Moments of the charge distribution

QUARTET meeting 7-8.11

Ben

The two languages:

1. H-He: From factors, Frier moments, two-photon-exchange , … ← Modern
2. Carbon onwards: Charge distribution, Barret moments, nuclear polarization, … ← Obsolete?
3. Li-B: *somewhere in between?*

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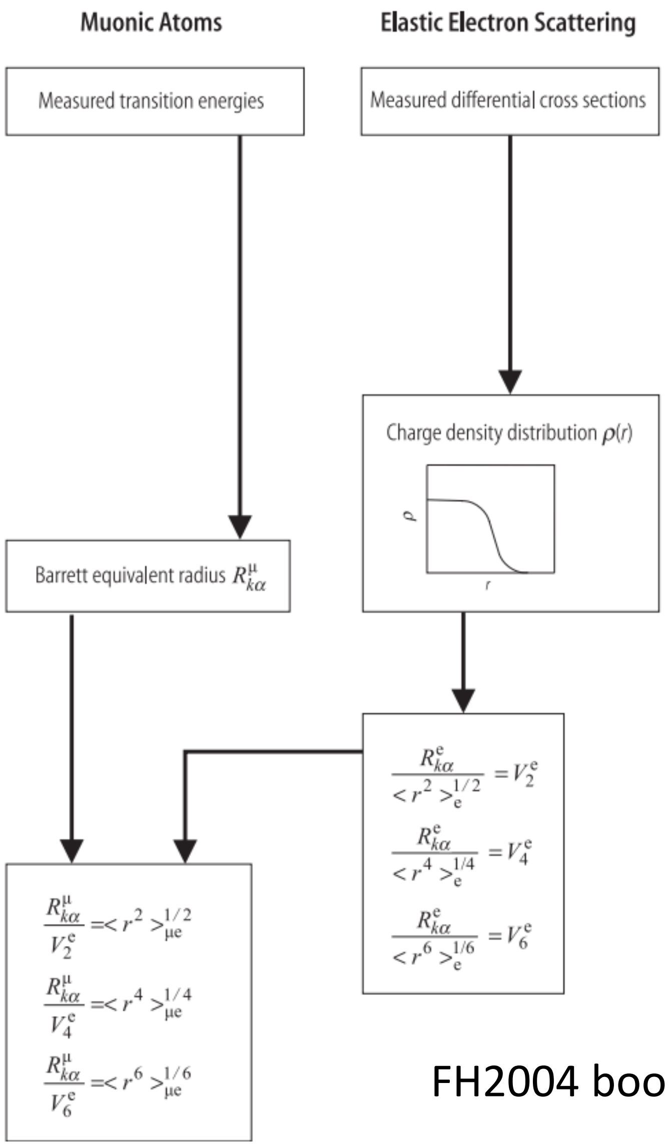
Today: understanding the Fricke & Heilig recipe

The Fricke & Heilig recipe (absolute radii):



FH2004 book

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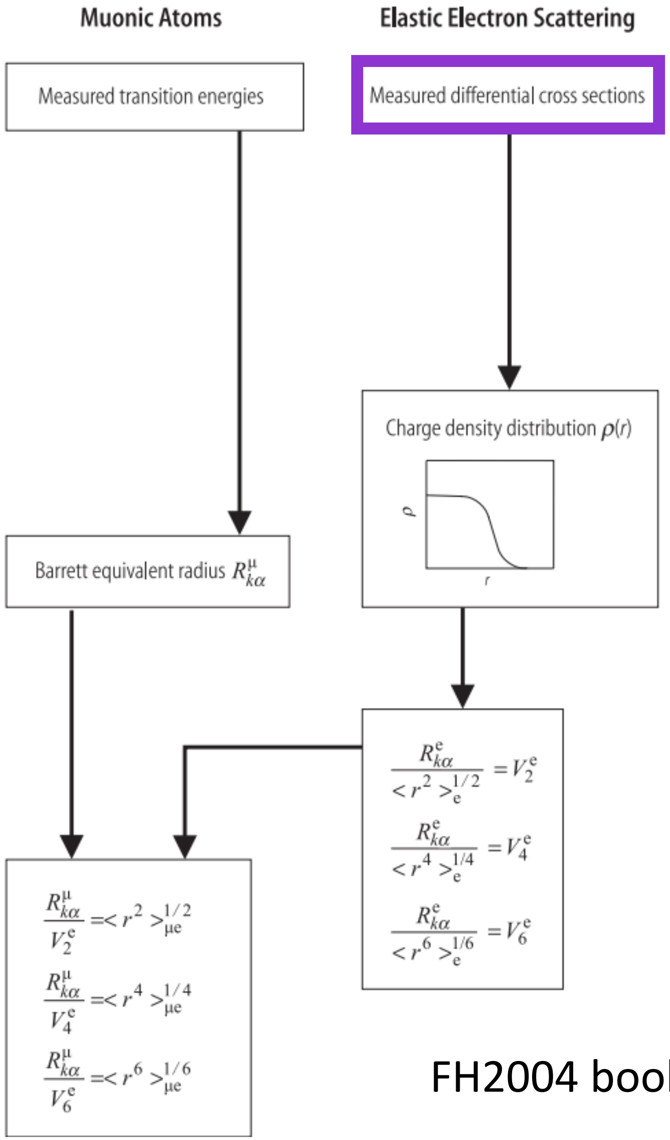

FH2004 book

Elastic electron scattering 101:

Measure elastic scattering cross sections as function of angle and energy

$$d\sigma / d\Omega = (1/(L_t \times t \times \Delta\Omega \times \Pi_i\, \varepsilon_i)) \times \text{Counts} \times R$$

Counts over background

Efficiencies

Radiative correction

Phase space

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What do these cross sections mean?

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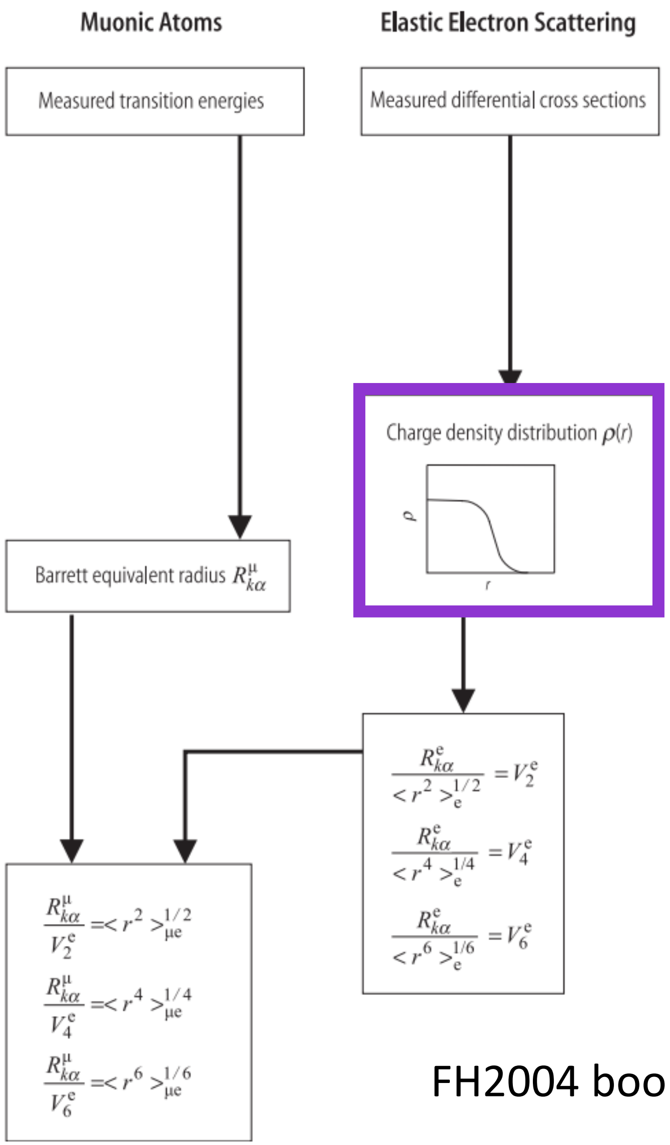

