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Compton scattering and nucleon polarisabilities

Judith McGovern

University of Manchester

Work done in collaboration with Harald Grießhammer, Daniel Phillips, Vadim Lensky, Vladimir Pascalutsa, Mike Birse, Jerry Feldman, Luke Myers, Bruno Strandberg, Arman Margaryan, Deepshikha Shukla, Vahe Sokhoyan, Edoardo Mornacchi, Evie Downie, Xiaqing Li, Mohammad Ahmed, Hayan Gao, Jordan Melendez, Dick Furnstahl, Alex Moore *et al.*

Prog. Nucl. Part. Phys. **67** 841 (2012) Eur. Phys. J. A **49** (2013) 12

Phys. Rev. Lett. **113** (2014) 262506 Eur. Phys. J. C **75** (2015) 604

Eur. Phys. J. A **52** (2016) 139 Eur. Phys. J. A **54** (2018) 37

Eur. Phys. J. A (2021) 57:81

(1) Compton Scattering and polarisabilities

(2) Bayesian analysis for experimental design

(3) The deuteron

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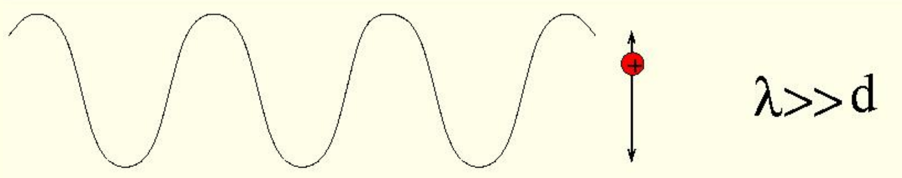
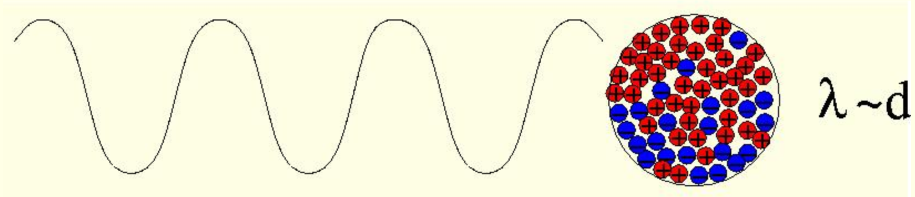
Compton Scattering

For large wavelengths, only sensitive to overall charge: Thomson scattering



But for smaller wavelengths, the target is polarised by the electric and magnetic fields



To leading order in photon energy

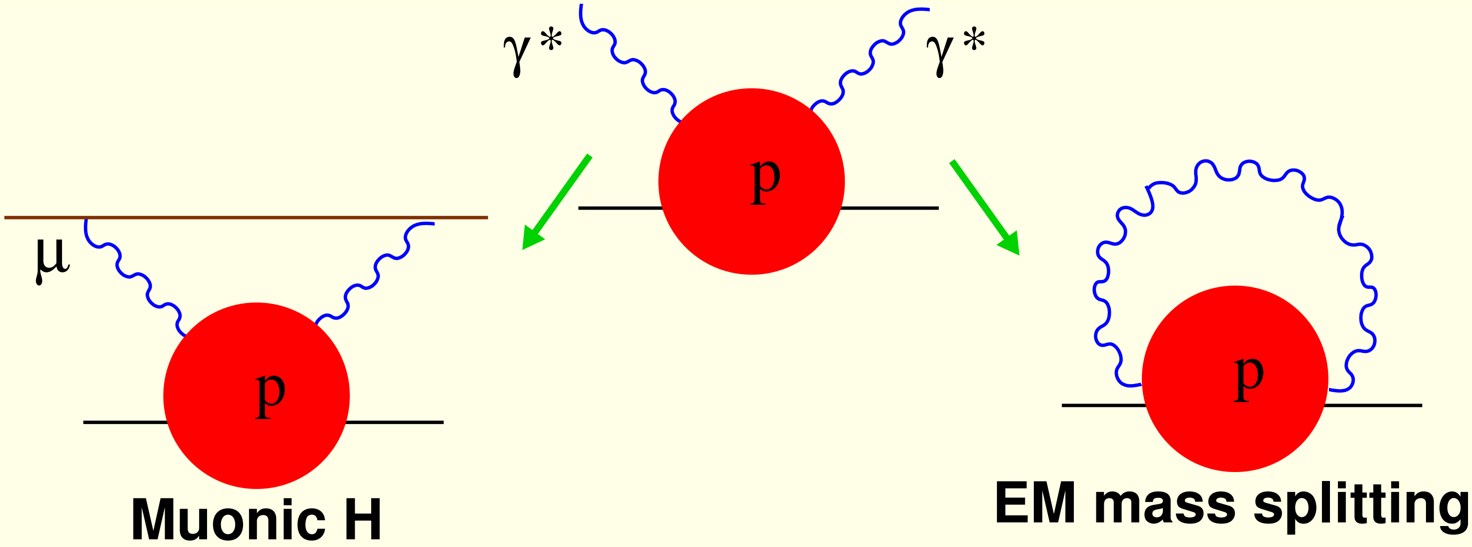
$$H_{\mathrm{eff}} = \frac{(\mathbf{p}-Q\mathbf{A})^2}{2m} + Q\phi - \frac{1}{2}4\pi\Big(\alpha_{E1}\vec{E}^2 + \beta_{M1}\vec{H}^2 + \gamma_{E1E1}\vec{\sigma}\cdot\vec{E}\times\dot{\vec{E}} + \gamma_{M1M1}\vec{\sigma}\cdot\vec{H}\times\dot{\vec{H}} - 2\gamma_{M1E2}E_{ij}\sigma_i H_j + 2\gamma_{E1M2}H_{ij}\sigma_i E_j\Big)$$

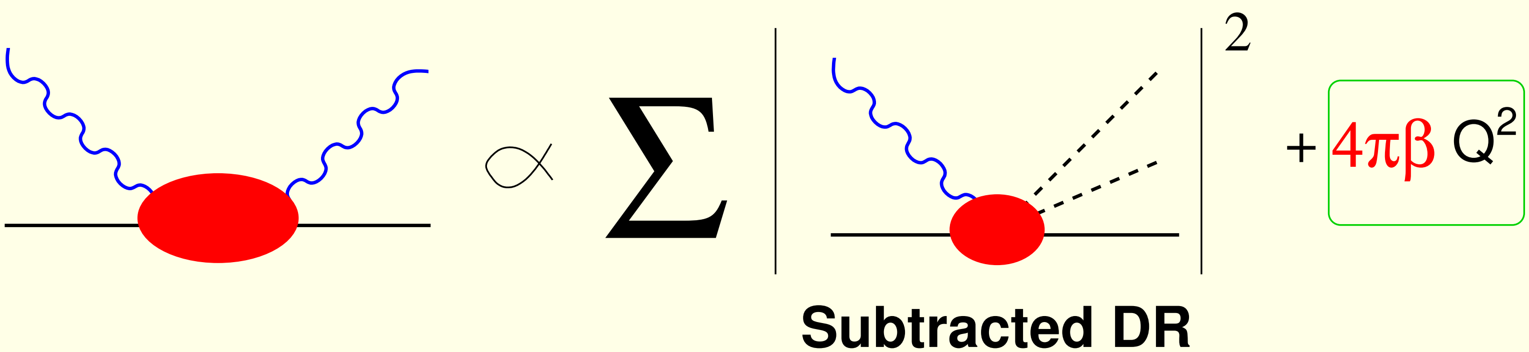
where $E_{ij} = \frac{1}{2}(\nabla_i E_j + \nabla_j E_i)$ and $H_{ij} = \frac{1}{2}(\nabla_i H_j + \nabla_j H_i)$

For real compton scattering, forward amplitude: $\sim(\alpha_{E1}+\beta_{M1})\omega^2$, $\sim\gamma_0\omega^3$.

Both well-constrained by sum rules

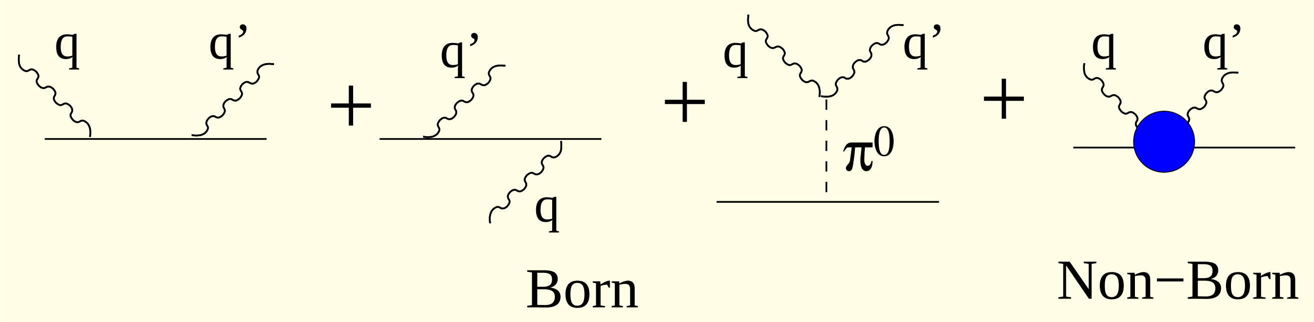
Why β matters



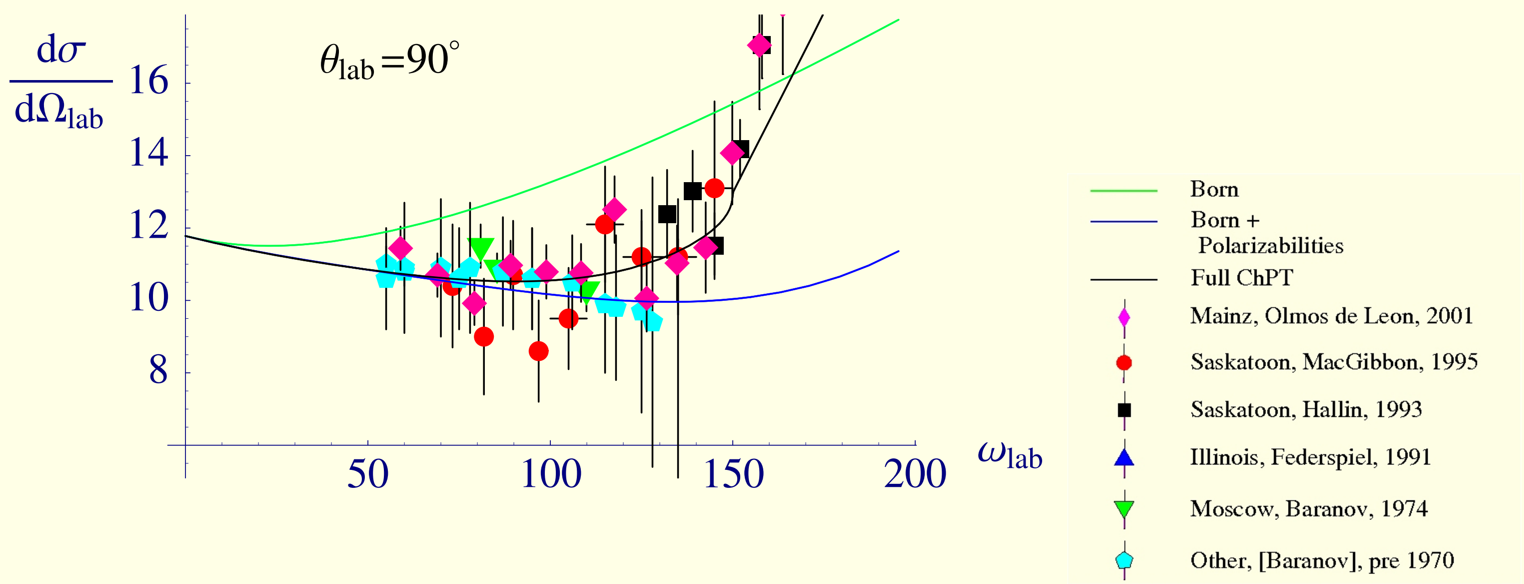


$$\overline{T}_1(\nu,Q^2) = -\nu^2 \int_{\nu_{th}^2}^{\infty} \frac{d\nu'^2}{\nu'^2} \frac{W_1(\nu',Q^2)}{\nu'^2-\nu^2} + 4\pi\beta Q^2 + O(Q^4)$$

Compton Scattering from the nucleon



The scattering amplitude has Born and non-Born pieces. The latter probe the structure of the nucleon; polarisabilities are leading signs of non-pointlike nucleons as we increase the photon energy.



Compton Scattering from the nucleon

The full non-Born contribution has 6 independent amplitudes for real photons

Two common methods: Dispersion relations and Chiral Perturbation Theory

Both consider pions as crucial source of energy-dependence in amplitudes (Delta resonance also captured)

DR uses partial wave analysis of $\gamma N \to \pi N$ data as input

Chiral Perturbation Theory is a field theory which treats pions and nucleons as basic degrees of freedom

Both have difficulties with parameter-free predictions; both can be used to fit Compton scattering data and extract polarisabiltiies.

Chiral Perturbation theory

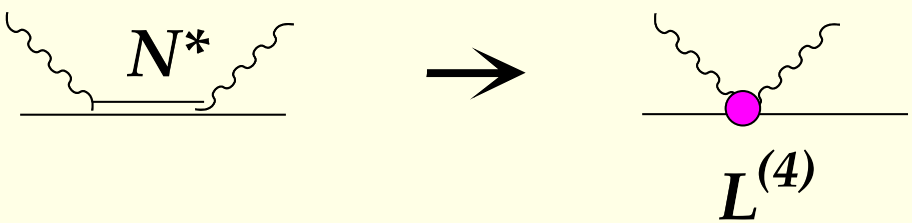
Effective field theory of QCD– relies on separation of scales

• pions are light ($m_\pi \ll m_\rho$)
• low-energy pions interact weakly with other matter ($L_{\pi NN} \propto \overline{N}\partial_\mu \pi N$).
Thus pion loops are suppressed by $\approx m_\pi^2/\Lambda^2$ where $\Lambda \approx m_\rho$.

The Lagrangian contains infinitely many terms:

$$\mathcal{L} = \sum_n \mathcal{L}^{(n)}(c_i^{(n)})$$

Non-pionic nucleon structure shows up in low energy constants $c_i^{(n)}$, but is suppressed by power of momentum: $(k/\Lambda)^n$:



Calculations to nth order involve vertices from $\mathcal{L}^{(n)}$ and pion loops with vertices from $\mathcal{L}^{(n-2)}$; truncation errors are $\sim (k/\Lambda)^{(n+1)}$.

χPT for Compton Scattering from the nucleon

We include nucleons, pions and the Delta in our Lagrangian.

$$\mathcal{L}_{\pi N}^{(4),\mathrm{CT}} = 2\pi e^2 H^\dagger \left[\left(\delta\beta^{(s)} + \delta\beta^{(v)}\tau_3 \right) \left(\tfrac{1}{2} g_{\mu\nu} - v_\mu v_\nu \right) - \left(\delta\alpha^{(s)} + \delta\alpha^{(v)}\tau_3 \right) v_\mu v_\nu \right] F^{\mu\rho} F^\nu{}_\rho H.$$

Counterterms shift α and β at 4th order. Counterterms for spin pols at 5th order.

$$\mathcal{L}_{\gamma N\Delta}^{\mathrm{PP},(2)} = \frac{3e}{2M_N(M_N+M_\Delta)} \left[\bar{\psi}(\mathrm{i} g_M \tilde{F}^{\mu\nu} - g_E \gamma_5 F^{\mu\nu}) \partial_\mu \Psi^3_\nu - \bar{\Psi}^3_\nu \overleftarrow{\partial}_\mu (\mathrm{i} g_M \tilde{F}^{\mu\nu} - g_E \gamma_5 F^{\mu\nu}) \psi \right],$$

$\Delta \equiv M_\Delta - M_N \approx 271$ MeV is a rather small scale. Traditionally it is counted as $\Delta/\Lambda_\chi \sim m_\pi/\Lambda_\chi$ (“SSE”). But in Compton scattering the pion is clearly important at lower energies than the Delta.

