \left[\mathrm{Re}\, T_1(\nu, Q^2) - \mathrm{Re}\, T_1^R(\nu, Q^2)\right]\Big|_{\nu\to\infty}$$

If information on C_∞ exists — a statement on LE subtraction can be made

Subtraction Function from FESR

DR for the difference: dispersion integral has support below $N(Q^2)$ where Regge sets in

$$\mathrm{Re}T_1(\nu,Q^2) = \bar{T}_1(0,Q^2) + \frac{\nu^2}{2\pi M}\int_{thr}^{\infty}\frac{d\nu' F_1(\nu',Q^2)}{\nu'(\nu'^2-\nu^2)}$$

$-$

$$\mathrm{Re}T_1^R(\nu,Q^2) = 0 + \frac{\nu^2}{2\pi M}\int_{0}^{\infty}\frac{d\nu' F_1^R(\nu',Q^2)}{\nu'(\nu'^2-\nu^2)} \Bigg|_{\nu\to\infty}$$

Express subtraction fn via C_∞

$$\bar{T}_1(0,Q^2) = C_\infty(Q^2) + \frac{1}{2\pi M}\int_{thr}^{N}\frac{d\nu}{\nu}F_1(\nu,Q^2) - \frac{1}{2\pi M}\sum_{i=P,f}\frac{c_i(Q^2)}{\alpha_i}[N(Q^2)]^{\alpha_i}$$

MG, Llanes-Estrada, Szczepaniak, Phys Rev A87 (2013)

C_∞ a.k.a. the J=0 pole; data suggest $C_\infty \approx 0$ —> clean prediction for subtraction fn!

Duality: the exact balance between the integral over data and over Regge
J=0 pole quantifies duality violation (some missing physics)

Jerry Miller suggested some unknown physics at sub-asymptotic Q^2

Miller et al, Phys Rev A84 (2011), Phys Rev C86 (2012)

Would show up in many places, including DVCS

Brodsky et al, Phys Rev D79 (2009)

2γ-box for 2S-2P Lamb Shift: FESR vs. the rest of the world

Outdated table from
MG, Llanes-Estrada, Szczepaniak, Phys Rev A87 (2013)

	FESR	Ref. [2]	Ref. [14]	Ref. [34]
ΔE^{subt}	3.3 ± 4.6	6.6	5.3 ± 1.9	9.0 ± 1.0
ΔE^{el}	-30.1 ± 1.2	-27.8	-29.5 ± 1.3	-29.5 ± 1.3
ΔE^{inel}	-13.0 ± 0.6	-13.9	-12.7 ± 0.5	-12.7 ± 0.5
ΔE	-39.8 ± 4.8	-35.1	-36.9 ± 2.4	-33 ± 2

Fazit: FESR result is consistent with LE-motivated approaches (natural)

Uncertainty is not very competitive: almost complete cancellation (90% or more!)
between Regge and integral over data; generic data uncertainty - few%,
but Regge and data are highly correlated —> would an updated analysis be of interest?

Duality is a fundamental concept but its realization is not well understood:
Duality in electron and neutrino scattering may follow different patterns
Kopeliovich et al, Prog Part Nucl Phys 68 (2013)