FH2004 book

Elastic electron scattering 101:

Point nuclei:

Rutherford
(nonrelativistic, no spins):

$$\sigma(\theta) = \frac{z^2 Z^2 e^4}{16E^2} \frac{1}{\sin^4 \frac{1}{2}\theta},$$

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For finite nuclei:

“for light nuclei which do not recoil”:

$$\sigma_s(\theta) = \left(\frac{Ze^2}{2E}\right)^2 \frac{\cos^2 \frac{1}{2}\theta}{\sin^4 \frac{1}{2}\theta} \left[\int_0^\infty \rho(r) \frac{\sin qr}{qr} 4\pi r^2 dr \right]^2 . \quad q = \frac{2E}{\hbar c} \sin \tfrac{1}{2}\theta = \frac{2}{\lambda} \sin \tfrac{1}{2}\theta$$

From factor: $F = \dfrac{4\pi}{q} \displaystyle\int_0^\infty \rho(r) \sin(qr) r dr$

Elastic electron scattering 101:

Elastic electron scattering measures form factor function: $F = \frac{4\pi}{q} \int_0^\infty \rho(r) \sin(qr) r dr$

In principal, can invert the transform to get charge dist.: $\rho(r) = \frac{1}{2\pi^2 r} \int_0^\infty F(q) \sin(qr) q dq.$

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Cutoff not infinite…

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Moments of the charge distribution: $r_k \propto \left(\int \rho(r') r'^{k+2} dr'\right)^{-k}$

e.g. for k=2, RMS charge radius: $a^2 \equiv r_2^2 \propto \int \rho(r') r'^4 dr'$

Where does the special role of the RMS radius comes from?

Relationship between form factor and radius (a) Hofstadter 56

TABLE I. In this table $\rho(r)$ is the charge density function; "a" is the root-mean-square radius of the charge distribution; $F(qa)$ is the form factor; $x=qa$.

Model number	Name of model	Expression for charge density $4\pi a^3\rho(r)$; $y=r/a$	$F(qa)$; $x=qa$
I	Point	δ function	1
II	Uniform	$\begin{cases} \frac{9}{5}\left(\frac{3}{5}\right)^{\frac{1}{2}} \text{ for } y\leqslant\left(\frac{5}{3}\right)^{\frac{1}{2}} \\ 0 \text{ for } y\geqslant\left(\frac{5}{3}\right)^{\frac{1}{2}} \end{cases}$	$5\left(\frac{5}{3}\right)^{\frac{1}{2}}x^{-3}\left[\sin\left(\frac{5}{3}\right)^{\frac{1}{2}}x-\left(\frac{5}{3}\right)^{\frac{1}{2}}x\cos\left(\frac{5}{3}\right)^{\frac{1}{2}}x\right]$
III	Gaussian	$3\left(\frac{6}{\pi}\right)^{\frac{1}{2}}\exp\left(-\frac{3}{2}y^2\right)$	$\exp(-x^2/6)$
IV	Exponential	$12\sqrt{3}\exp(-(12)^{\frac{1}{2}}y)$	$\left(1+\frac{x^2}{12}\right)^{-2}$
V	Shell	$\delta(y-1)$	$x^{-1}\sin x$
VI	Hollow exponential	$\frac{200}{3}y\exp(-(20)^{\frac{1}{2}}y)$	$\left(1-\frac{x^2}{60}\right)\left(1+\frac{x^2}{20}\right)^{-3}$
VII	···	$\frac{75}{2}(30)^{\frac{1}{2}}y^2\exp(-(30)^{\frac{1}{2}}y)$	$\left(1-\frac{x^2}{30}\right)\left(1+\frac{x^2}{30}\right)^{-4}$
VIII	Yukawa I	$\sqrt{2}y^{-2}\exp(-\sqrt{2}y)$	$\sqrt{2}x^{-1}\tan^{-1}(x/\sqrt{2})$
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XI	Generalized shell model	$\begin{cases} \frac{8}{\sqrt{\pi}}\frac{k^3}{(2+3\alpha)}(1+\alpha k^2y^2)\exp(-k^2y^2) \\ \text{where } k=\left[\frac{3(2+5\alpha)}{2(2+3\alpha)}\right]^{\frac{1}{2}} \end{cases}$	$\left[1-\frac{\alpha x^2}{2k^2(2+3\alpha)}\right]\exp\left(-\frac{x^2}{4k^2}\right)$
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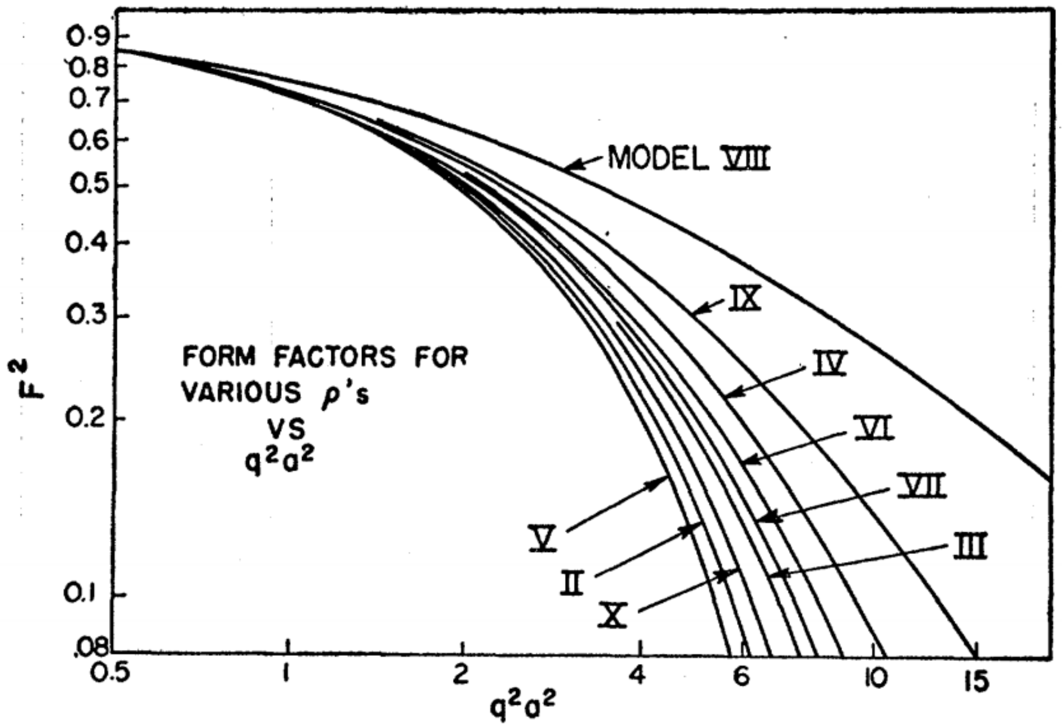