Alternative: count $$\frac{m_\pi}{\Delta} \sim \frac{\Delta}{\Lambda_\chi} \quad \Rightarrow \quad \delta^2 \equiv \left(\frac{\Delta}{\Lambda_\chi} \right)^2 \sim \frac{m_\pi}{\Lambda_\chi}$$

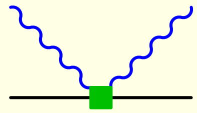
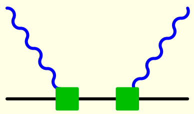
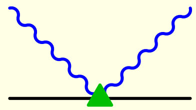
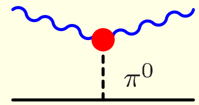
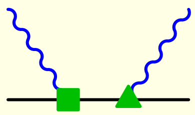
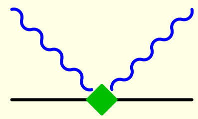
Then graphs with one Δ propagator are one order of δ higher than the corresponding nucleon graphs in low energy region.

Pascalutsa and Phillips, Phys. Rev. C67 (2003) 055202

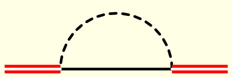
Different counting in resonance region, $\omega \sim \Delta$; we work to at least NLO in both.

Tree graphs

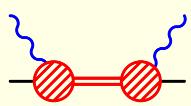
Born terms give the Thomson term and spin-dependent LETs (ensured by gauge and Lorentz invariance)

contribution with typical size		$\omega \sim m_\pi$	$\omega \sim \Delta$
(i)		$e^2\delta^0$ (LO)	$e^2\delta^0$
(ii)	(a)  (b)  (c) 	$e^2\delta^2$	$e^2\delta^1$
(iii)	(a)  (b) 	$e^2\delta^4$	$e^2\delta^2$

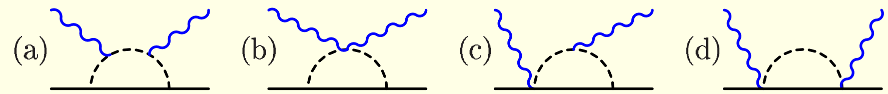
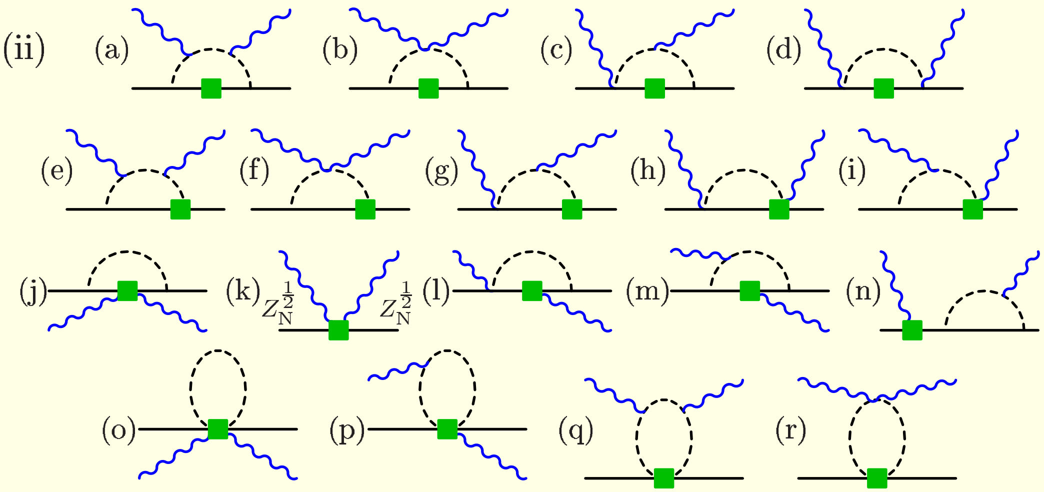
In resonance region Delta-pole graph dominates: width from resuming self-energy



$$\Rightarrow \quad S_\Delta \sim \frac{1}{\omega-(M_\Delta-M_N)+i\Gamma(\omega)}$$

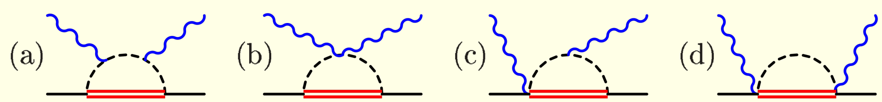
(i)		$e^2\delta^3$	$e^2\delta^{-1}$ (LO)

Loops

contribution with typical size		$\omega \sim m_\pi$	$\omega \sim \Delta$
(i)		$e^2\delta^2$	$e^2\delta^1$
(ii)		$e^2\delta^4$	$e^2\delta^2$

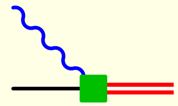
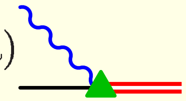
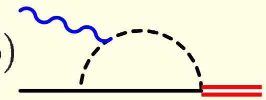
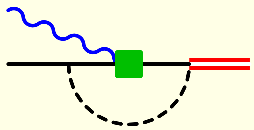
At 4th order we have $1/M$ corrections and c_i contributions

Delta loops are less important in low-energy region

(ii)		$e^2\delta^3$	$e^2\delta^1$

Important: predicts full energy-dependent amplitudes, not just polarisabilities

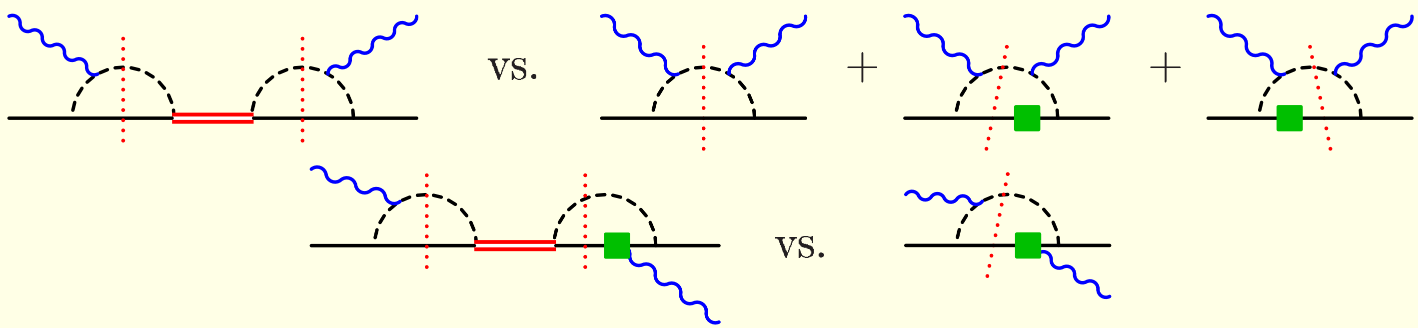
Running of γNΔ vertex

contribution with typical size		$\omega \sim m_\pi$	$\omega \sim \Delta$
(i)		$e\delta^2$	$e\delta^1$
(ii)	(a)  (b) 	$e\delta^4$	$e\delta^2$
(iii)		$e\delta^6$	$e\delta^3$

The inclusion of the imaginary part of running vertices satisfies Watson's theorem
- cancellation of $I = 3/2$ loops at resonance

(i)

(ii)



Multipoles

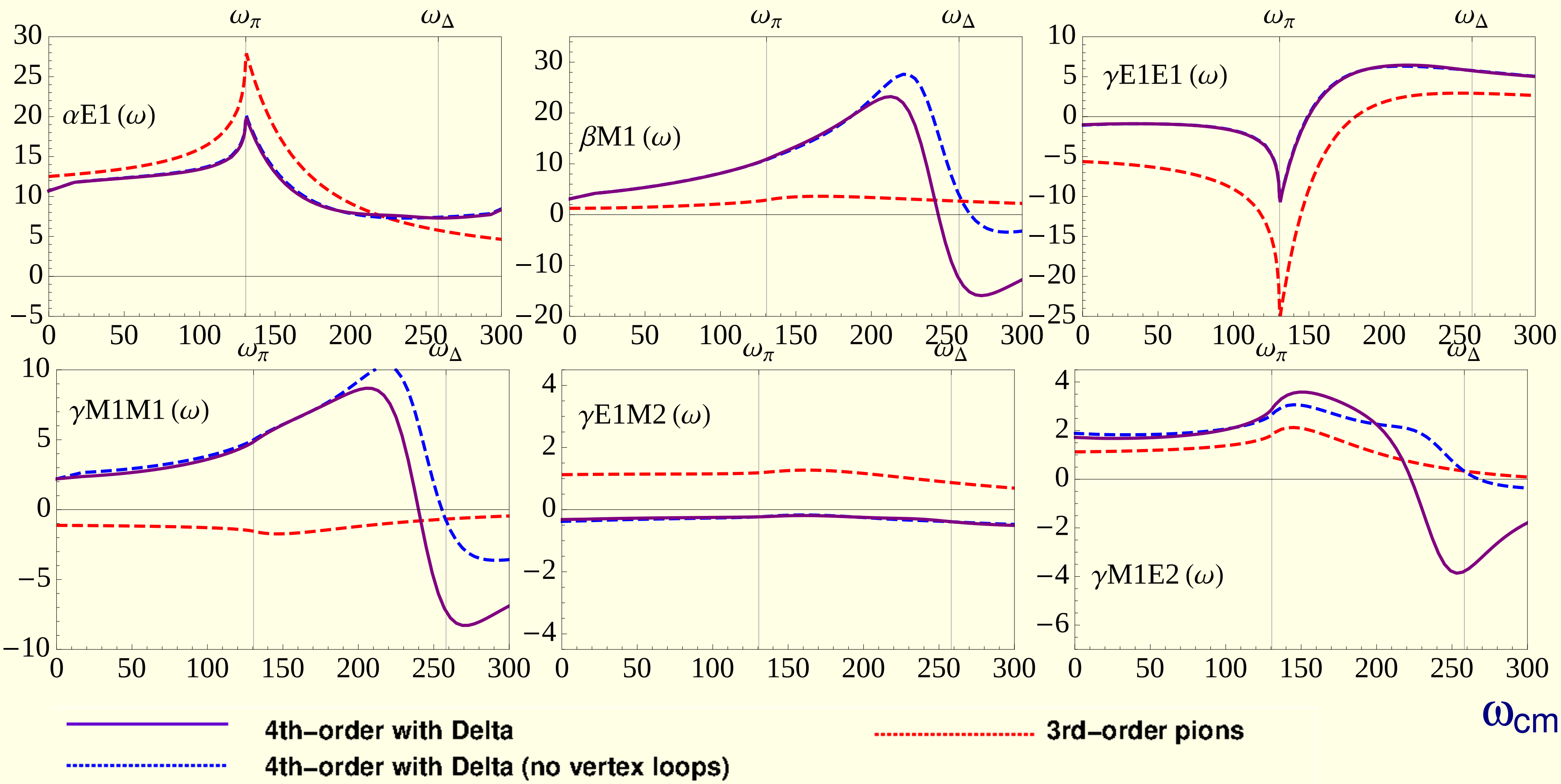
Restricting to lowest photon angular momentum, but at finite photon energy, we can write the effective Hamiltonian

$$
\begin{aligned}
H_{\text{eff}} &= \frac{(\mathbf{p} - Q\mathbf{A})^2}{2m} + Q\phi - \frac{1}{2} 4\pi \Big(\alpha_{E1}(\omega)\vec{E}^2 + \beta_{M1}(\omega)\vec{H}^2 \\
&+ \gamma_{E1E1}(\omega)\vec{\sigma}\cdot\vec{E}\times\dot{\vec{E}} + \gamma_{M1M1}(\omega)\vec{\sigma}\cdot\vec{H}\times\dot{\vec{H}} \\
&- 2\gamma_{M1E2}(\omega)E_{ij}\sigma_i H_j + 2\gamma_{E1M2}(\omega)H_{ij}\sigma_i E_j \Big)
\end{aligned}
$$

with $\alpha \equiv \alpha_{E1}(0)$ etc

Multipoles

We can predict the full energy-dependence of the amplitudes, and only the value at the origin for α, β and γ_{M1M1} are fitted.

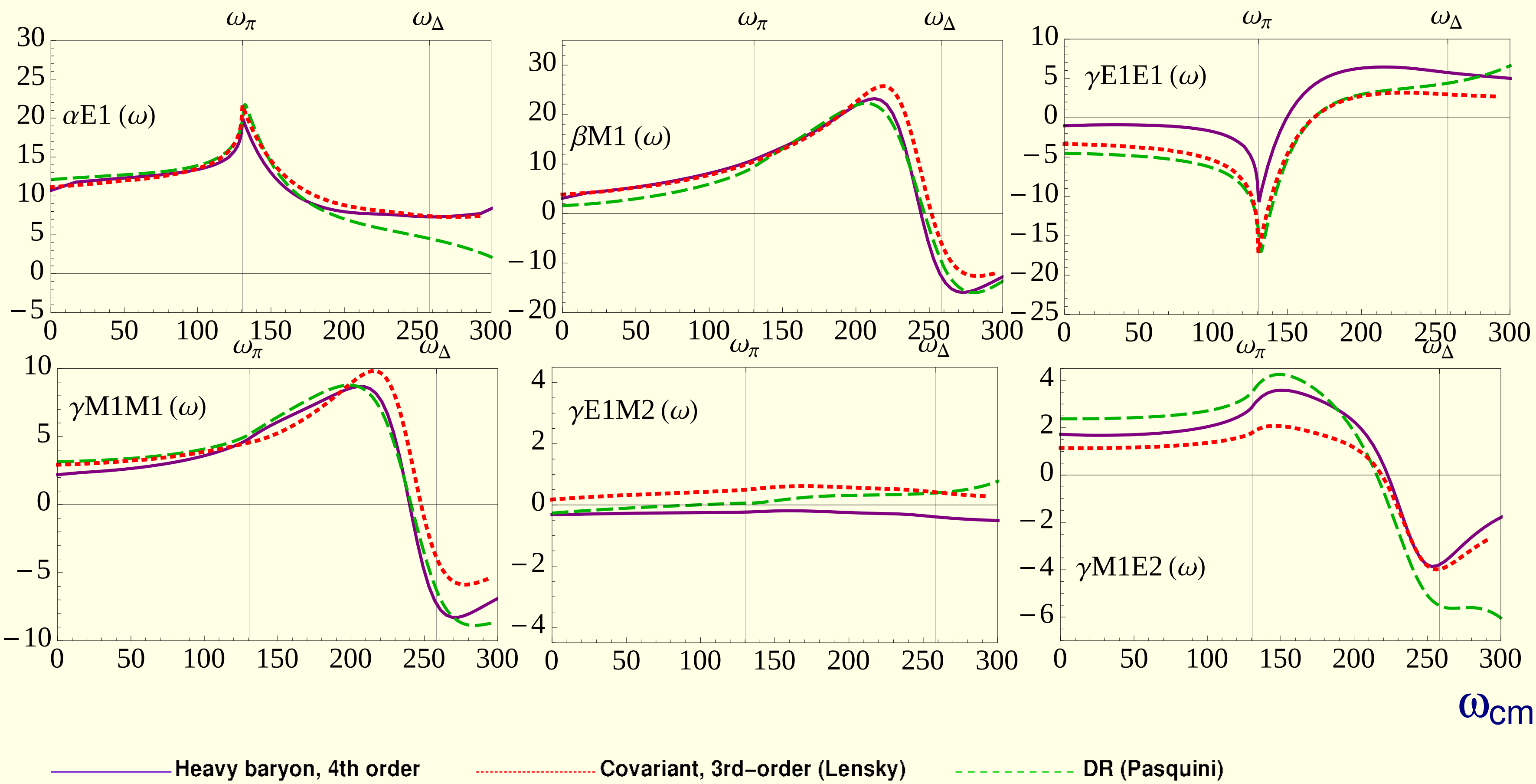


Note contribution of Delta, and also of the running of the γNΔ vertex.