Absence of J=0 pole assumed — reasonable but not proven

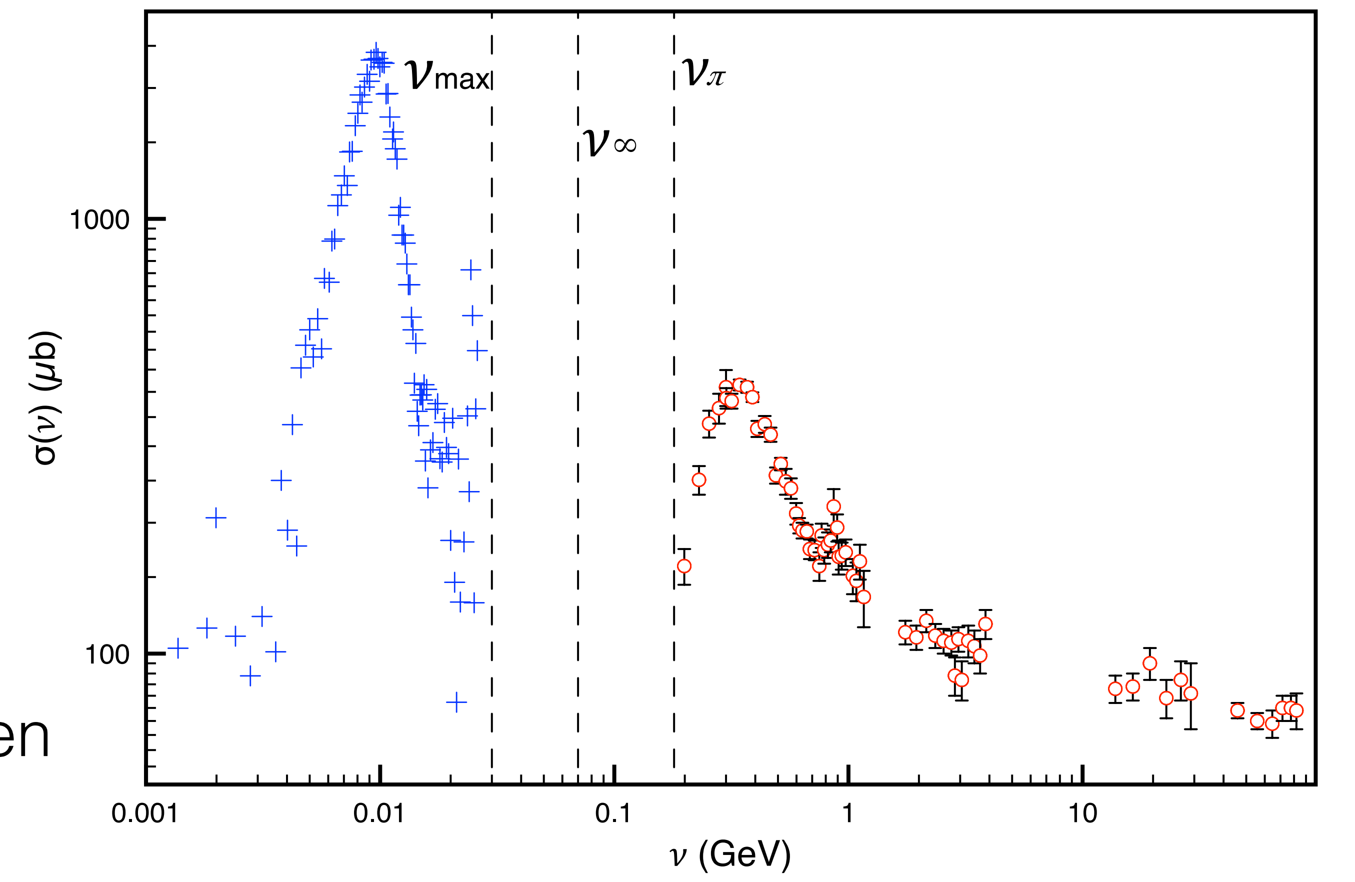
FESR for Subtraction Function in Nuclei

Duality in nuclei:

Strength of photoabsorption in nuclear range
Is fixed by number, charge and mass of
elementary scatterers - nucleons

Advantage over FESR on nucleons:
Presence of an hierarchy of scales

$\nu_{Nucl} \sim 10\,\mathrm{MeV}, \quad \nu_{hadr} \sim 300\,\mathrm{MeV}$ gap in between