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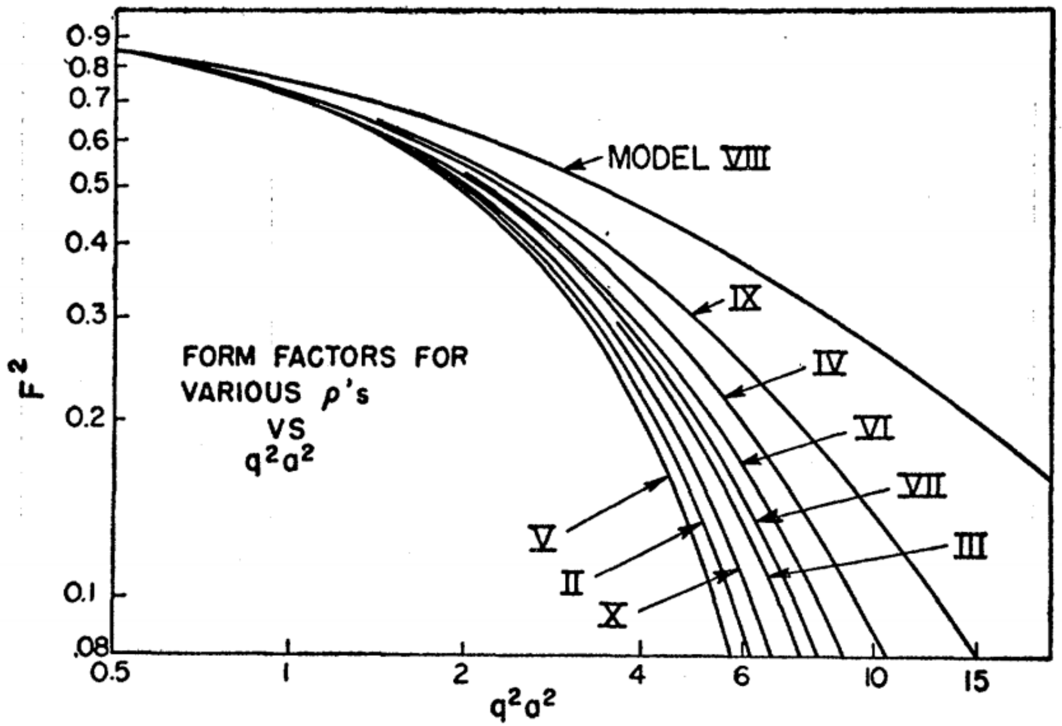


If qa is small, where a is the root-mean-square radius, all form factors reduce to the simple expansion

$$F=1-(q^2a^2/6)+\cdots. \qquad (19)$$

Modern (1970's) analysis of scattering exp.:

Two "model independent" methods, Sum Of Gaussians (SOG) and Fourier Bessel (FB):

Expand distribution in series: $$\rho(r) = \sum a_\nu j_0(\nu\pi r/R) \quad \text{for} \quad r \leq R,$$

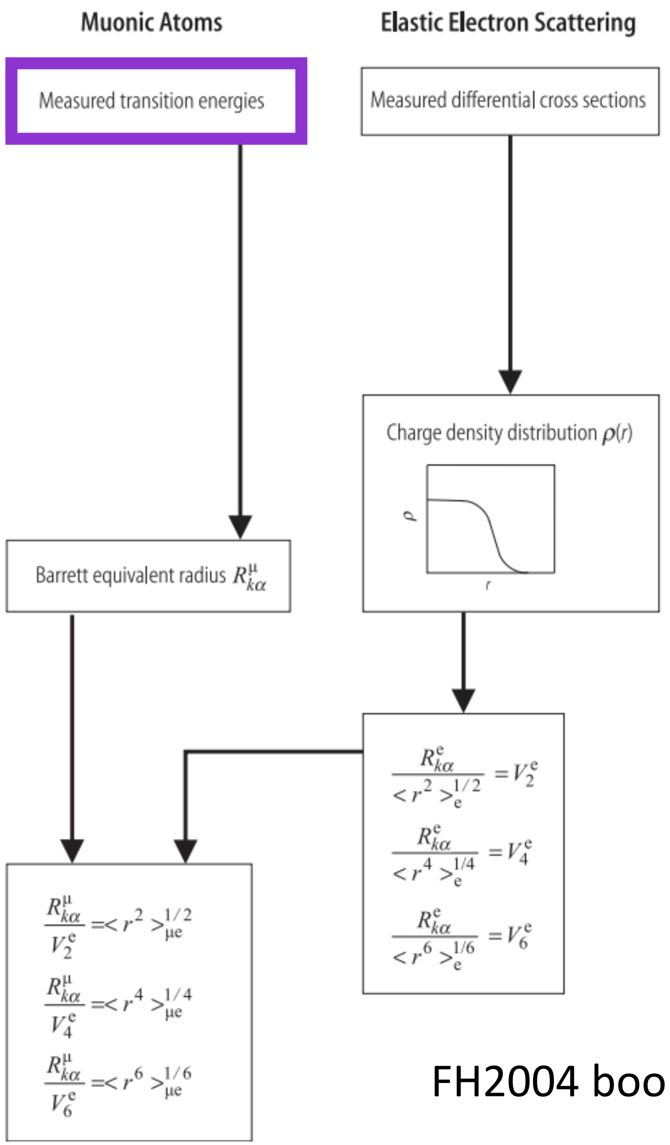
By choosing a cutoff radius beyond which charge distribution would be zero, we get

$$F(q) = \frac{1}{Z}\int_0^R \sum_{n=1}^{\infty} a_n \frac{\sin(\frac{n\pi r}{R})}{(\frac{n\pi r}{R})} \frac{\sin(q.r)}{qr} 4\pi r^2 dr$$

$$= \frac{8}{3q}[a_1 \frac{8\pi\sin(8q)}{\pi^2 - 64q^2} - a_2 \frac{8\pi\sin(8q)}{4\pi^2 - 64q^2} + a_3 \frac{8\pi\sin(8q)}{9\pi^2 - 64q^2} - a_4 \frac{8\pi\sin(8q)}{16\pi^2 - 64q^2} + ...]$$

Fit scattering data to get FB coefficients, then calculate distribution.

The Fricke & Heilig recipe (absolute radii):



FH2004 book

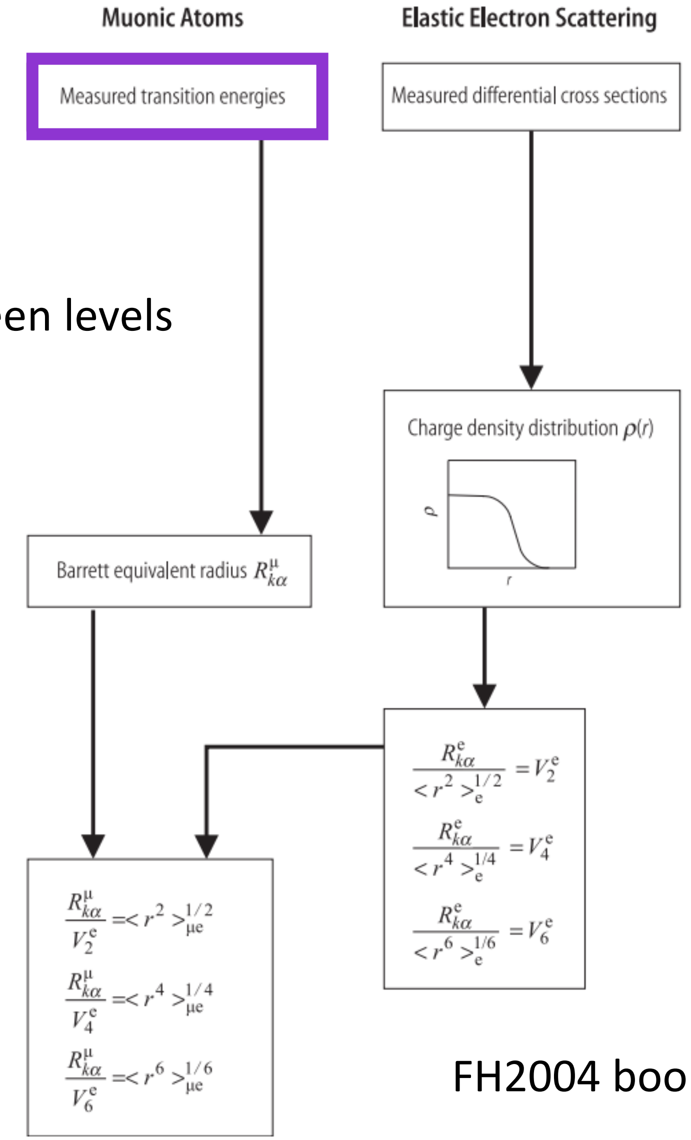
The Fricke & Heilig recipe (absolute radii):