H. Grießhammer *et al.* , Eur. Phys. J. A **54** (2018) 37

Comparison of theoretical predictions for Multipoles

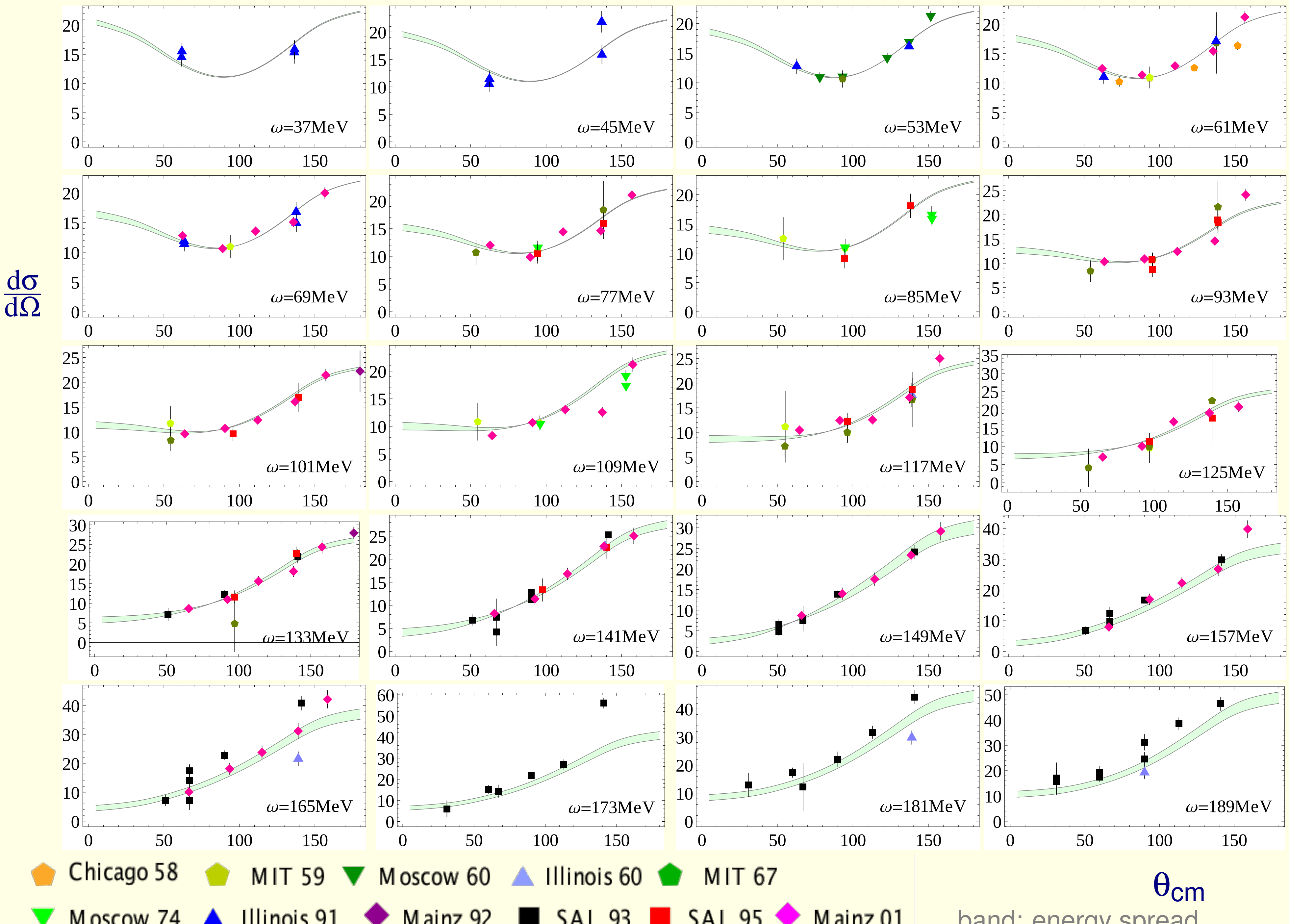
Different predictions do not fully agree on the physical origins of the polarisabilities.
But Chiral and DR predictions agree very well for the shape of the energy dependence of corresponding multipoles

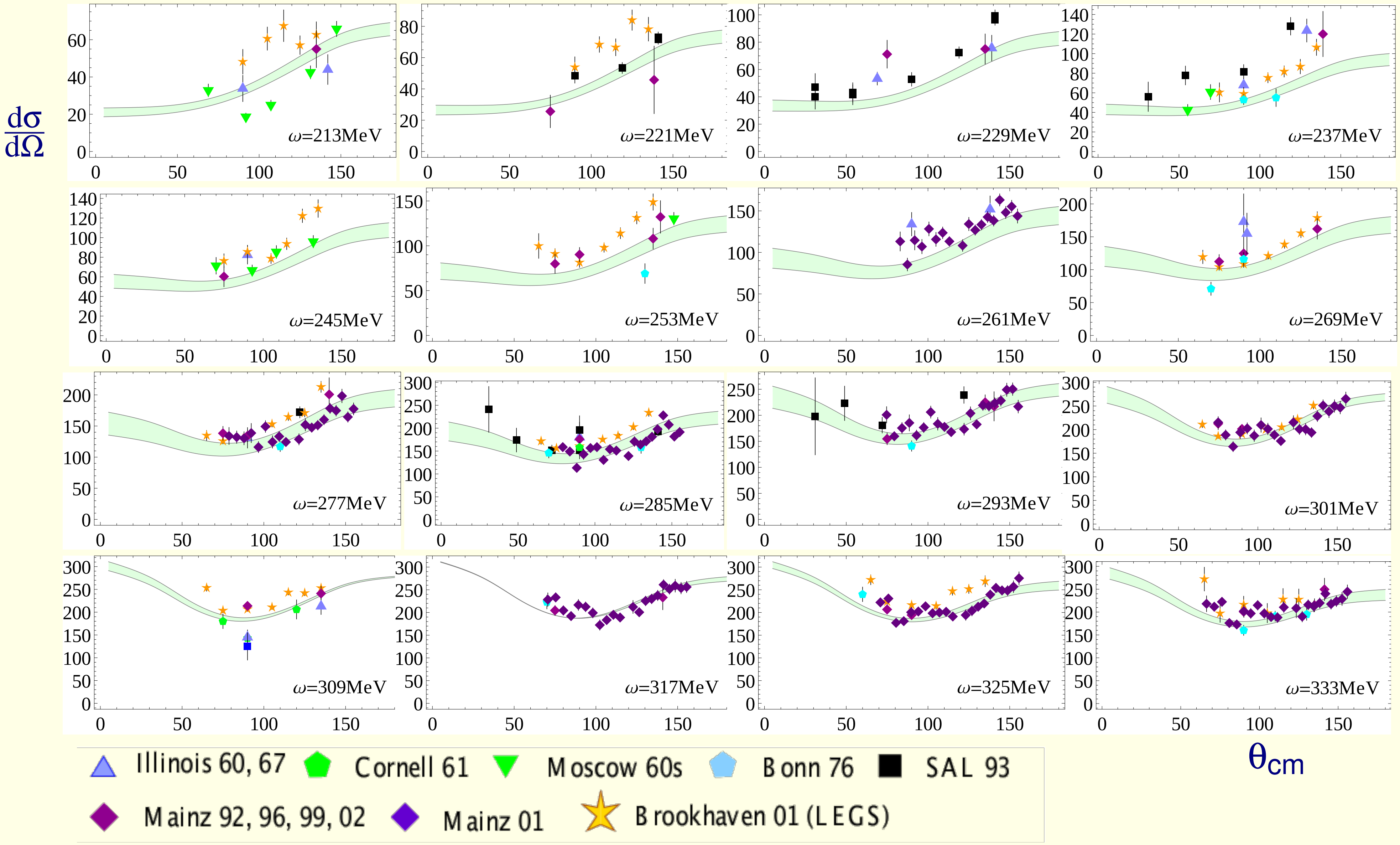


DR: Hildebrandt *et al.* , Eur. Phys. J. A **20** 293 (2004) Chiral: V Lensky *et al.* EPJC **75** 604 (2015)

Our strategy: Static polarisabilities best obtained from Compton scattering.

Fitting the proton data





band gives spread of theory curve due to energy binning

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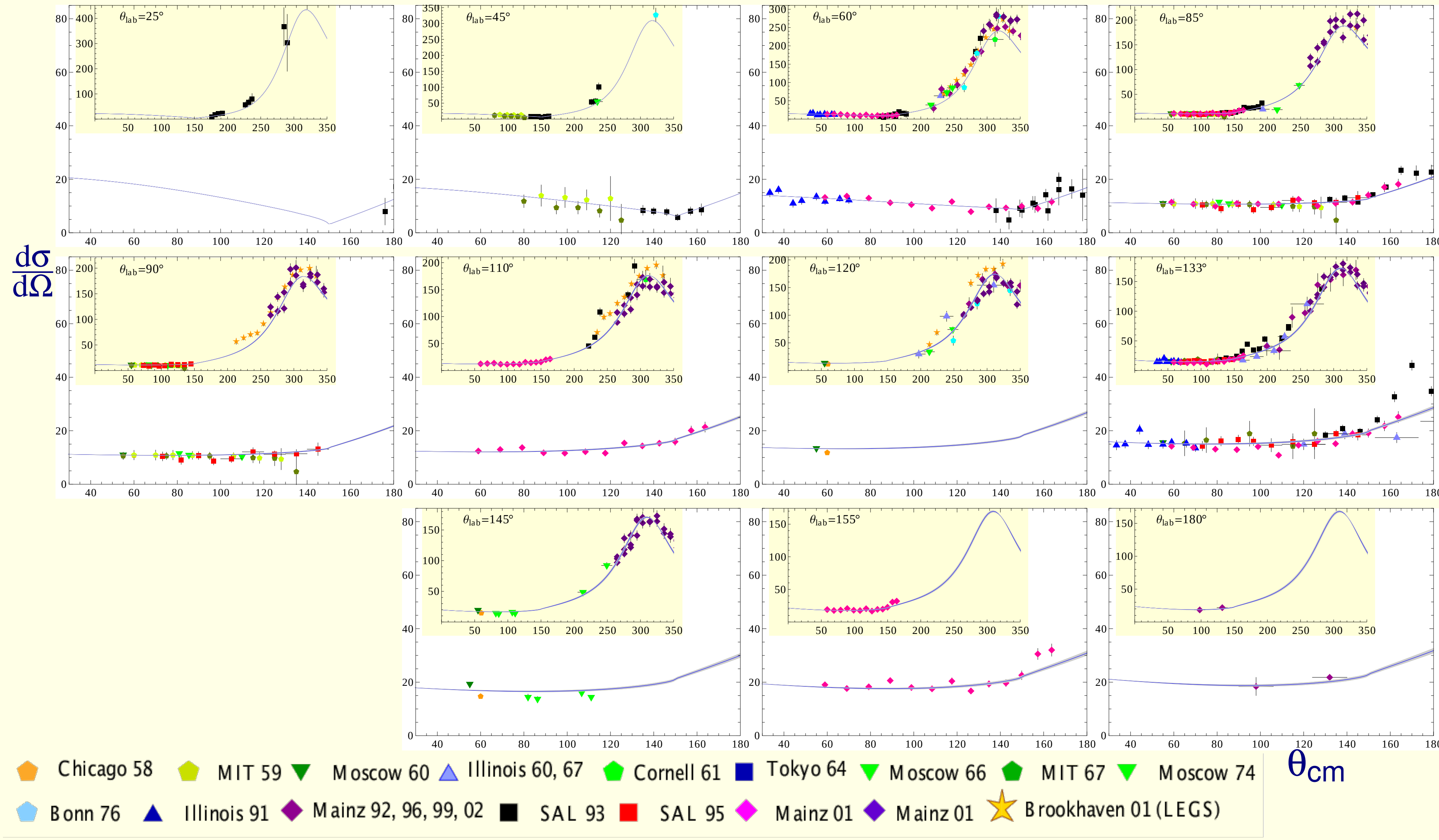


Constraining $\alpha+\beta$ with Baldin Sum rule and fitting consistent data set up to 170 MeV:

$$\alpha_p = (10.65 \pm 0.35(\text{stat}) \pm 0.2(\text{Bald}) \pm 0.3(\text{theory})) \times 10^{-4}\,\text{fm}^3$$

$$\beta_p = (3.15 \mp 0.35(\text{stat}) \pm 0.2(\text{Bald}) \mp 0.3(\text{theory})) \times 10^{-4}\,\text{fm}^3$$

Comparison

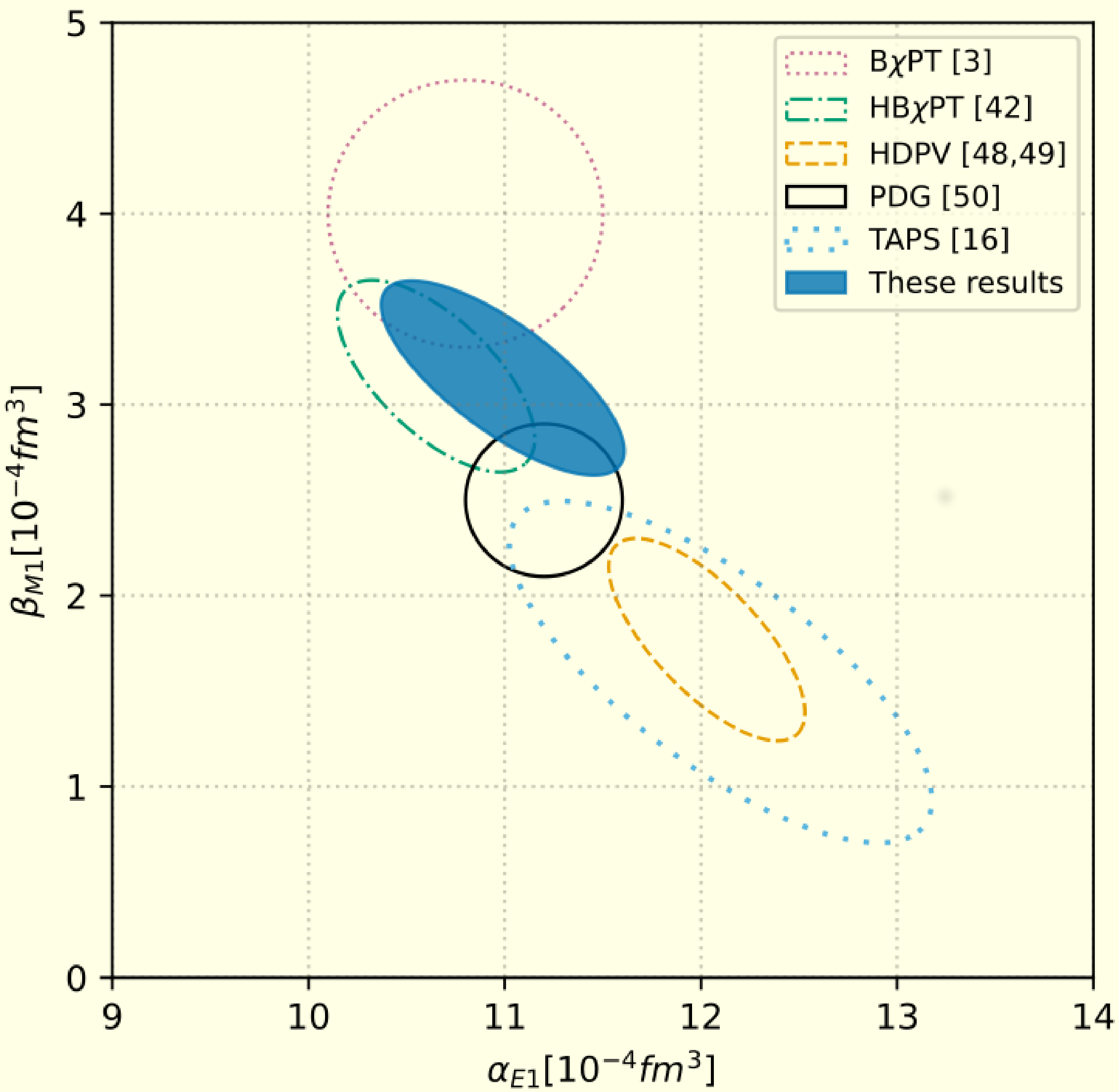


figure from E. Mornacchi *et al.* (A2) - ”these results”, see Edoardo’s talk

Doubly-Virtual Compton Scattering

Same diagrams give $T^{\mu\nu}(\nu, q^2)$, both elastic and polarization.
Calculate $\Delta E^{2\gamma}$ directly? previous talks

But χPT only valid for μ, $q \ll m_\rho \sim 700$ MeV. Wrong asymptotic behaviour.
And only feasible at leading order, contribution from 4th order LEC $\delta\beta$ diverges

We chose to use experimental input where it exists: $F_{D,P}(q^2)$, $W_{1,2}(\nu, q^2)$.
only using χPT to constrain $\overline{T}_1(0, q^2)$

First subtract elastic (Born) contribution calculated to same order.
Low energy theorem:

$$\overline{T}_1(0, Q^2) = 4\pi\beta Q^2 + O(Q^4) \equiv 4\pi\beta Q^2 f_\beta(Q^2)$$