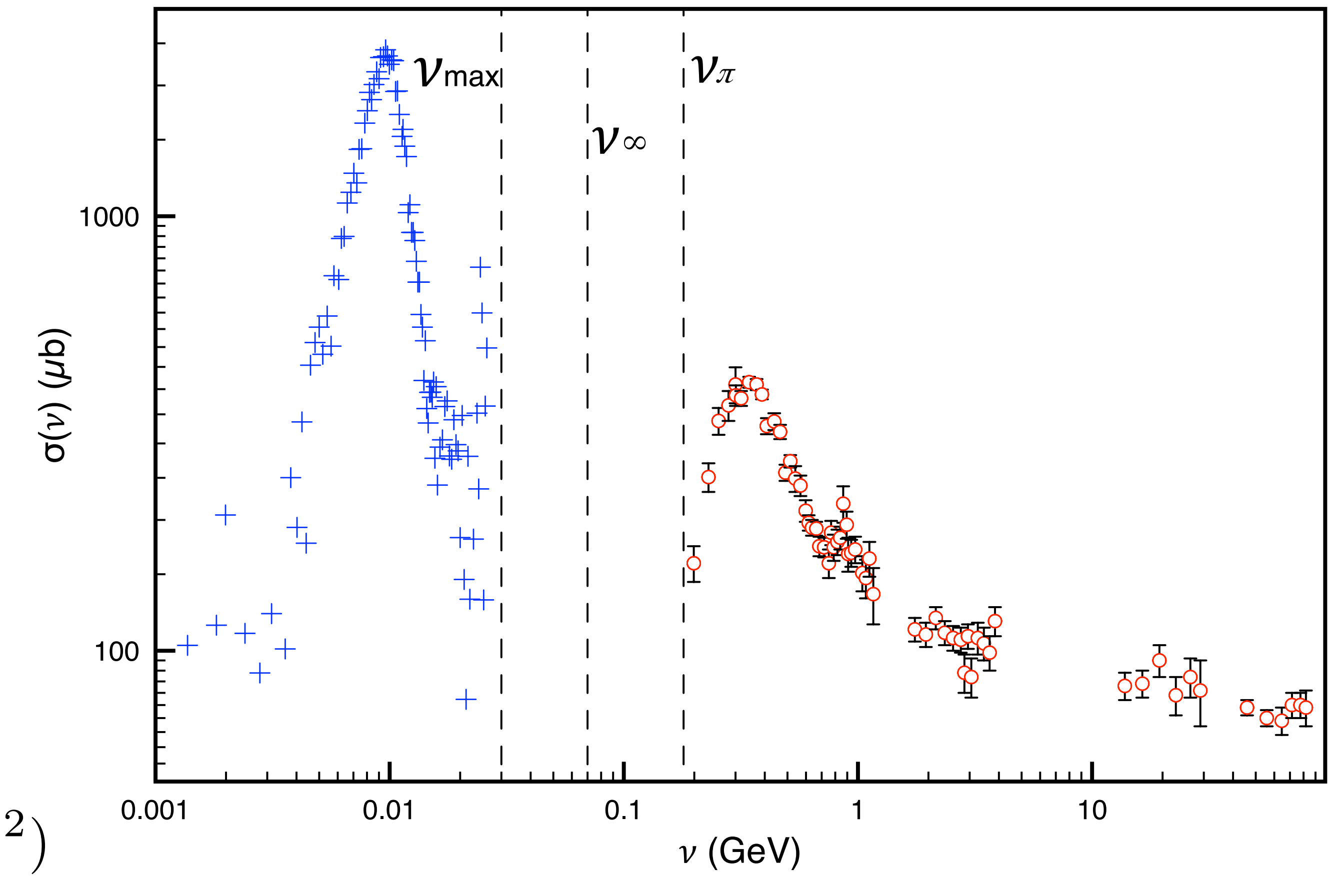
In the nuclear range nucleons are unresolved and elementary
In the hadronic range nucleon structure is fully resolved
In the gap: ~no photoabsorption —> scattering on nucleons with internal structure (size, polarizabilities)

C_∞ is known and given by the LEX of the nucleon Compton amplitude!

FESR for Subtraction Function in Nuclei

LEX of $T_1^{np}(0,Q^2)$: nuclear LE constants

$$T_1^{np}(0,Q^2) = -\frac{\alpha_{em}}{M}\frac{Z^2F_C^2(Q^2)}{Z+N} + Q^2\beta_M^{nucl}(Q^2)$$



LEX of $T_1(\nu_\infty, Q^2)$: nucleon LEX (duality!)

$$\mathrm{Re}T_1^{np}(\nu_\infty,Q^2) = -Z\frac{\alpha_{em}}{M}F_D^{p\,2}(Q^2) - N\frac{\alpha_{em}}{M}F_D^{n\,2}(Q^2)$$
$$+ZQ^2\beta_M^p(Q^2) + NQ^2\beta_M^n(Q^2) \quad + \frac{2\alpha_{em}\nu_\infty^2}{M}\int_{\nu_\pi}^{\infty}\frac{d\nu}{\nu^3}\left[ZF_1^p(\nu,Q^2)+NF_1^n(\nu,Q^2)\right]$$

General dispersion representation for $T_1(\nu_\infty, Q^2)$

$$\mathrm{Re}\,T_1^{np}(\nu_\infty,Q^2) = T_1^{np}(0,Q^2) - \frac{2\alpha_{em}}{M_T}\int_{\nu_{min}}^{\nu_{max}}\frac{d\nu}{\nu}F_1(\nu,Q^2) + \frac{2\alpha_{em}\nu_\infty^2}{M_T}\int_{\nu_\pi}^{\infty}\frac{d\nu}{\nu^3}F_1(\nu,Q^2) + \frac{\alpha_{em}}{M_T}\mathcal{P}\int_{\nu_{max}}^{\nu_\pi}\frac{d\nu^2\nu_\infty^2F_1(\nu,Q^2)}{\nu^2(\nu^2-\nu_\infty^2)}$$