Finite size effects on muonic atoms:

$$\delta E_{\mathrm{if}} = 4\pi \int \delta\rho(r)\left[V_\mu{}^{\mathrm{i}}(r) - V_\mu{}^{\mathrm{f}}(r)\right] r^2\,\mathrm{d}r$$

Difference in potential of the muon between levels

Interesting part of transition energy



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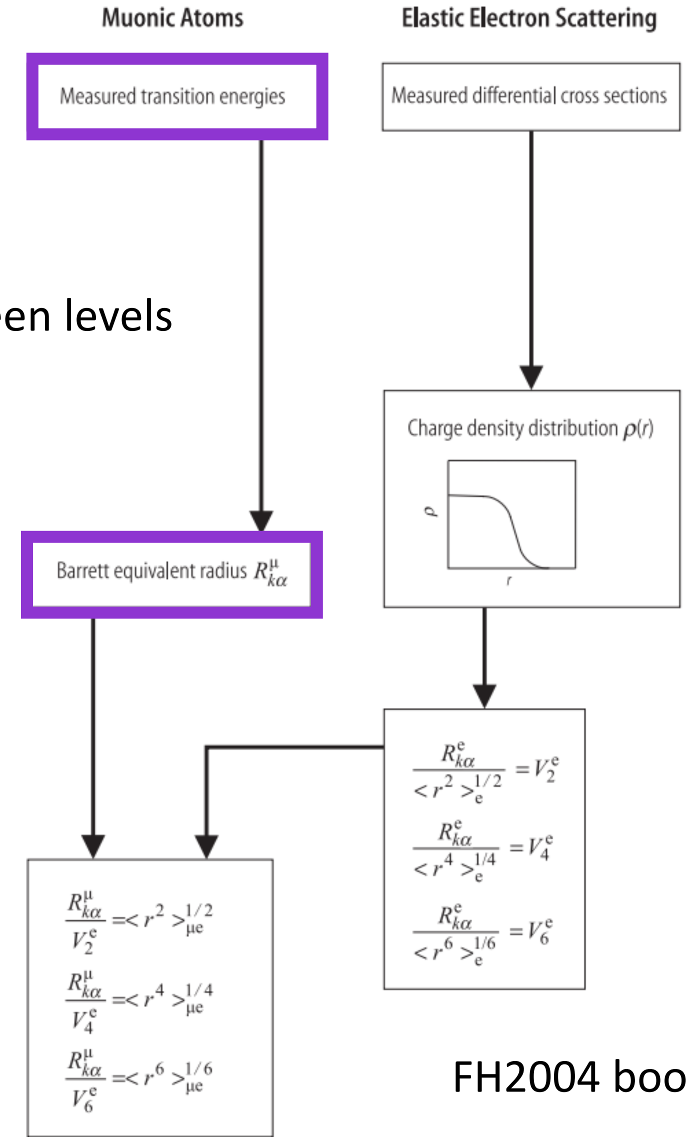
Difference in potential of the muon between levels

Perturbation theory: $V(r) \sim Br^{\mathrm{k}}\mathrm{e}^{-\alpha r}$

So: $\delta E_{\mathrm{if}} \ \propto \ \left\langle r^k e^{-\alpha r}\right\rangle = \frac{4\pi}{Ze}\int \rho_{\mathrm{N}}(r)\, r^k e^{-\alpha r} r^2 \,\mathrm{d}r$

Notice, (k,alpha) depend on transition



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Where to find (K,Alpha)?

Barret 1970:

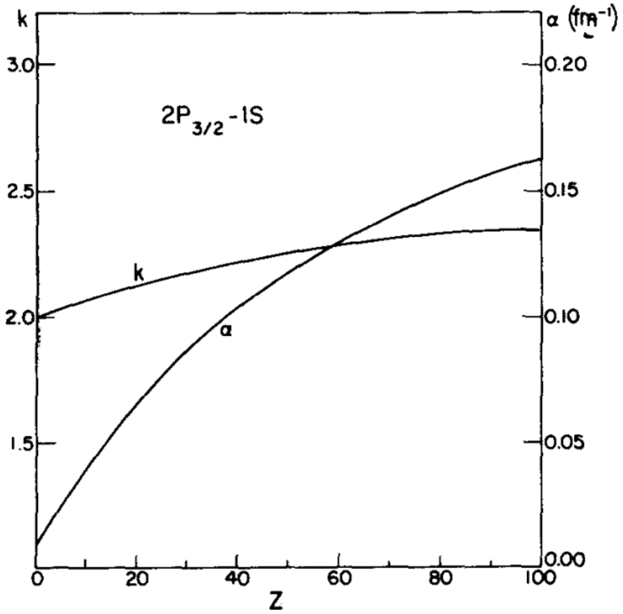


Fig. 3. Parameters α and k in the function $r^k \exp(-\alpha r)$ for the $2P_{3/2}$ - 1S transition against atomic number Z.