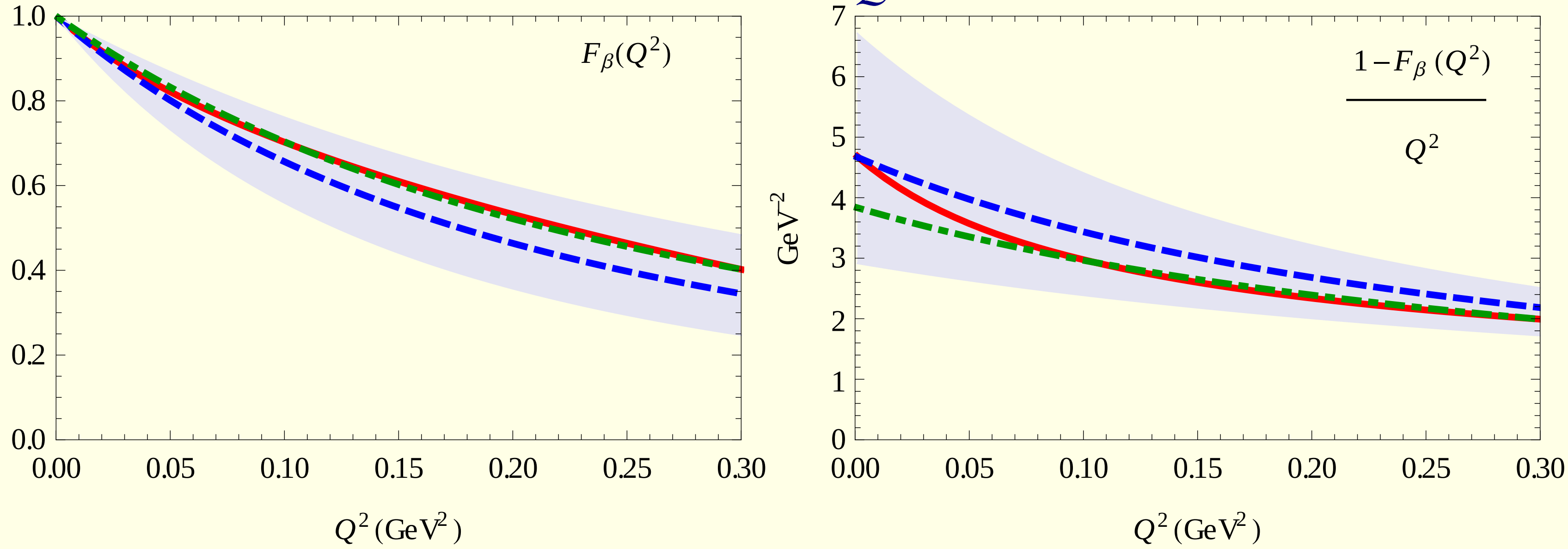
High-Q^2:

$$\overline{T}_1(0, Q^2) \sim Q^{-2} \implies f_\beta(Q^2) \sim \frac{1}{(1 + Q^2/2M_\beta^2)^2}$$

Choose M_β so that χPT and quadrupole form factors match.

Form factor

$$4\pi\beta f_\beta(Q^2) \equiv \frac{\overline{T}_1(0,Q^2)}{Q^2}$$



Match χPT form at origin: $M_\beta = 462$ MeV
at $Q^2 \sim m_\pi^2$: $M_\beta = 510$ MeV
Estimated error band: $M_\beta = 385$ - 585 MeV.

Use $M_\beta = 485 \pm 100 \pm 40 \pm 25$ MeV,
Errors from

- generous allowance for higher-order effects and errors in input parameters
- $\beta = 3.1 \pm 0.5$
- matching uncertainty

Energy shift

$$\Delta E_{\text{sub}} = \frac{\alpha_{\text{EM}}\,\phi(0)^2}{4\pi m}\int_0^\infty \mathrm{d}Q^2\,\frac{\overline{T}_1(0,Q^2)}{Q^2}\times\left[1+\left(1-\frac{Q^2}{2m^2}\right)\left(\sqrt{\frac{4m^2}{Q^2}+1}-1\right)\right],$$

For constant f_β this would diverge
but with quadrupole form, 90% comes from $Q^2 < 0.3$ GeV^2

Insensitive to M_β: for $M_\beta = 485 \pm 110$ MeV,

$$\int \mathrm{d}Q^2\, f_\beta(Q^2)\times[\cdots] = 0.114 \pm 0.013\ \text{GeV}^2$$

Main error from $\beta = 3.1 \pm 0.5$:

Result: $\Delta E_{\text{sub}} = -0.0042(10)$ meV

Broadly compatible with previous results

M. C. Birse and J. McG, Eur. Phys. J. A **48** (2012) 120

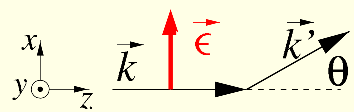
Spin-dependent Compton scattering

$$H_{\text{eff}} = \frac{(\mathbf{p}-Q\mathbf{A})^2}{2m} + Q\phi - \frac{(Q+\kappa)}{2m}\sigma\cdot H - \frac{1}{2}4\pi\left(\alpha\vec{E}^2+\beta\vec{H}^2 + \gamma_{E1E1}\vec{\sigma}\cdot\vec{E}\times\dot{\vec{E}} + \gamma_{M1M1}\vec{\sigma}\cdot\vec{H}\times\dot{\vec{H}} - 2\gamma_{M1E2}E_{ij}\sigma_i H_j + 2\gamma_{E1M2}H_{ij}\sigma_i E_j\right)$$

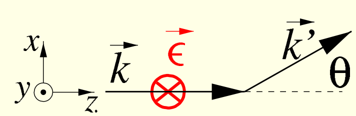
Spin-polarisabities have most influence if the beam or target or both are polarised.

Linearly polarised beam $\Sigma_3 = \dfrac{\sigma^{\parallel}-\sigma^{\perp}}{\sigma^{\parallel}+\sigma^{\perp}}$

$\left[\dfrac{d\sigma}{d\Omega}\right]^{\text{lin}}_x$:

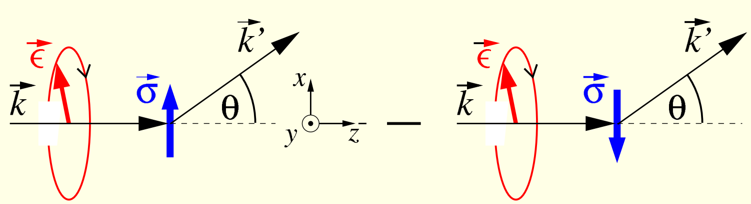


$\left[\dfrac{d\sigma}{d\Omega}\right]^{\text{lin}}_y$:

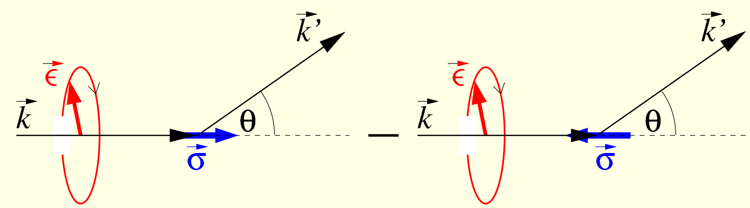


Circular beam, polarised target $\qquad \Sigma_{2x} = \dfrac{\sigma^R_\perp - \sigma^L_\perp}{\sigma^R_\perp + \sigma^L_\perp} \qquad \Sigma_{2z} = \dfrac{\sigma^R_\parallel - \sigma^L_\parallel}{\sigma^R_\parallel + \sigma^L_\parallel}$

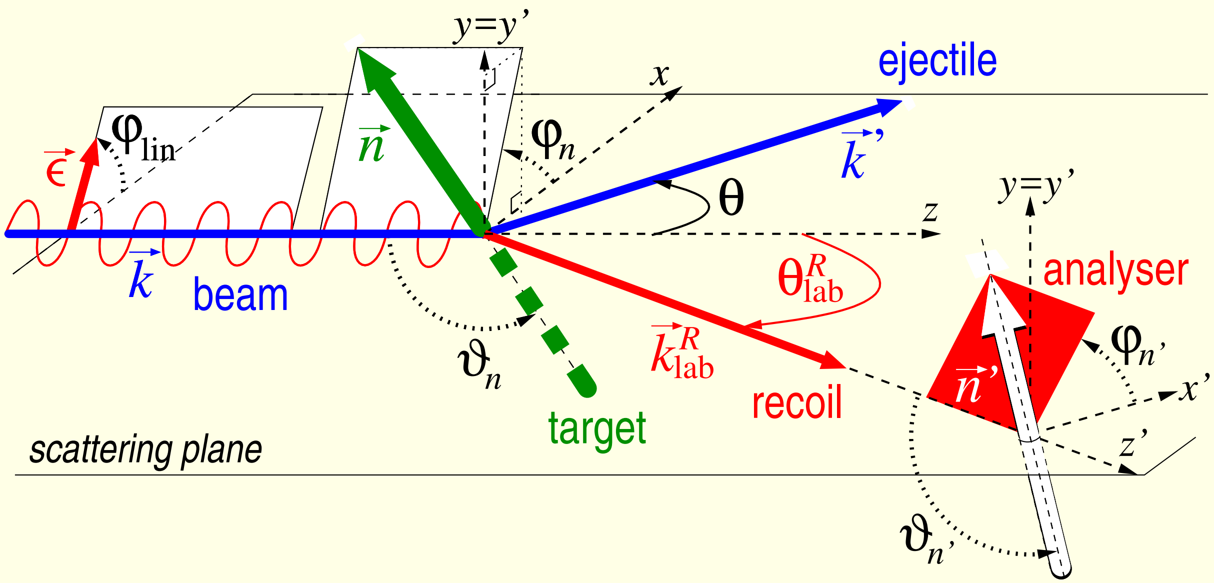
Δ^{circ}_x :



Δ^{circ}_z :



Other asymmetries and polarisability transfer observables



$$\Sigma_{2x}$$

Numerical index: polarisation of light

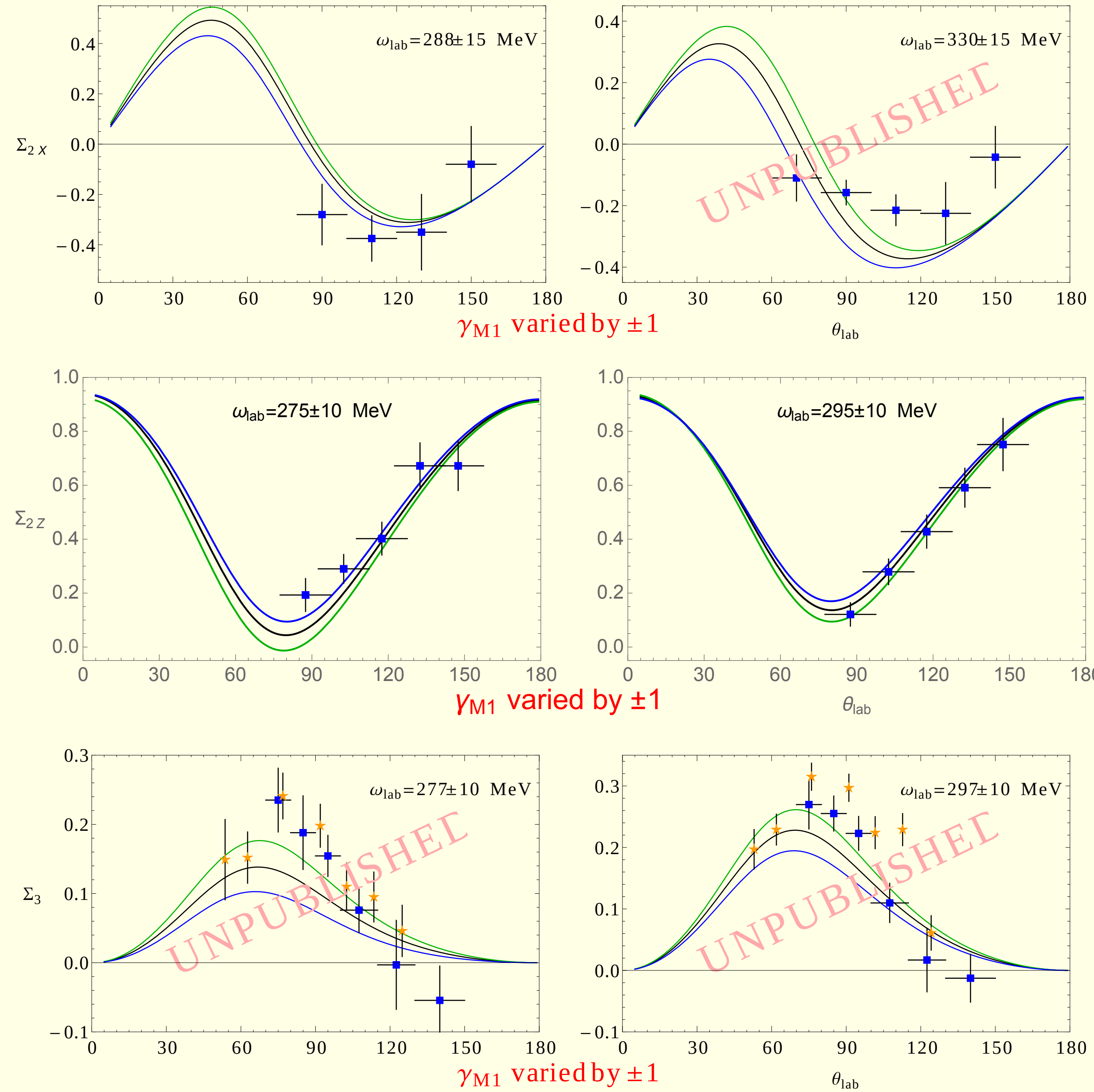
- 3: linear, 0 or π
- 1: linear, $\pm\frac{\pi}{2}$
- 2: right/left circular

Cartesian index: polarisation of nucleon

- z: along beam
- y: $\perp$ to reaction plane
- x: in reaction plane, $\perp$ to z

Prime on either indicates scattered photon or nucleon: polarisation transfer.
polarised scattered nucleon might be detectable.

High-energy data from MAMI



■ P. Martell, Phys. Rev. Lett. 114, 112501 and PhD thesis; D. Paudyal, Phys. Rev. C 102, 035205; C. Collicott, PhD thesis; ☆ LEGS data

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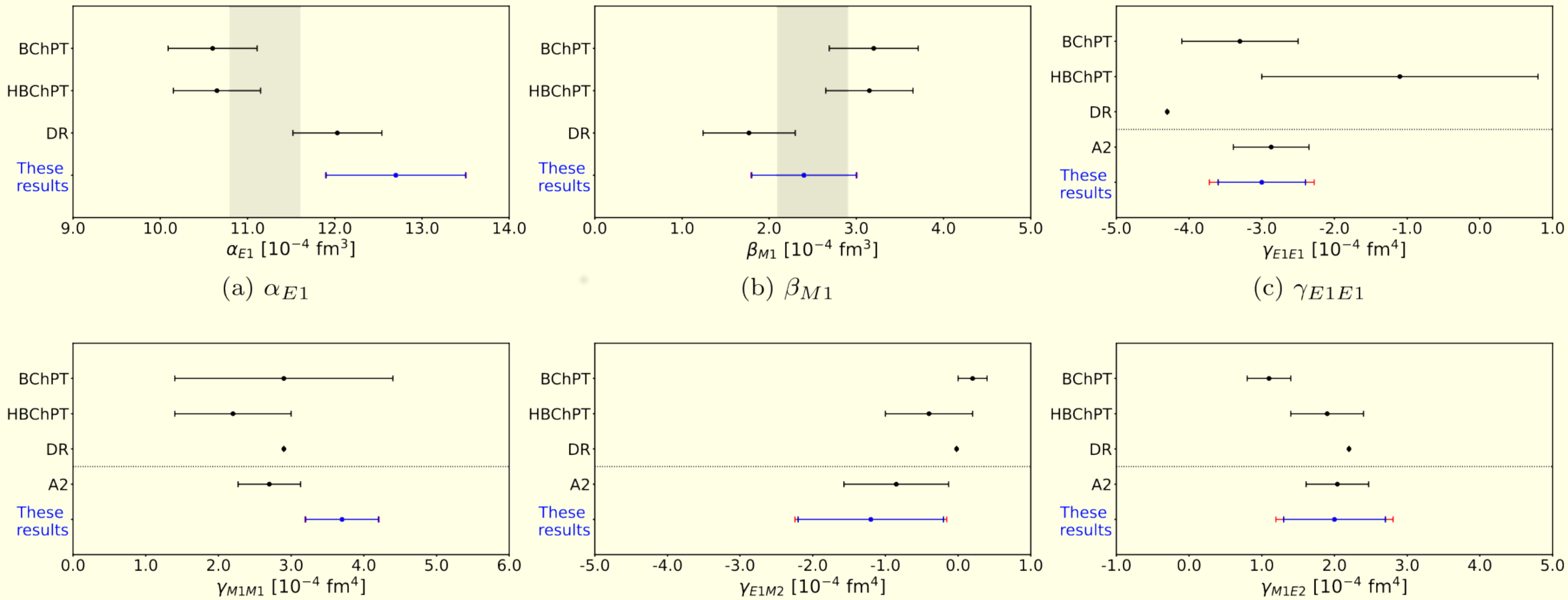
Results for polarisabilities, fits and predictions



(a) α_{E1} (b) β_{M1} (c) γ_{E1E1}

(d) γ_{M1M1} (e) γ_{E1M2} (f) γ_{M1E2}

Figure reproduced from E. Mornacchi, S. Rodini, B. Pasquini, and P. Pedroni Phys. Rev. Lett. 129, 102501

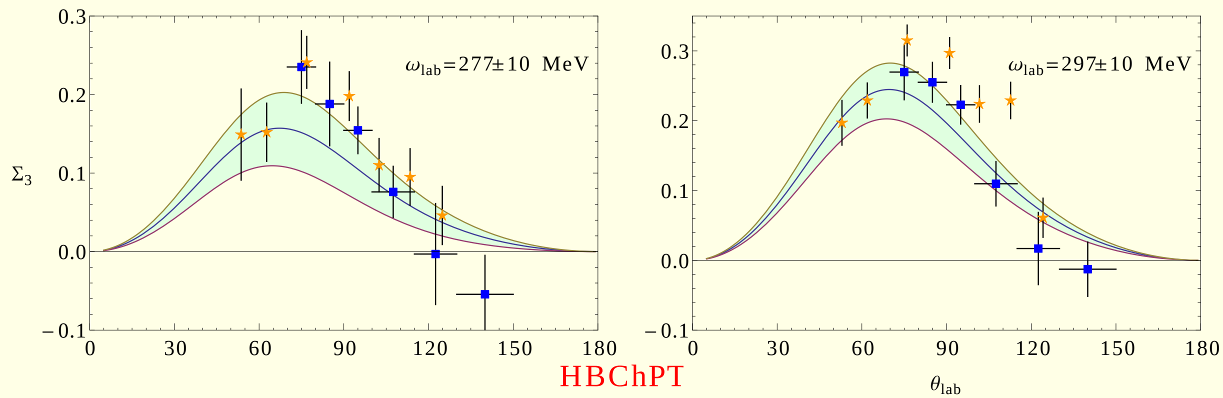
”These results” refer to a fit using DR to unpolarised and polarised data including new low-energy MAMI data - see Edoardo’s talk!