Combine terms at powers of ν_∞ —> exact balance of nuclear and nucleon contributions

FESR for Subtraction Function in Nuclei

Constant term for $Q^2 = 0$

$$ZN = 2\int_{\nu_{min}}^{\nu_{max}} \frac{d\nu}{\nu} F_1(\nu, 0)$$

Integrated strength of nuclear photoabsorption is fixed by the number of nucleons
Recall: Thomas-Reiche-Kühn sum rule in QM (integrated strength = number of oscillators)

First derivative at $Q^2 = 0$: FESR for nuclear magnetic polarizability

MG, Phys Rev Lett 115 (2015)

$$\beta_M^{nucl} = \frac{2\alpha_{em}}{M}\int_{\nu_{thr}}^{\nu_{max}} \frac{d\nu}{\nu}\frac{d}{dQ^2} F_1(\nu, Q^2)\Big|_{Q^2=0}$$
$$- \frac{Z^2\alpha_{\rm em}}{(Z+N)M}\frac{R^2_{\rm ch}}{3} + \frac{\alpha_{em}(ZR_p^2 + NR_n^2)}{3M_p} + Z\beta_M^p + N\beta_M^n$$

Caveat: assumed a perfect nuclear-hadronic scale separation - p,n unbound

$$\int_{\nu_\pi}^{\infty} \frac{d\nu}{\nu^3}\left[\frac{M}{M_T}F_1(\nu, Q^2) - ZF_1^p(\nu, Q^2) - NF_1^n(\nu, Q^2)\right] + \frac{M}{M_T}\mathcal{P}\int_{\nu_{max}}^{\nu_\pi} \frac{d\nu F_1(\nu, Q^2)}{\nu(\nu^2 - \nu_\infty^2)} = 0$$

shadowing + PV-integral over the gap region

FESR for β_M: check for the deuteron

Deuteron: β_M known theoretically

EFT (lowest order): β_M^d = 0.068 fm^3 — Chen et al., 2002
Potential models (LO): β_M^d = 0.068 fm^3 — Friar 1997, Khriplovich 1979, …
Potential models (NLO): β_M^d = 0.078 fm^3 — Friar 1997

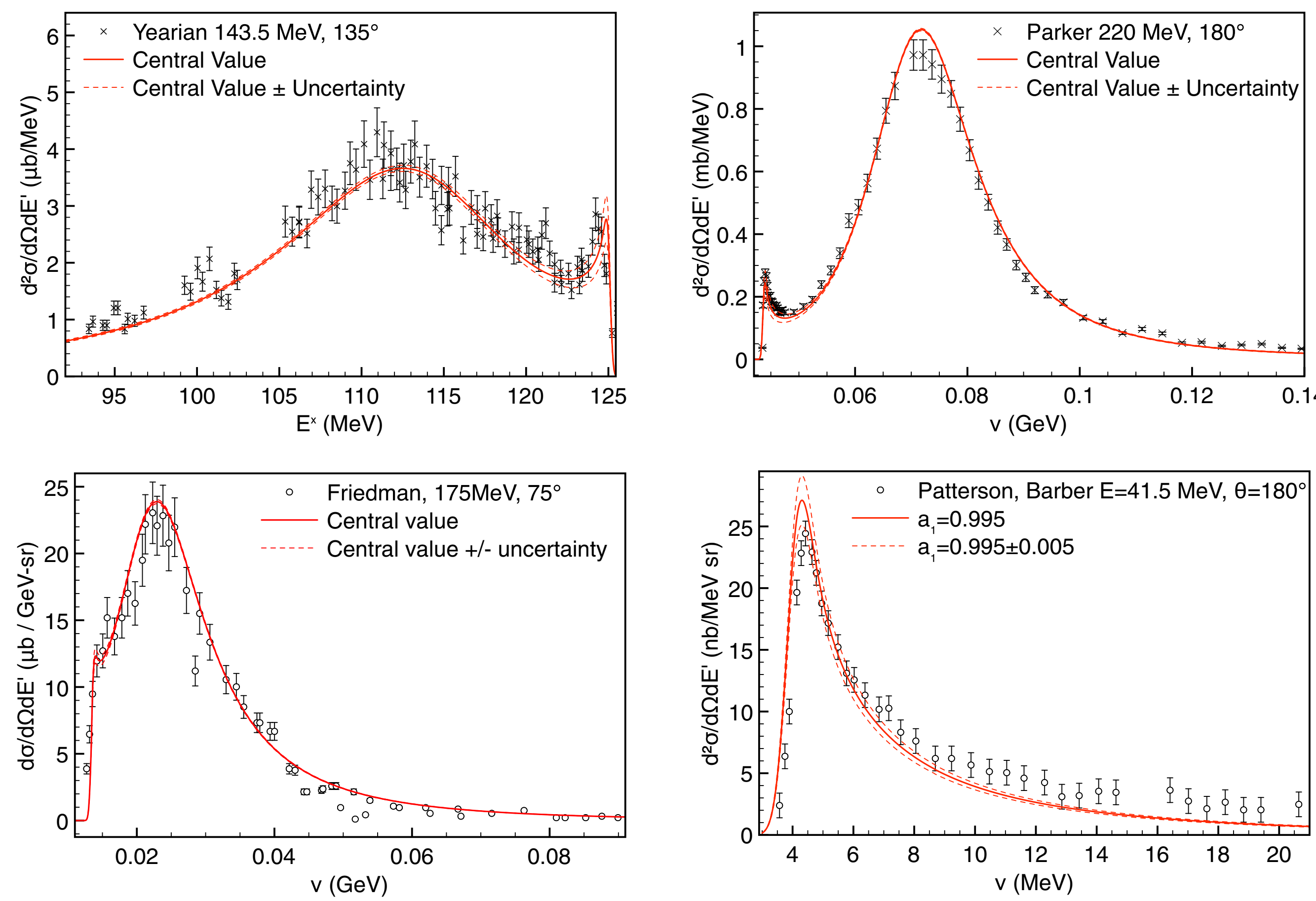
Fit to virtual photoabsorption on the deuteron — ***Carlson, MG, Vanderhaeghen, Phys Rev A89 (2014)***