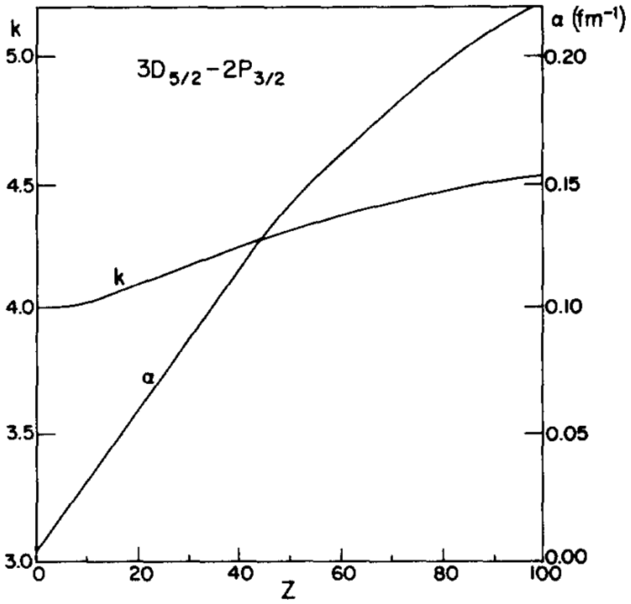
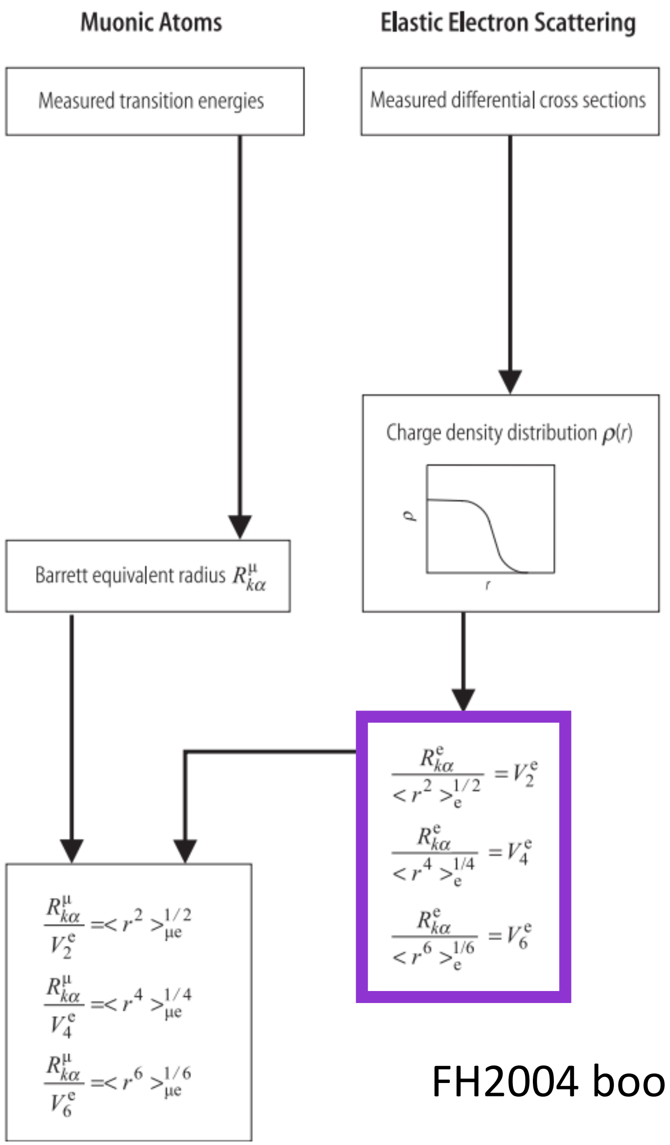


Fig. 4. Parameters α and k for the $3D_{5/2}$ - $2P_{3/2}$ transition plotted against Z.

FH2004:

A	E_{exp} [keV]	E_{theo} [keV]	$Npol$ [keV]	c [fm]	$<r^2>^{1/2}_{model}$ [fm]	real	α [1/fm]	k	C_z [10^{-3}fm/eV]	$R^{\mu}_{k\alpha}$ [fm]	Ref.
39	713.118(32)	713.118	0.119	3.6542(19)	3.435	3.435	0.0572	2.0866	–0.050	4.4077(16;18)	WSH81
41	712.769(28)	712.769	0.132	3.6815(17)	3.452		0.0571	2.0869	–0.050	4.4303(14;20)	WSH81

The Fricke & Heilig recipe (absolute radii):

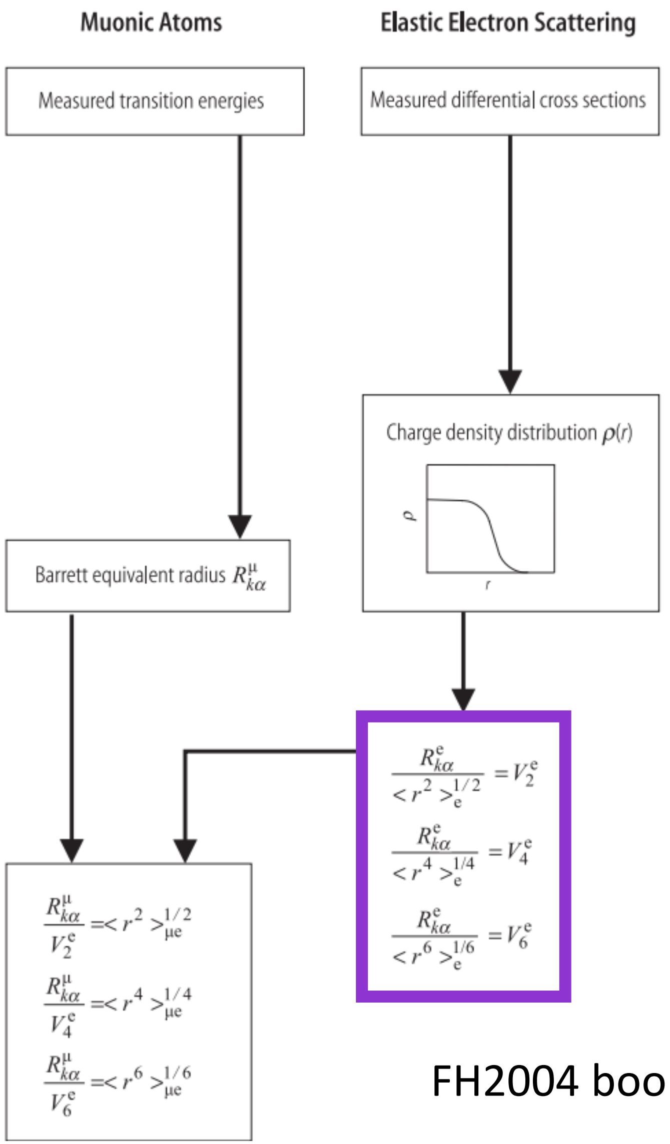


FH2004 book

The Fricke & Heilig recipe (absolute radii):

$$r_{\mu e} = \frac{R_\mu}{\nu_e} = R_\mu \frac{r_e}{R_e}$$

- Take Charge distribution from elec. Scat
- Calculate $\nu_e = \frac{r_e}{R_e} \rightarrow \sqrt{\frac{5}{3}}\,(\text{sphere})$
- Can be obtained with much smaller uncertainty than e.g. r_e



FH2004 book

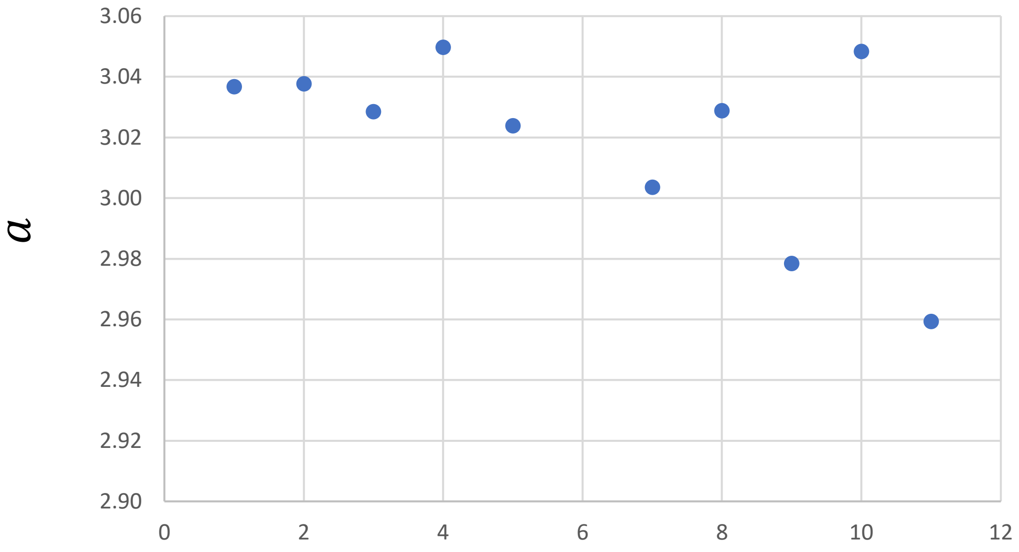
Example, v-factor vs. a in ^{20}Ne

VJV87:

Take different neon distributions:

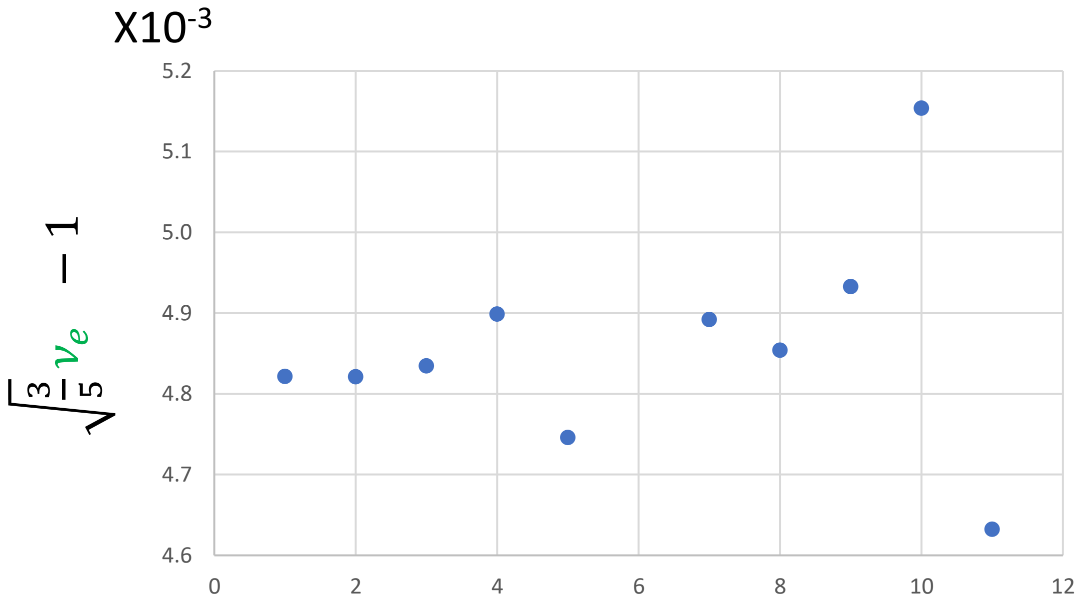
				c or a [fm]	z or α [fm]	w	q-range [fm^{-1}]	ref.
^{20}Ne*	2pF	3.040(25)	3.037(14)	2.805(15)	0.571(5)		0.22 - 1.04	Mo71
	2pF	3.004(25)	3.0036	2.740(46)	0.572(17)		0.21 - 1.12	Kn81
maybe adjusted?	3pF	2.992(8)	2.9486	2.791(9)	0.698(5)	-0.168(8)	0.49 1.80	Be85

Calculate radius for each (including uncertainties)



Deviations of the radius about 3%

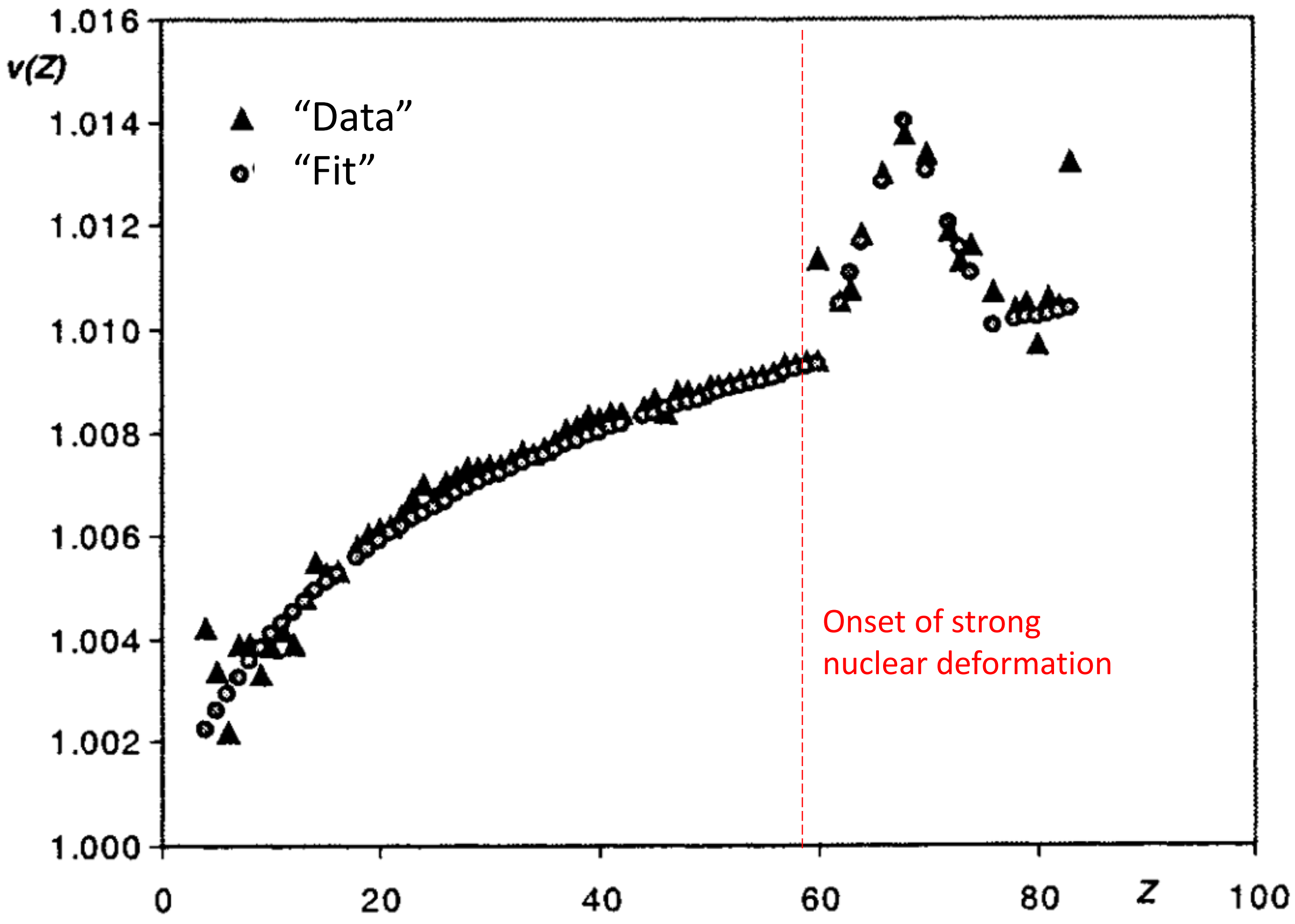
Calculate v-factor, and look at deviations:



Deviates around 5e-4

Abuse of the Fricke's recipe:

Angeli 2002



Basis for all later reviews…

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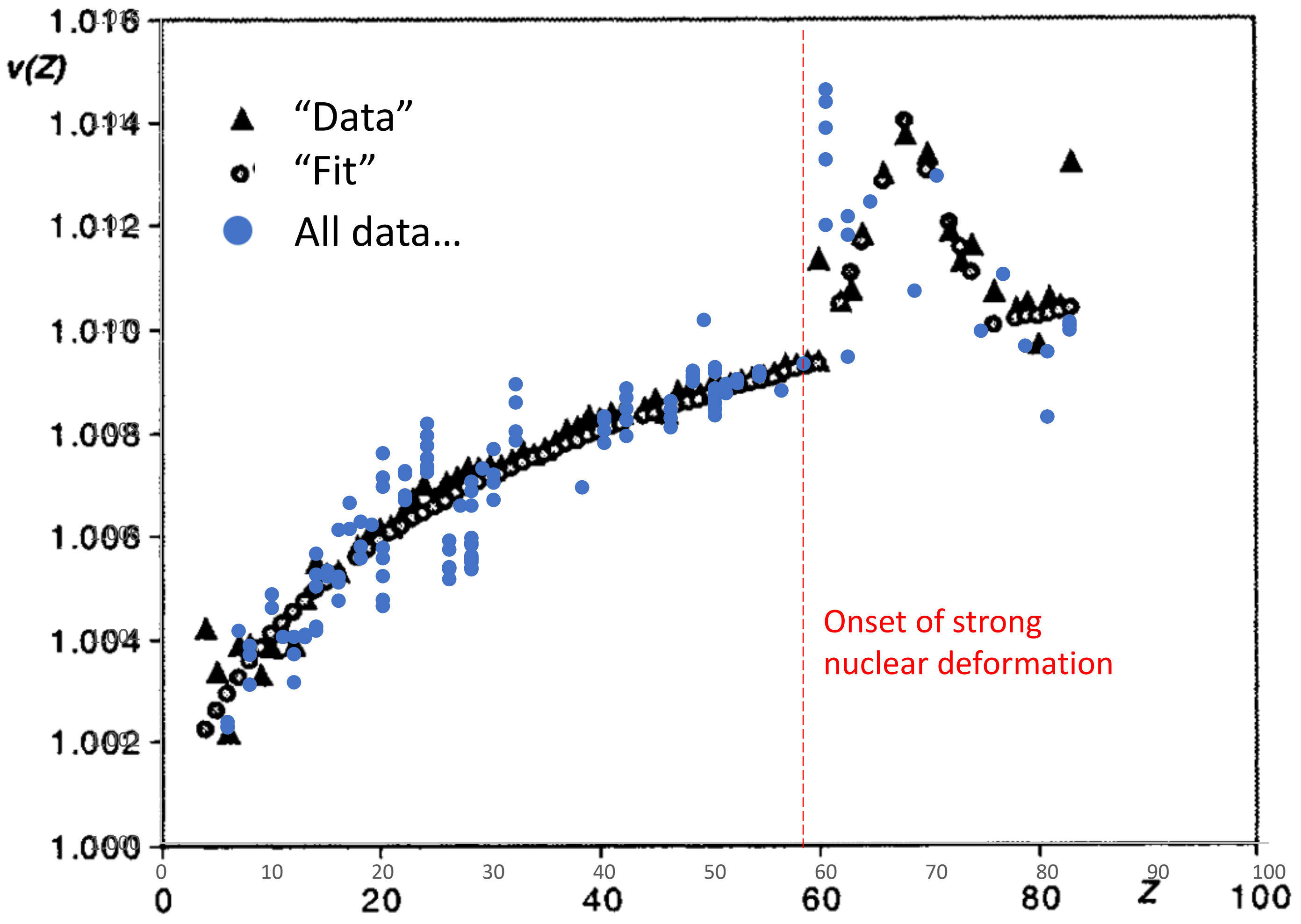

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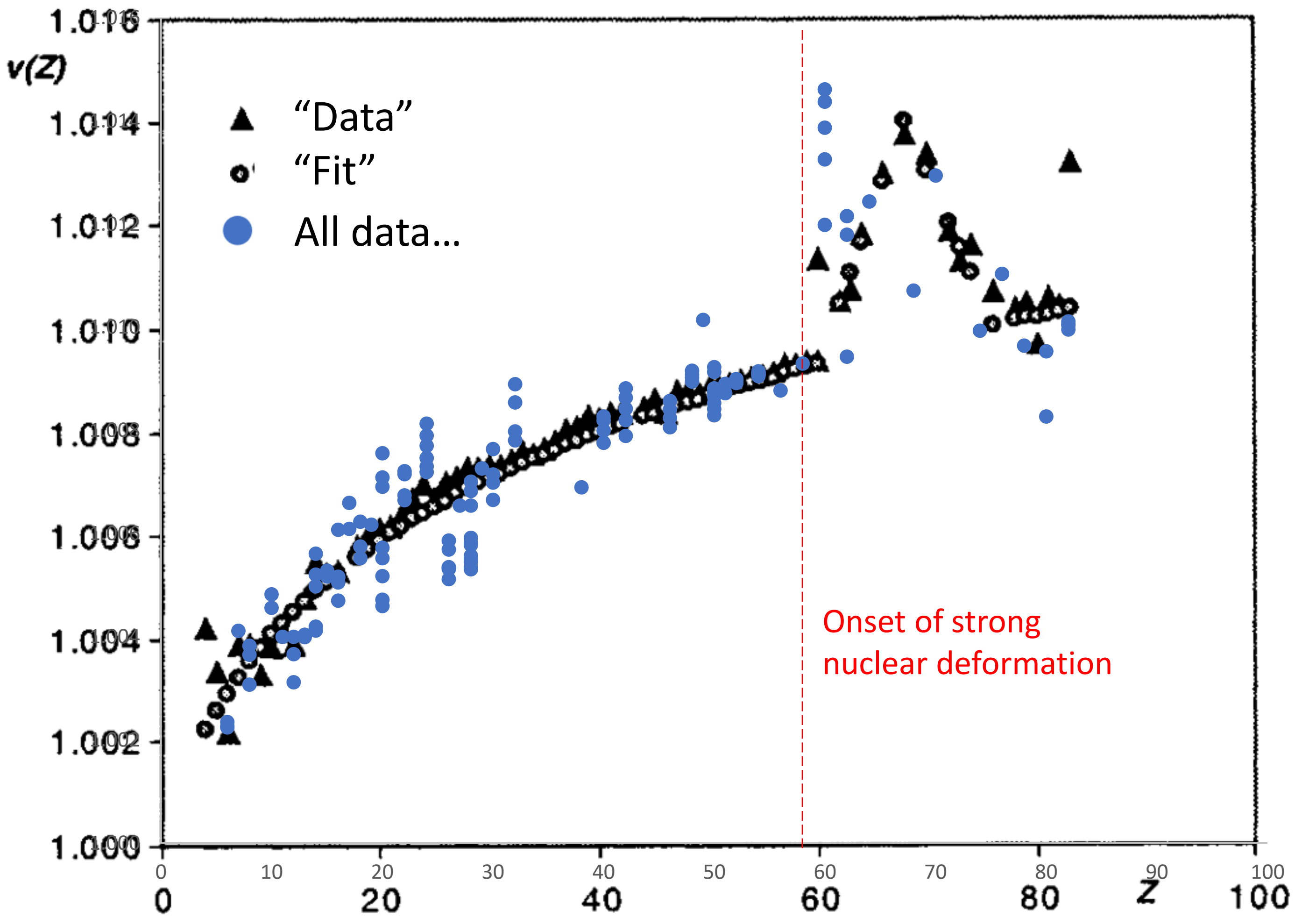