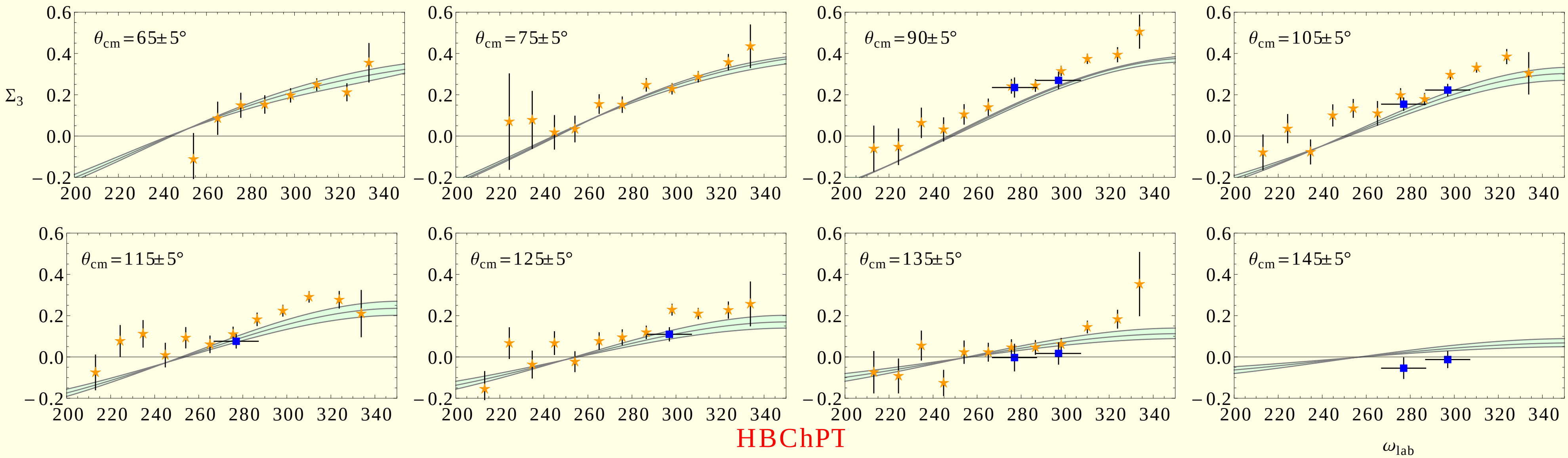
But: uncertainties more than just polarisabilities

Σ_3: Varying theory details and $\gamma N\Delta$ coupling:

But: uncertainties more than just polarisabilities

Σ_3: Varying theory details and $\gamma N \Delta$ coupling:





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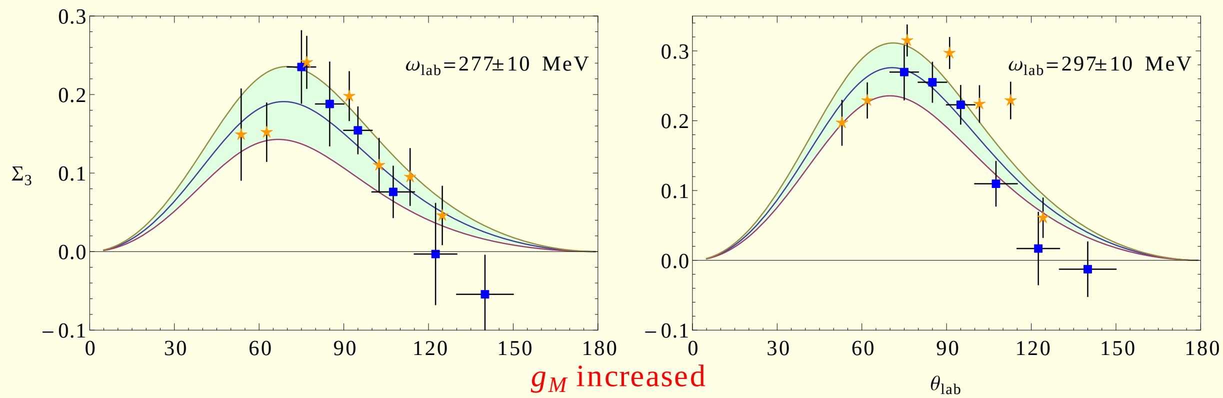
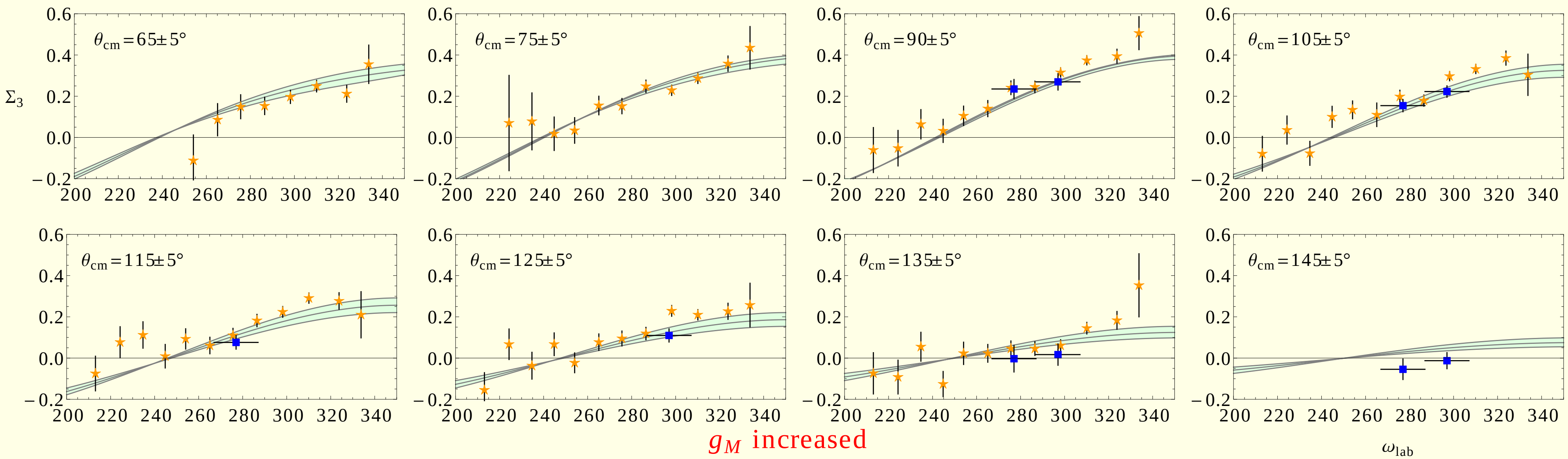
But: uncertainties more than just polarisabilities

Σ_3: Varying theory details and $\gamma N\Delta$ coupling:





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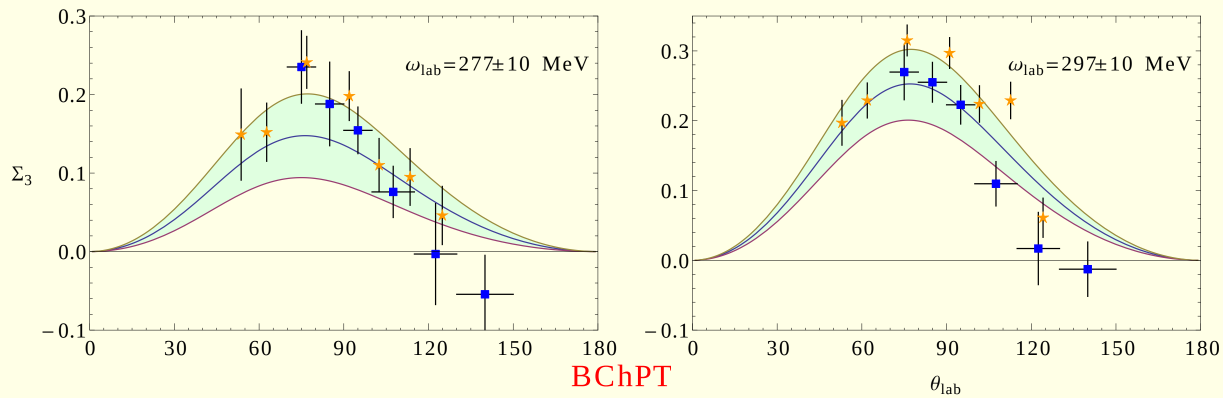
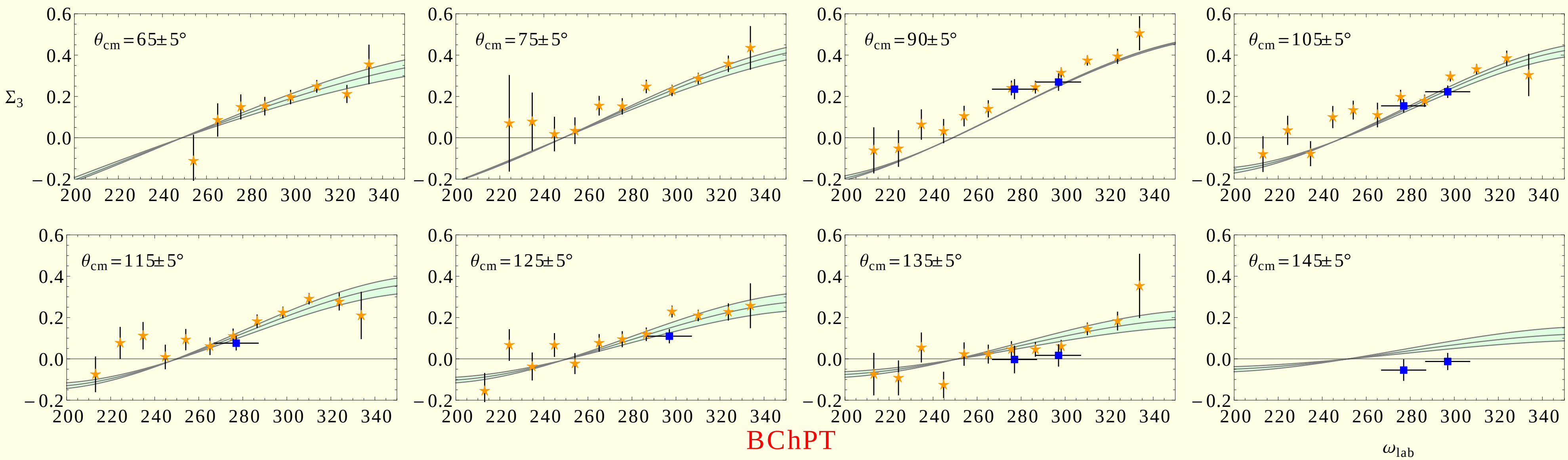


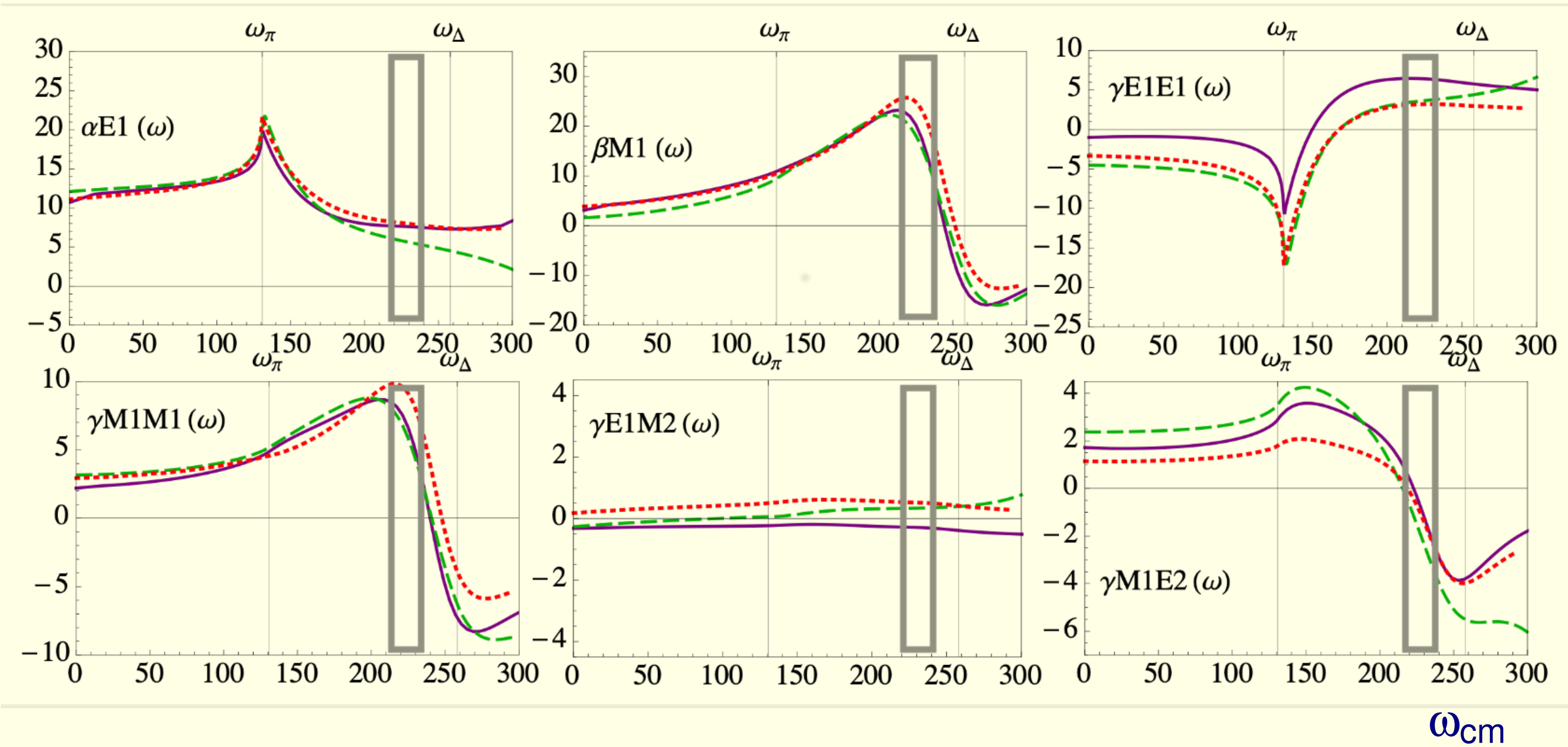


BChPT: V. Lensky *et al.* , EPJC **75** 604 (2015)

Multipoles again

This MAMI data is taken well into the resonance region....