β_M from fit + FESR (unconstrained by theory)

$$\beta_M^d = 0.096(15)\,\mathrm{fm}^3$$

A valuable check for theory uncertainty!
E.g., above uncertainty seems low

Can impose the theory constraint on β_M^d
to ensure self-consistence fit <—> LEX



Lamb Shift in μ-D with FESR

With FESR can go beyond just β_M: reconstruct full Q^2-dependence of subtraction function

$$T_1^{np}(0,Q^2) - T_1^{np}(0,0) = \frac{2\alpha_{\rm em}}{M_T}\int_{\nu_{\rm min}(Q^2)}^{\nu_{\rm max}(Q^2)} \frac{d\nu}{\nu}\left[F_1(\nu,Q^2) - F_1(\nu,0)\right]$$

Subtraction function contribution to the 2S level energy

$$\Delta E_{2S}^{\rm subt} = 4\alpha_{\rm em}\phi_{2S}^2(0)\int_0^\infty dQ\gamma_1(\tau_l)\frac{T_1^{np}(0,Q^2) - T_1^{np}(0,0)}{Q^2}$$

Sum rule with and without FESR for 2S-2P splitting in μ-D

ΔE_{2S}^i	DR w FESR	DR w/o FESR	Nucl. Mod.
$\Delta E_{2S}^{\rm inel}$	$-2.294(740)$	$-2.357(740)$	$\cdots$
$\Delta E_{2S}^{\rm subt}$	$0.505(35)(40)$	$0.763(40)$	$\cdots$
$\Delta E_{2S}^{\rm tot}$	$-1.945(740)^*$	$-1.750(740)^*$	$-1.709(15)$

MG, Phys Rev Lett 115 (2015)
Carlson, MG, Vanderhaeghen, Phys Rev A89 (2014)
Hernandez, Bacca, Dinur, Barnea, Phys Lett B736 (2014)

The use of FESR does matter: significant shift!
But the overall DR uncertainty is ~ 50 times larger that that of nuclear models!
The subtraction is under control, but the sum rule part is not!

Deuteron Electroabsorption Data and Lamb Shift in μ-D

Elastic e.m. form factors: measured over a wide range of Q^2 range

$$\Delta E_{n0}^{el} = \frac{m\alpha^2}{M_d\left(M_d^2 - m^2\right)}\phi_{n0}^2(0)\int_0^\infty \frac{dQ^2}{Q^2}\left\{\frac{2}{3}G_M^2(1+\tau_d)\left(\frac{\gamma_1(\tau_d)}{\sqrt{\tau_d}} - \frac{\gamma_1(\tau_l)}{\sqrt{\tau_l}}\right) - \left(\frac{\gamma_2(\tau_d)}{\sqrt{\tau_d}} - \frac{\gamma_2(\tau_l)}{\sqrt{\tau_l}}\right)\left[\frac{G_C^2}{\tau_d} + \frac{2}{3}G_M^2 + \frac{8}{9}\tau_d G_Q^2\right]\right\}$$

Possible problem: good χ^2 does not guarantee correct charge radius ***Vadim's talk?***

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Possible problem: good χ^2 does not guarantee correct charge radius ***Vadim's talk?***

Inelastic structure functions: measured over $0.005\,\mathrm{GeV}^2 \le Q^2 \le 3\,\mathrm{GeV}^2$

$$\Delta E_{n0}^{\mathrm{inel}} = -\frac{2\alpha^2}{M_d m}\phi_{n0}^2(0)\int_0^\infty \frac{dQ^2}{Q^2}\int_{\nu_{thr}}^\infty \frac{d\nu}{\nu}\left[\tilde{\gamma}_1(\tau,\tau_l)F_1(\nu,Q^2) + \frac{M_d\nu}{Q^2}\tilde{\gamma}_2(\tau,\tau_l)F_2(\nu,Q^2)\right]$$

Problem: the extrapolation of the fit function to $0 \le Q^2 \le 0.005\,\mathrm{GeV}^2$ is unconstrained

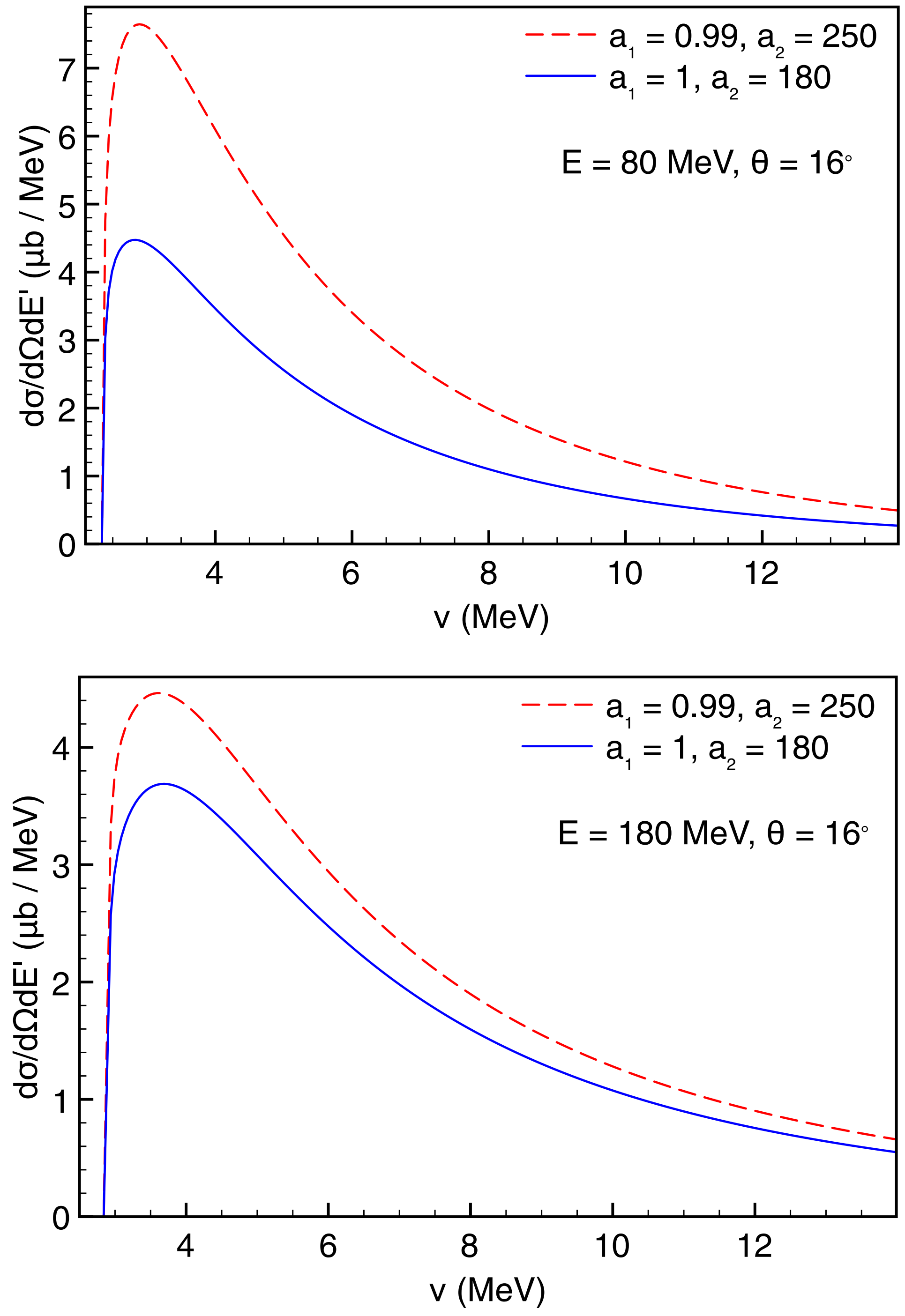
Unfortunately, the dispersion integral is heavily weighted towards the lowest Q^2!