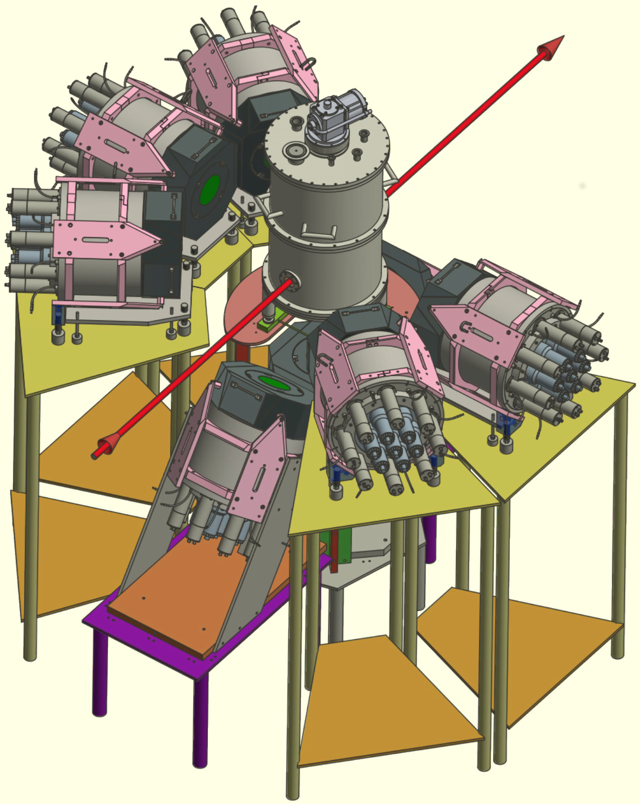
Not ideal for extracting zero-energy polarisabilities!

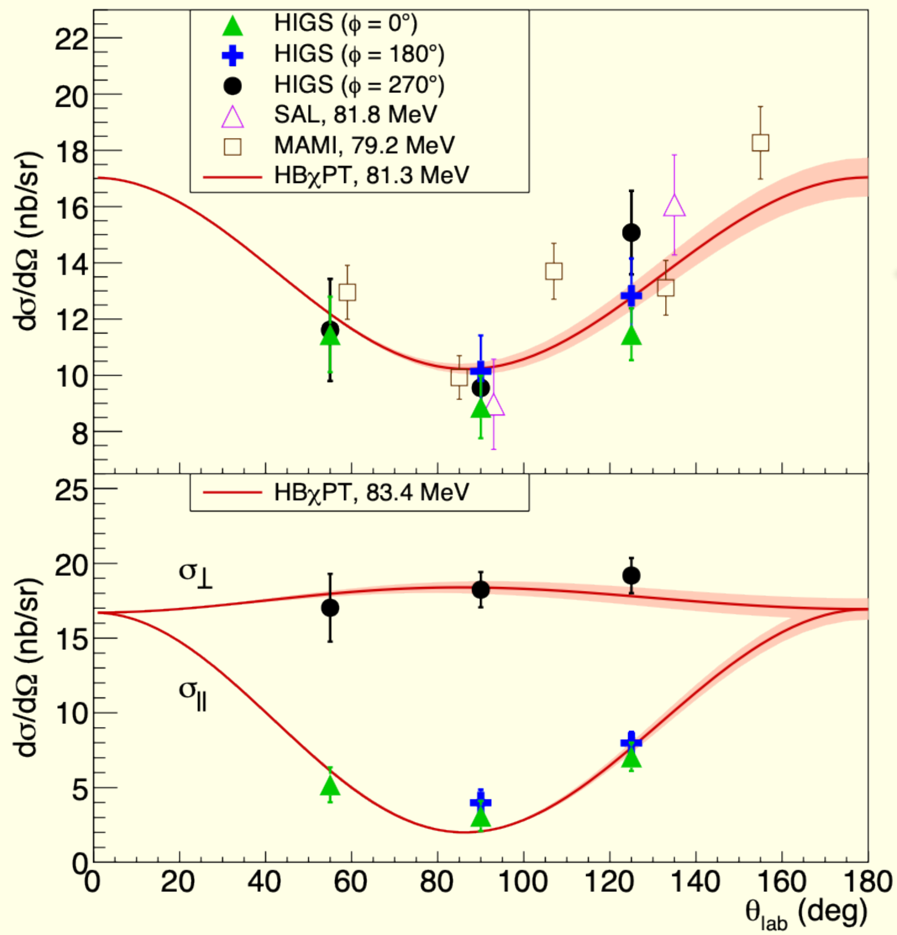


Heavy baryon, 4th order — Covariant, 3rd-order (Lensky) — DR (Pasquini)

Recent lower energy data: HIGS

HIINDA array of NaI detectors:



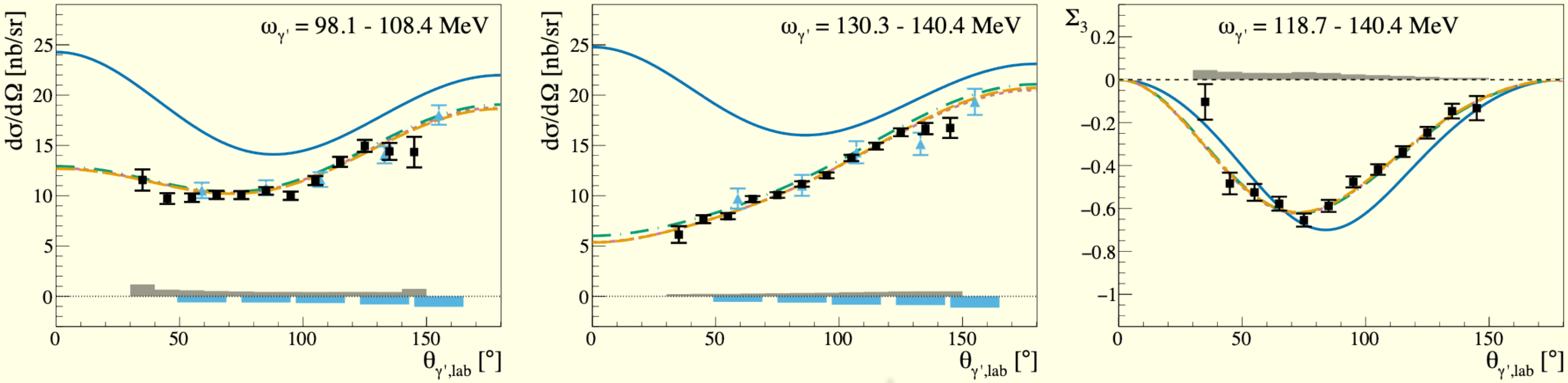


$\alpha_p = 13.8 \pm 1.2$(stat) $\pm\, 0.1$(Bald) $\pm\, 0.3$(theory)

$\beta_p = 0.2 \mp 1.2$(stat) $\pm\, 0.1$(Bald) $\mp\, 0.3$(theory)

X. Li et al. Phys. Rev. Lett. 128 (2022) 132502 see Evies's talk

Recent lower energy data: MAMI



(a) Unpolarized differential cross-section. (b) Beam asymmetry.

$\alpha_p = 11.0 \pm 0.5\text{(expt)} \pm 0.4\text{(model)}$

$\beta_p = 3.1 \pm 0.3\text{(expt)} \pm 0.4\text{(model)}$

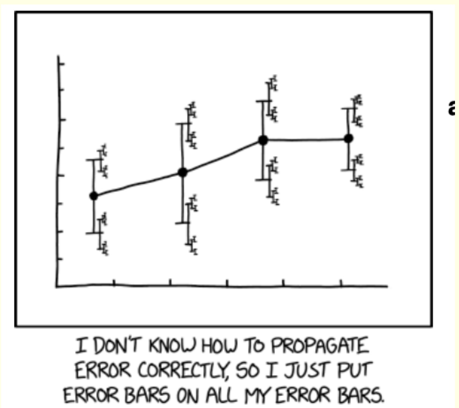
Figure reproduced from E Mornacchi *et al.* (A2), Phys. Rev. Lett. 128 (2022) 13 - see Edoardo's talk

Designing Optimal Experiments

There are 13 observables (if recoil proton polarisation is measured), beam energy and detector angle can be varied... Which experiments will maximise the (reliable) information gained about polarisabilities?

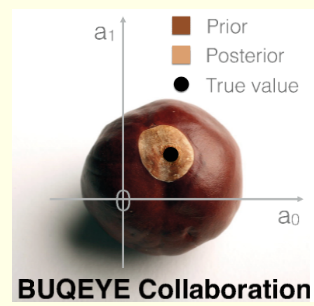
In any fit, the sensitivity to amplitude terms like $\delta\beta\omega^2$ or $\delta\gamma_{M1M1}\omega^3$ increases rapidly with energy.

BUT theory uncertainties in an EFT also rise rapidly as we approach the breakdown scale, and also due to power-counting rearrangement for $\omega \sim \Delta$



xkcd 13.02.2019

Need to quantify theory errors to answer this.



"Buqeye Collaboration"

- Bayesian Uncertainty Quantification: Errors in Your EFT

R. J. Furnstahl, N. Klco, D. R. Phillips, and S. Wesolowski, Phys. Rev. C 92, 024005

Theory errors

HBChPT, fit to data beyond LO:

$\alpha - \beta = 11.5(\text{LO}) - 3.5(\text{NLO}) - 0.1(\text{N}^2\text{LO}) + ????$

Expansion in powers of $\delta \sim \frac{m_\pi}{m_\Delta - m_N} \sim \frac{m_\Delta - m_N}{\Lambda_\chi} \sim 0.4$:

$\alpha - \beta = c_0 + c_1\delta + c_2\delta^2 + c_3\delta^3 \ldots$

Conservative estimate of $|c_3|$: $\text{Max}(|c_0|, |c_1|, |c_2|)$ gives $|c_3|\delta^3 \sim 0.7$ - theory error on extraction at N^{2}LO.

But how confident can we be? What is the probability that the next terms is larger than we've estimated? That depends on our prior belief in the distribution of the c_k - our priors

Bayes' thorem: $\text{pr}(A|B) = \dfrac{\text{pr}(B|A)\text{pr}(A)}{\text{pr}(B)}$

Need to assume something about the distribution of the c_i relative to the maximum c_i in the series, call it $\bar{c}$, eg uniform prior $\text{pr}(c_i|\bar{c})] \propto \Theta(\bar{c} + c_i)\Theta(\bar{c} - c_i)$.
Then marginalise over $\bar{c}$, typically with a log-uniform initial prior: $\text{pr}(\bar{c}) \propto \frac{1}{\bar{c}}$, using information about actual coefficients calculated:

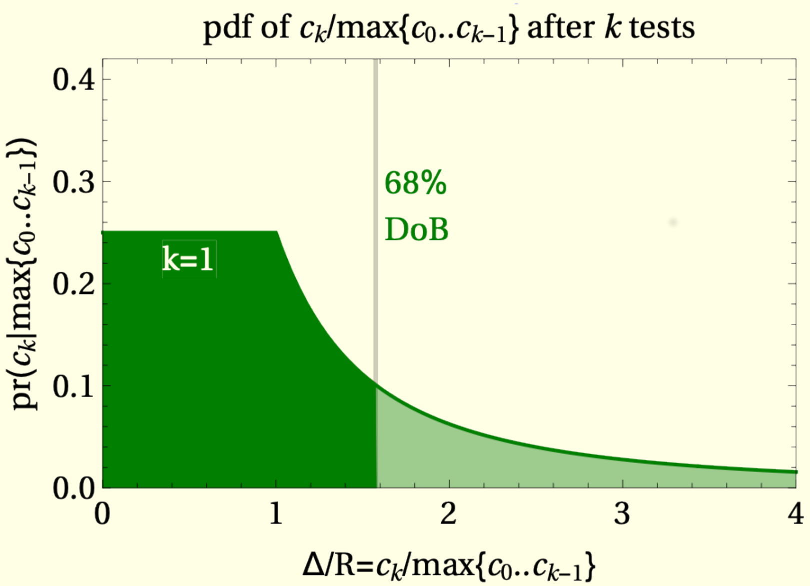
Theory errors

$$\begin{aligned}
\mathrm{pr}(c_k|c_0, c_1 \ldots c_{k-1}) &= \int d\bar{c}\, \mathrm{pr}(c_k|\bar{c})\mathrm{pr}(\bar{c}|c_0, c_1 \ldots c_{k-1}) \\
&= \frac{\int d\bar{c}\, \mathrm{pr}(c_k|\bar{c})\mathrm{pr}(c_0, c_1 \ldots c_{k-1}|\bar{c})}{\mathrm{pr}(c_0, c_1, \ldots c_{k-1})} \\
&\propto \int d\bar{c}\, \mathrm{pr}(c_k|\bar{c}) \prod_{i=0}^{k-1} \mathrm{pr}(c_i|\bar{c})
\end{aligned}$$

Theory errors

$$\mathrm{pr}(c_k|c_0,c_1\dots c_{k-1}) = \int \mathrm{d}\bar{c}\,\mathrm{pr}(c_k|\bar{c})\mathrm{pr}(\bar{c}|c_0,c_1\dots c_{k-1})$$
$$= \frac{\int \mathrm{d}\bar{c}\,\mathrm{pr}(c_k|\bar{c})\mathrm{pr}(c_0,c_1\dots c_{k-1}|\bar{c})}{\mathrm{pr}(c_0,c_1,\dots c_{k-1})}$$
$$\propto \int \mathrm{d}\bar{c}\,\mathrm{pr}(c_k|\bar{c})\prod_{i=0}^{k-1}\mathrm{pr}(c_i|\bar{c})$$

Defining Δ as the actual (unknown) truncation error and R as the estimate given by $c_k\delta^k$:



Priors: all c_n "equally likely", "any upper bound" $\bar{c}$.

order	DoB in $\pm R$	σ: 68%	$\Delta(95\%)$
LO	$\frac{1}{2} = 50\%$	$1.6\,R$	$11R = 7\sigma$
Gauß	68.27%	$1.0\,R$	2.0σ

Laplace's Law of Succession (flat prior, T/F): Chance that next coefficient $<$ max is $\dfrac{k}{k+1}$.

Theory errors

$$\text{pr}(c_k|c_0,c_1\ldots c_{k-1}) = \int \mathrm{d}\bar{c}\,\text{pr}(c_k|\bar{c})\text{pr}(\bar{c}|c_0,c_1\ldots c_{k-1})$$
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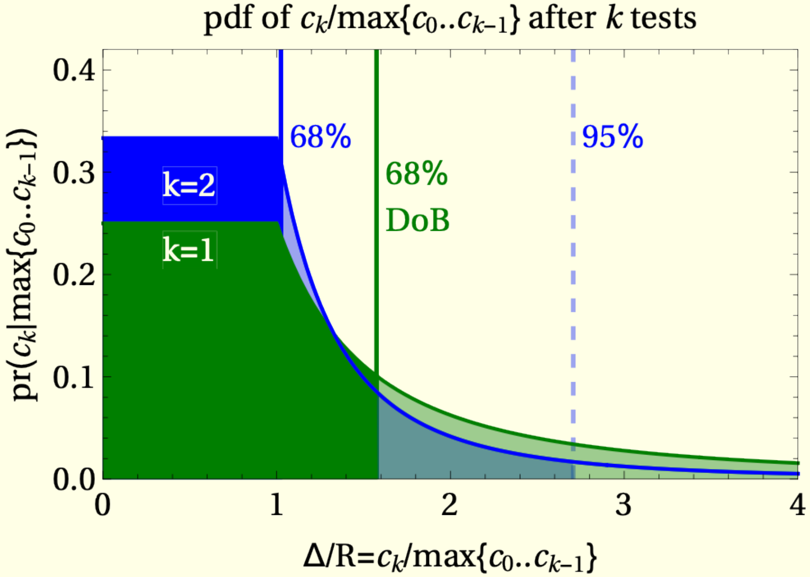

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LO	$\frac{1}{2} = 50\%$	$1.6\,R$	$11R = 7\sigma$
NLO	$\frac{2}{3} = 66.7\%$	$1.0\,R$	$2.7R = 2.6\sigma$
Gauß	68.27%	$1.0\,R$	2.0σ

Laplace's Law of Succession (flat prior, T/F): Chance that next coefficient $< \max$ is $\dfrac{k}{k+1}$.