Result: huge uncertainty due to lack of data where they're most needed

Taming Uncertainties of the DR Calculation

Ask experimentalists for new data

$E_{\rm lab}, \theta_{\rm lab}$	Expt. precision	$\delta(\Delta E^{\mu D}_{2S-2P})$ in μeV	$\delta(\Delta E^{eD}_{1S-2S})$ in kHz
180 MeV, 30°	2%	740	12
	1%	370	6
180 MeV, 22°	2%	390	6.32
	1%	195	3.16
180 MeV, 16°	2%	211	3.36
	1%	110	1.68
80 MeV, 16°	2%	67	1.08
	1%	48	0.78



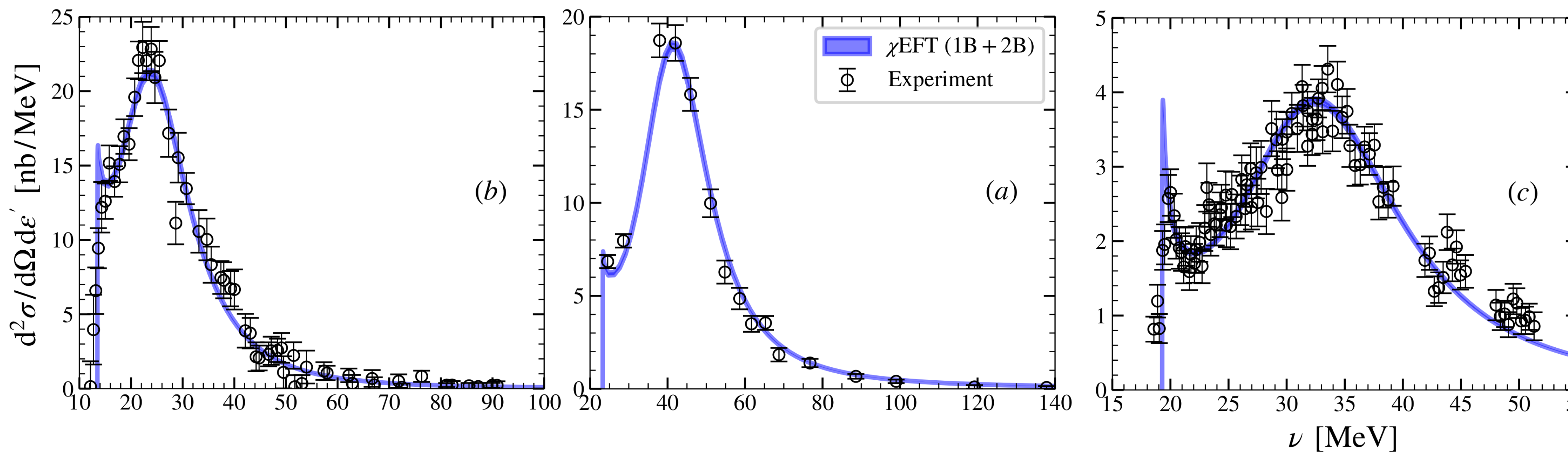
High precision at difficult kinematics needed — hard!

Taming Uncertainties of the DR Calculation

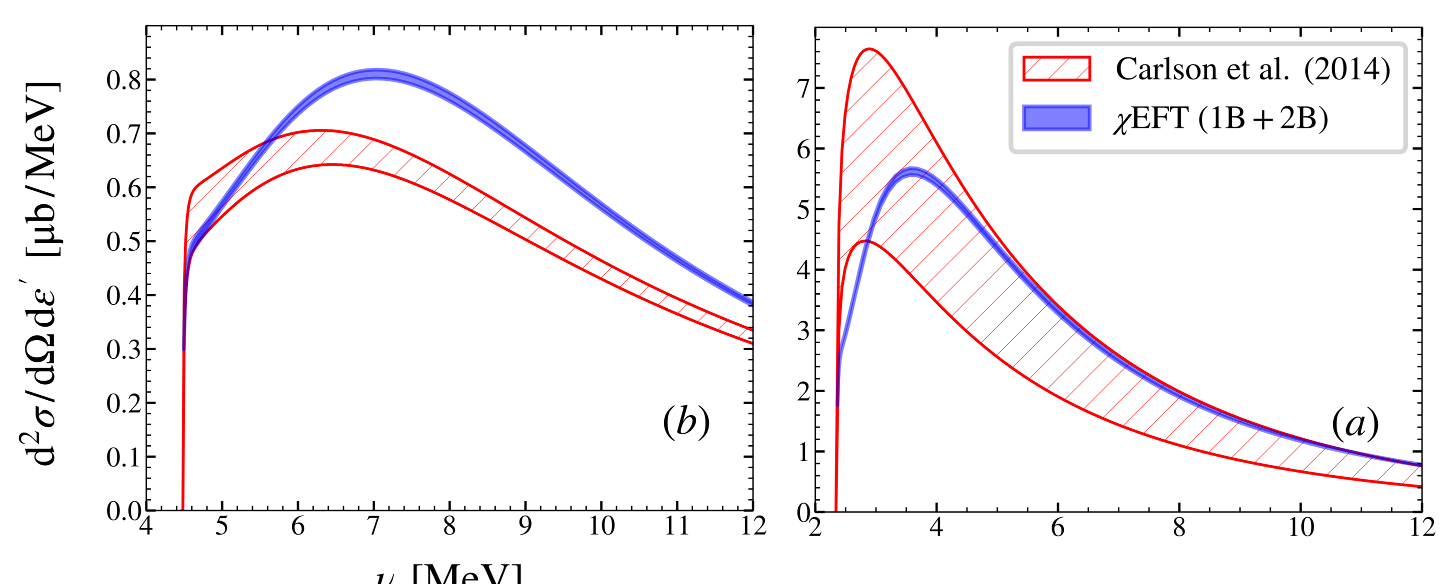
Ask nuclear theorists for theory constraints on the extrapolation

Acharya, Lensky, Bacca, Gorchtein, Vanderhaeghen, Phys Rev C103 (2021)

Great description of the available data



Significant improvement on the extrapolation



Combined χEFT + DR theory for Lamb Shift

New χEFT + DR formalism incorporates the advantages of both methods:

χEFT informed on NR dynamics even where no data exist;

DR naturally connects different energy scales, is fully relativistic, obeys all symmetries

		$\Delta E_{2S}^{\mathrm{TPE}}$ [meV]
	This work	
χEFT+DR	— 1B+2B	-1.695(13)
	— Siegert	-1.703(15)
AV18	Ref. [8]	-1.680(16)
	Ref. [9]	-1.717(20)
	Ref. [11]	-1.690(20)
χEFT	Ref. [12]	-1.712(21)
Rosenfelder	Ref. [13]	-1.703
DR	Ref. [14]	-2.011(740)

DR as Unifying Framework for Precision Tests of SM With Input from Nuclear Theory, Lattice QCD, pQCD, …

DR + other inputs — relevant in several other precision tests of the SM where EW boxes play a central role in defining the uncertainty!

γW-box correction to Fermi part of β-decay

Free neutron decay: combine pQCD + lattice QCD in the DR formalism

Feng, MG, Jin, Ma, Seng, PRL124 (2020)
Seng, Feng, MG, Jin, PR D104 (2020)

Unified formalism for hadronic and nuclear corrections!

MG, Seng et al., PRD 101 (2019); PRL 123 (2019)

Nuclear β-decays: incorporate χEFT input in DR integrals

Collaborations with
S. Pastore (GFMC) P. Navratil (NCSM)

γW-box correction to Gamow-Teller strength (g_A):

Analogous expression as HFS (but very different weighting due to heavy boson)
Free nucleon result exists ***MG, Seng, JHEP 10 (2021)***

Future applications to nuclear mirror decays

γZ-box contribution to nuclear weak charges and anapole moments
γZ-box contribution to weak charges and strange FF in PV electron scattering

Summary

Dispersion relations: well-established framework

Obeys all symmetries and limiting cases dictated by theory

Allows to incorporate input from data, nuclear theory, lattice, pQCD…

Ensures correct matching of different regimes

Contributed to the definition of low-energy precision tests:

Proton Radius Puzzle, CKM unitarity, weak mixing angle …