Theory errors

$$\mathrm{pr}(c_k|c_0,c_1\dots c_{k-1}) = \int \mathrm{d}\bar{c}\,\mathrm{pr}(c_k|\bar{c})\mathrm{pr}(\bar{c}|c_0,c_1\dots c_{k-1})$$
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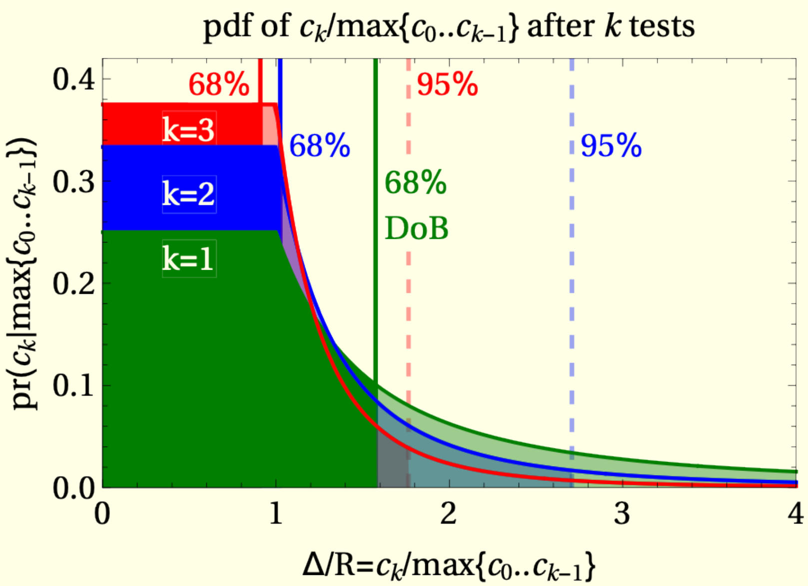

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NLO	$\frac{2}{3}=66.7\%$	$1.0\,R$	$2.7R=2.6\sigma$
N^2LO	$\frac{3}{4}=75\%$	$0.9\,R$	$1.8R=1.9\sigma$
N^{k-1}LO k terms	$\frac{k}{k+1}$	$0.68\frac{k+1}{k}R(k\geq 2)$	
Gauß	68.27%	$1.0\,R$	2.0σ

$\Longrightarrow$ Interpret all theory uncertainties with these priors: "$A\pm\sigma$": 68% *DoB interval* $[A-\sigma;A+\sigma]$.

Figure courtesy of H Grießhammer.

Processes rather than LECS

What about errors on, say, asymmetry or cross section for Compton Scattering? Different θ and ω are not independent. $y(x) = c_0(x) + c_1(x)\delta + c_2(x)\delta^2 + ...$
Here the coefficients are not the LECs (polarisabilities) but functions of them.

The c_i are not considered random variables, but random processes, model by Gaussian processes, defined by mean and correlation $c_k = c_0 \mathcal{GP}[m(x), \kappa(x, x'; \bar{c}, l)]$

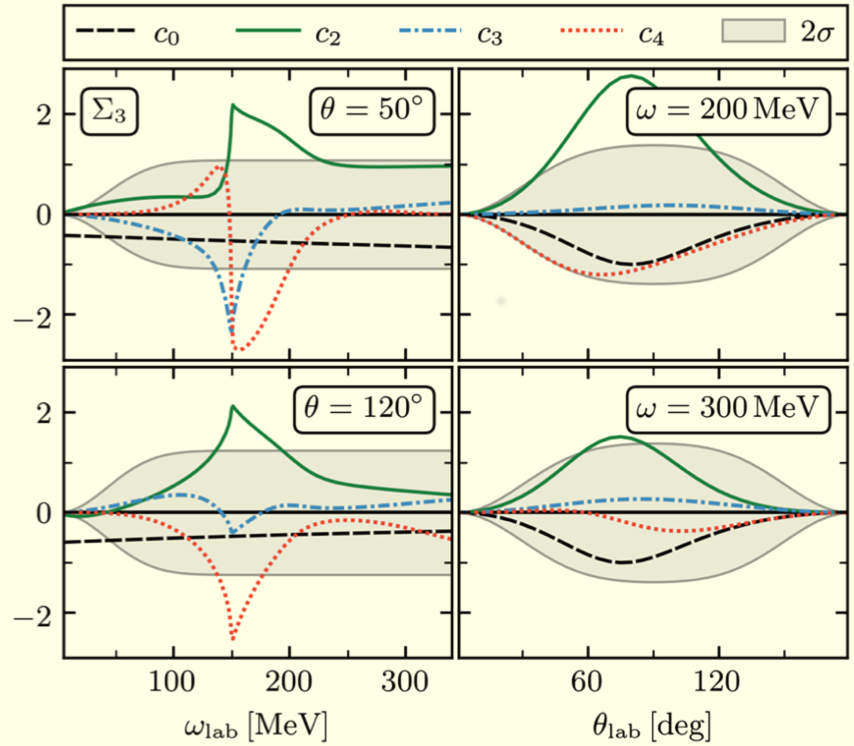
For Compton scattering, set $m(x) = 0$ and set scale using LO prediction, so $c_0 = 1$ for correlation function use

$$\kappa(\omega, \omega', \theta, \theta'; \bar{c}, l_\omega, l_\theta) = \bar{c}^2 \exp\left(-\frac{(\omega - \omega')^2}{l_\omega^2} - \frac{(\theta - \theta')^2}{l_\theta^2} \right)$$

“Train” GP parameters on known $c_0(x) \ldots c_{k-1}(x)$ to inform estimate of $c_k(x)$ which gives the uncertainty estimate.

GP parameters for Compton Scattering

	Proton			Neutron		
	$\bar{c}$	ℓ_ω	ℓ_θ	$\bar{c}$	ℓ_ω	ℓ_θ
$d\sigma$	0.9	56	64	3.2	40	78
Σ_{1x}	0.76	35	46	0.68	58	49
Σ_{1z}	0.48	35	56	0.4	56	42
Σ_{2x}	0.59	42	37	0.7	50	39
Σ_{2z}	1.5	50	45	2.1	46	52
Σ_3	0.7	49	35	0.5	70	44
Σ_y	0.63	41	52	0.57	61	44
Σ_{3y}	0.83	36	45	0.66	60	45



The $\bar{c}$'s are natural and the correlations lengths sensible.

With the ability to model theory uncertainties we can move on to the experimental design Melendez *et al.* , Eur. Phys. J. A (2021) 57:81

Experimental design

Which processes and kinematics will give the greatest improvement on the current knowledge of polarisabilities?

Depends on

- Experimental capability given event rate (eg measure cross section to 4%, asymmetry to ± 0.06)

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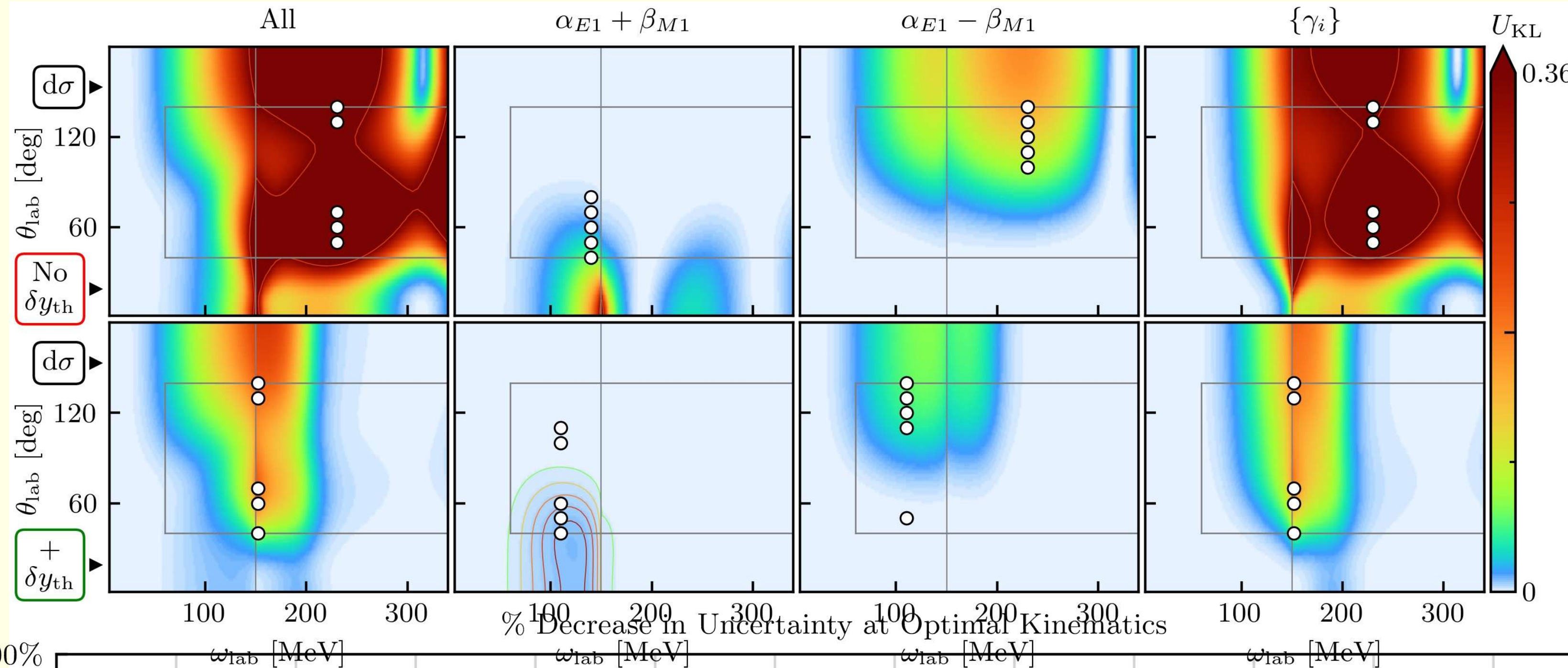
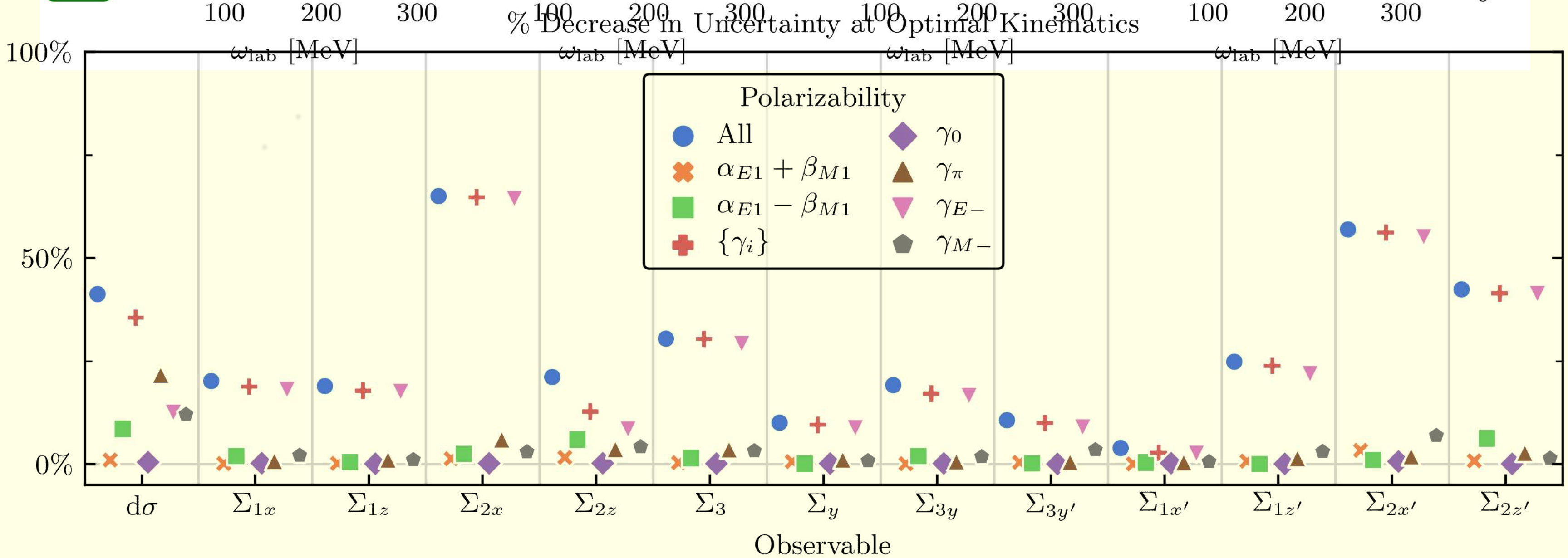
For a particular set of polarisabilities $\{a_1, a_2 \ldots\}$ and a particular design $\mathbf{d}$ measurement (fixed ω, a set of angles, luminosity, run-time,) what improvement in errors can we achieve?

"Utility" as reduction in volume of 1-sigma error hyperellipsoid, depends on unknown outcome data $\mathbf{y}$, modelled by theory and marginalised over:

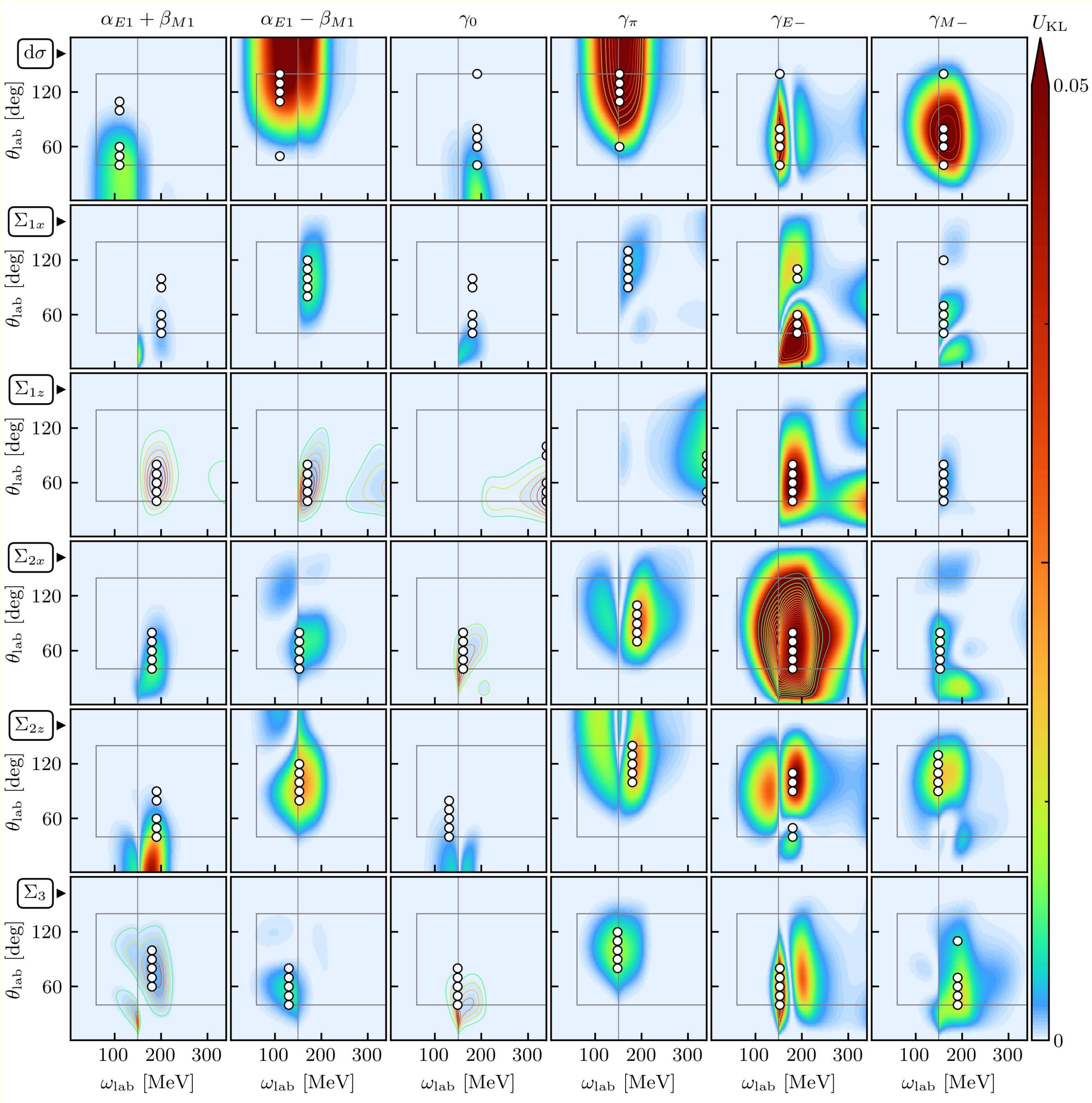
$$U_{\mathrm{KL}}(\mathbf{d}) = \int \left\{ \ln\left[\frac{\mathrm{pr}(\vec{a}\,|\,\mathbf{y},\mathbf{d})}{\mathrm{pr}(\vec{a})}\right] \mathrm{pr}(\vec{a}\,|\,\mathbf{y},\mathbf{d})\mathrm{d}\vec{a} \right\} \mathrm{pr}(\mathbf{y}\,|\,\mathbf{d})\mathrm{d}\mathbf{y}$$
$$= \frac{1}{2}\ln\frac{|V_0|}{|V(\mathbf{d})|} \equiv \ln \mathcal{S}(\mathbf{d}) \geq 0\,,$$

Truncation uncertainty; summary





Set 1



$\alpha_{E1}+\beta_{M1}$
$\alpha_{E1}-\beta_{M1}$
γ_0
γ_π
γ_{E-}
γ_{M-}
$U_{\rm KL}$
0.05
0
dσ
Σ_{1x}
Σ_{1z}
Σ_{2x}
Σ_{2z}
Σ_3
$\theta_{\rm lab}$ [deg]
120
60
100 200 300
$\omega_{\rm lab}$ [MeV]

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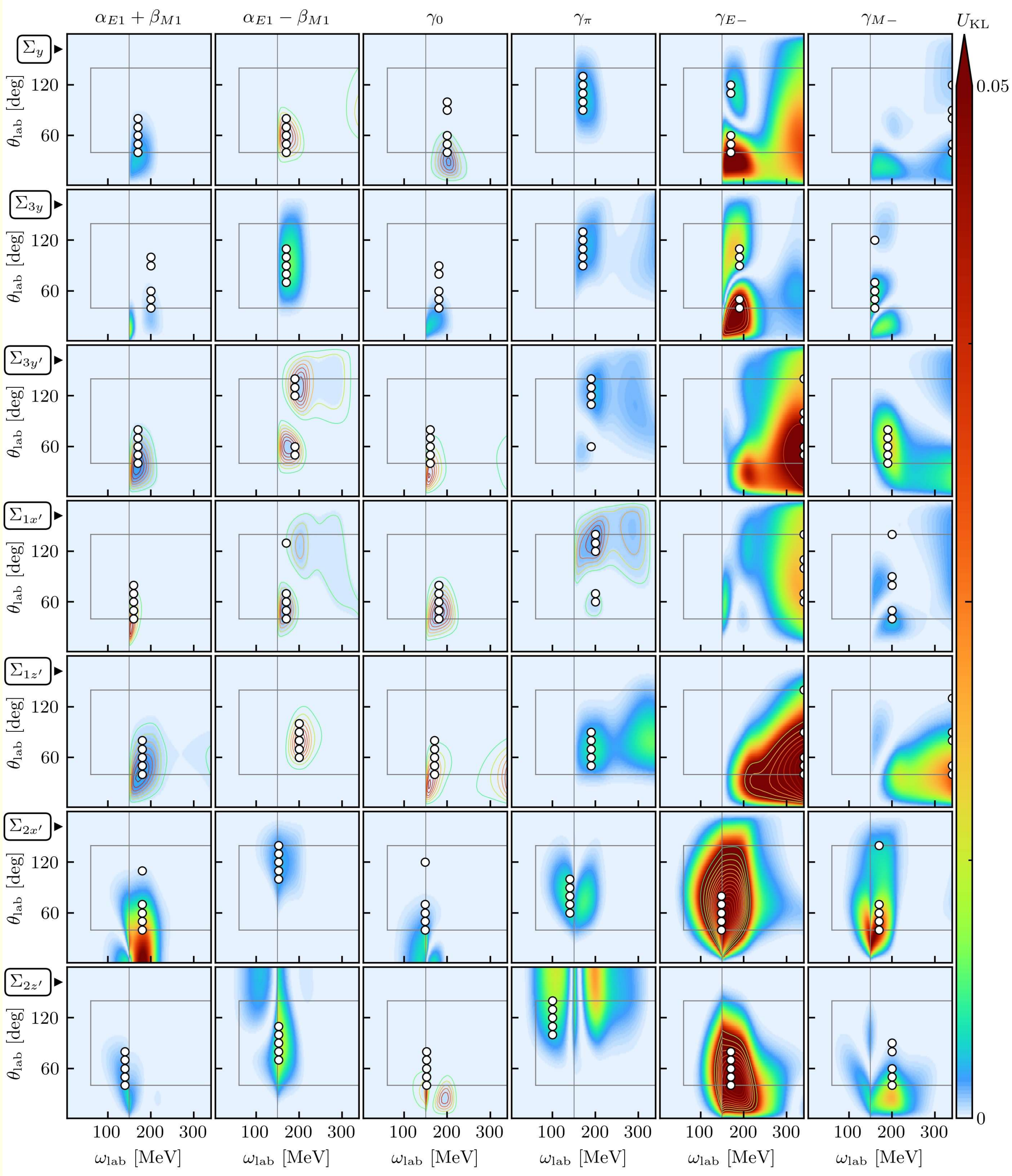
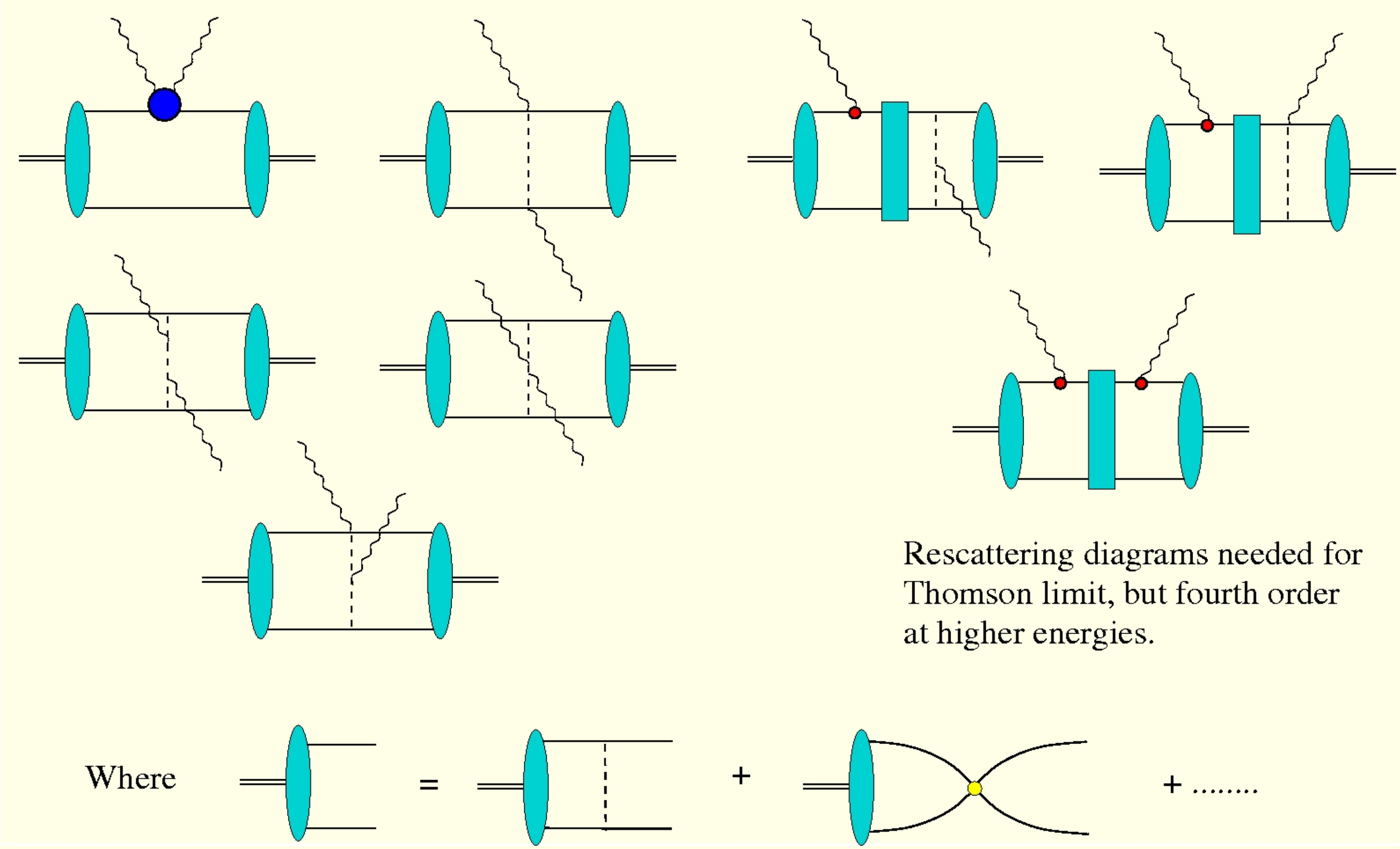
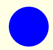
Set 2



Deuteron

Consistent treatment of one- and two-body diagrams



The Δ only enters in  at this order.

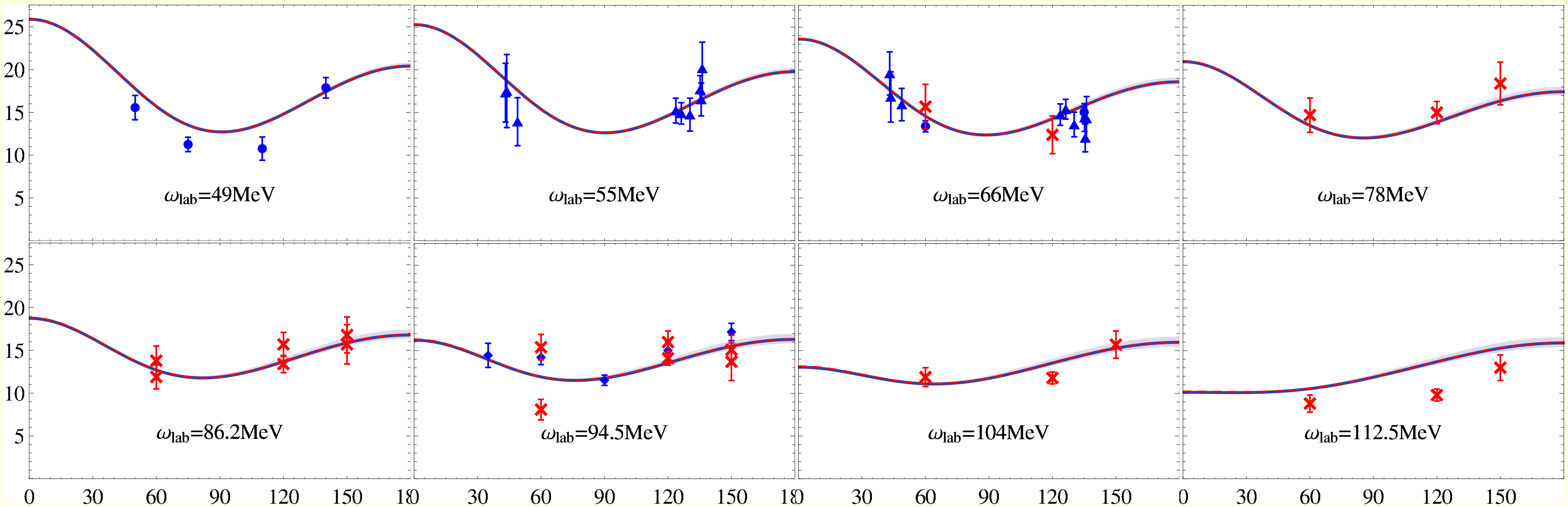
Ensuring correct Thomson limit for deuteron is important even at 50-60 MeV.

Extraction of isoscalar polarisabilities

So far only $O(Q^3)$; further work required to go above pion threshold.

Older data from Illinois ●, Saskatoon, ◆ and Lund ▲ (29 pts in total)

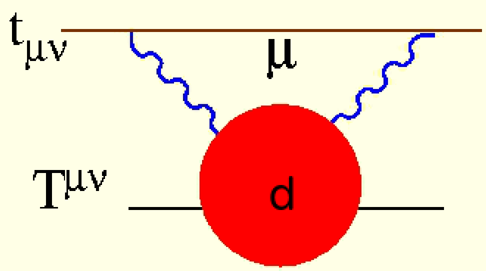
More comprehensive data from Lund ×, 23 points. Myers *et al.* , Phys. Rev. Lett. **113,** 262506 (2014)

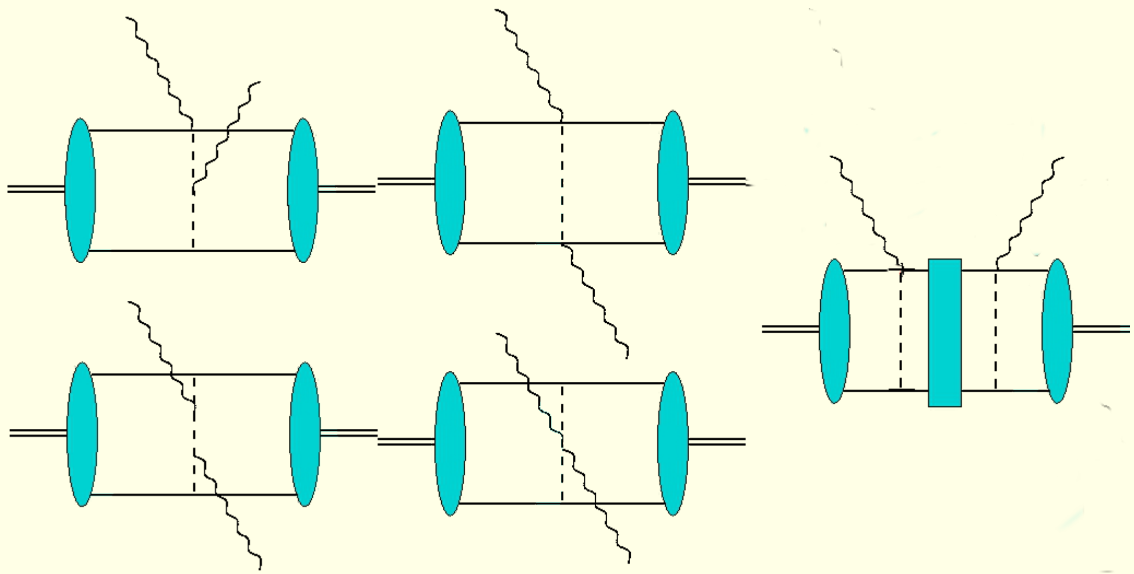


$$\alpha_s = 11.1 \pm 0.6(\text{stat}) \pm 0.2(\text{BSR}) \pm 0.8(\text{th})$$
$$\beta_s = 3.4 \mp 0.6(\text{stat}) \pm 0.2(\text{BSR}) \mp 0.8(\text{th}).$$

$$\alpha_n = 11.65 \pm 1.25(\text{stat}) \pm 0.2(\text{BSR}) \pm 0.8(\text{th})$$
$$\beta_n = 3.55 \mp 1.25(\text{stat}) \pm 0.2(\text{BSR}) \mp 0.8(\text{th})$$

Lamb Shift





	$L_{\rm max}=0$	$L_{\rm max}=1$	$L_{\rm max}=2$
δ_L^-	$-0.210(2)$	$-1.631(5)$	$-1.669(8)$
δ_L^+	$0.007\,92(1)$	$0.0204(2)$	$0.0225(8)$
$\delta^\mu_{\phi\phi2}$	-	$0.0026(2)$	$0.0026(2)$
$\delta^\gamma_{\phi\phi2}$	-	$-0.012(1)$	$-0.012(1)$
$\delta_{\phi\phi3}$	-	$-0.000\,71(3)$	$-0.000\,78(3)$
$\delta_{\phi\phi4\rm ke}+\delta_p$	-	$-0.001\,70(6)$	$-0.001\,30(5)$
$\delta_{\phi\phi4\rm pot}+\delta^\gamma_{2N}$	-	$-0.000\,85(5)$	$-0.000\,85(5)$
δ^μ_{2N}	-	$0.0030(1)$	$0.0030(1)$
δ_{1N}	-0.0216	-0.0216	-0.0216
δ_{pol}	-0.224	-1.64	-1.677

Lamb shift for deuterium in meV. The part of the two-body graphs not fixed by current conservation is new. Alex Moore, PhD thesis