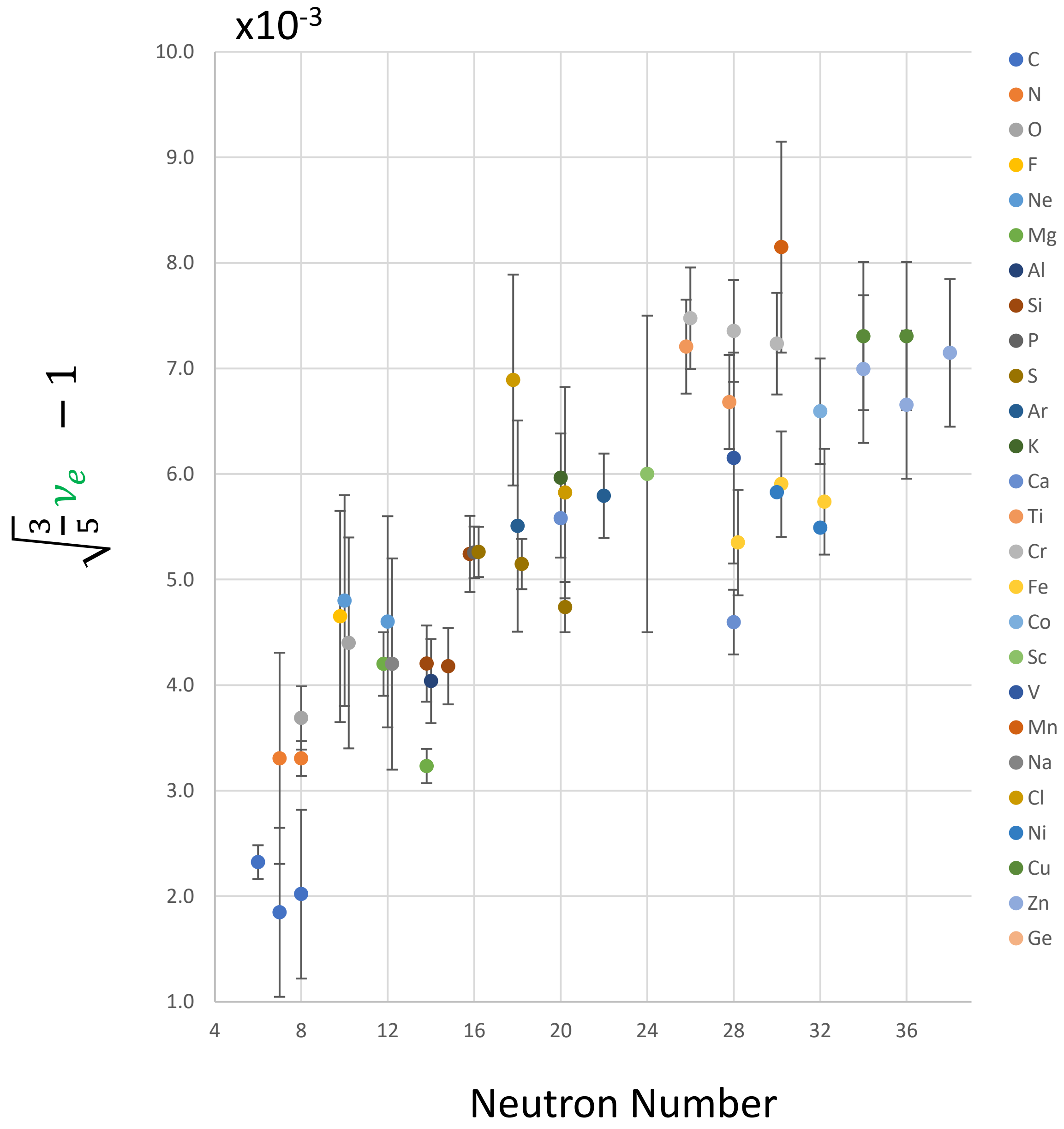
Isotope dependence of v-factor as high as **4e-3**

Reported absolute radii Uncertainties as low as **2e-4**

Situation if even worse for radii differences…

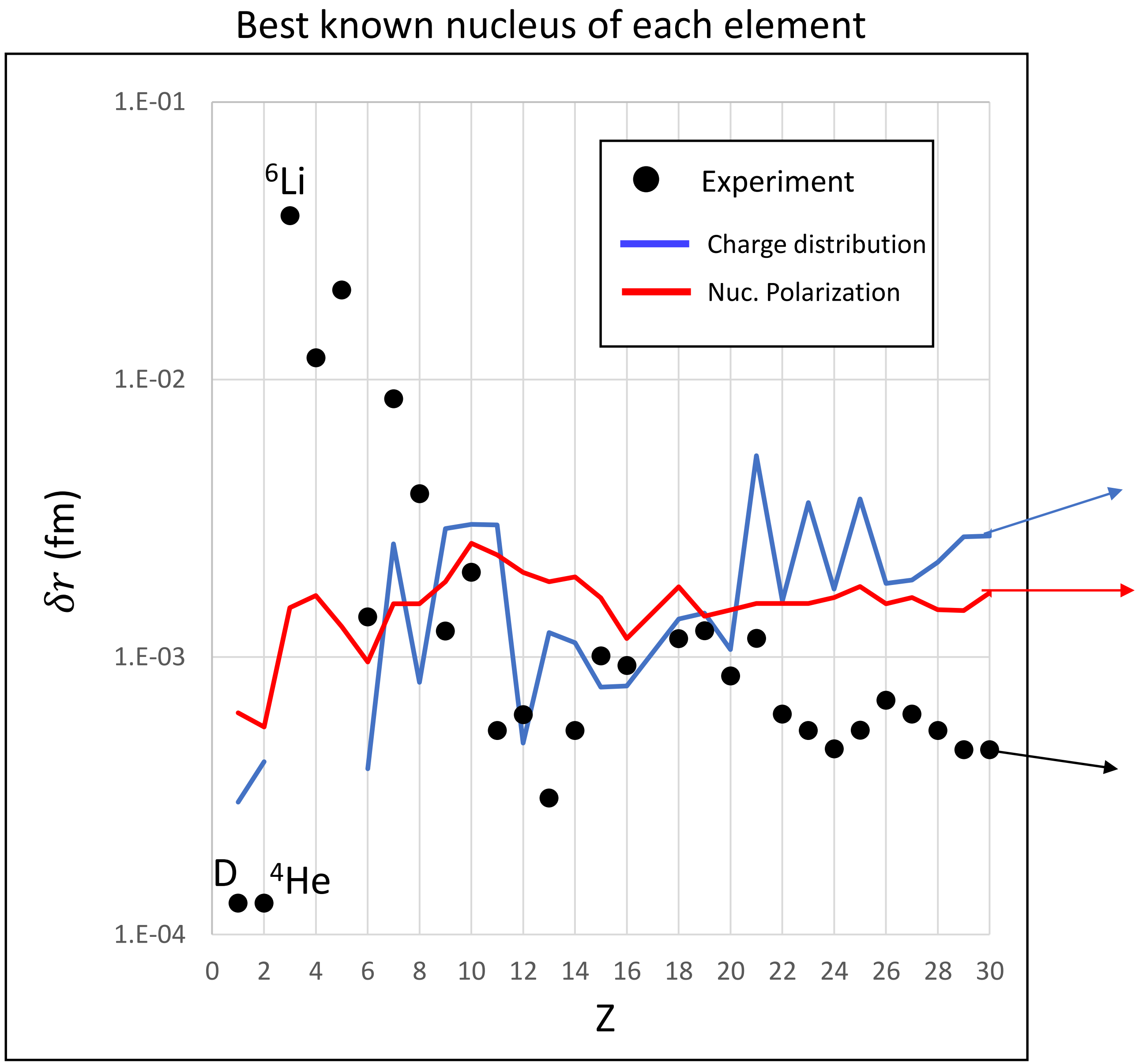
Basis for all later reviews…

Ongoing work:



1. Make the most out of the FH recipe.
2. Compare v factors for different distributions
3. Differentiate between reliable and less reliable estimations
4. More informed inter-/extrapolations

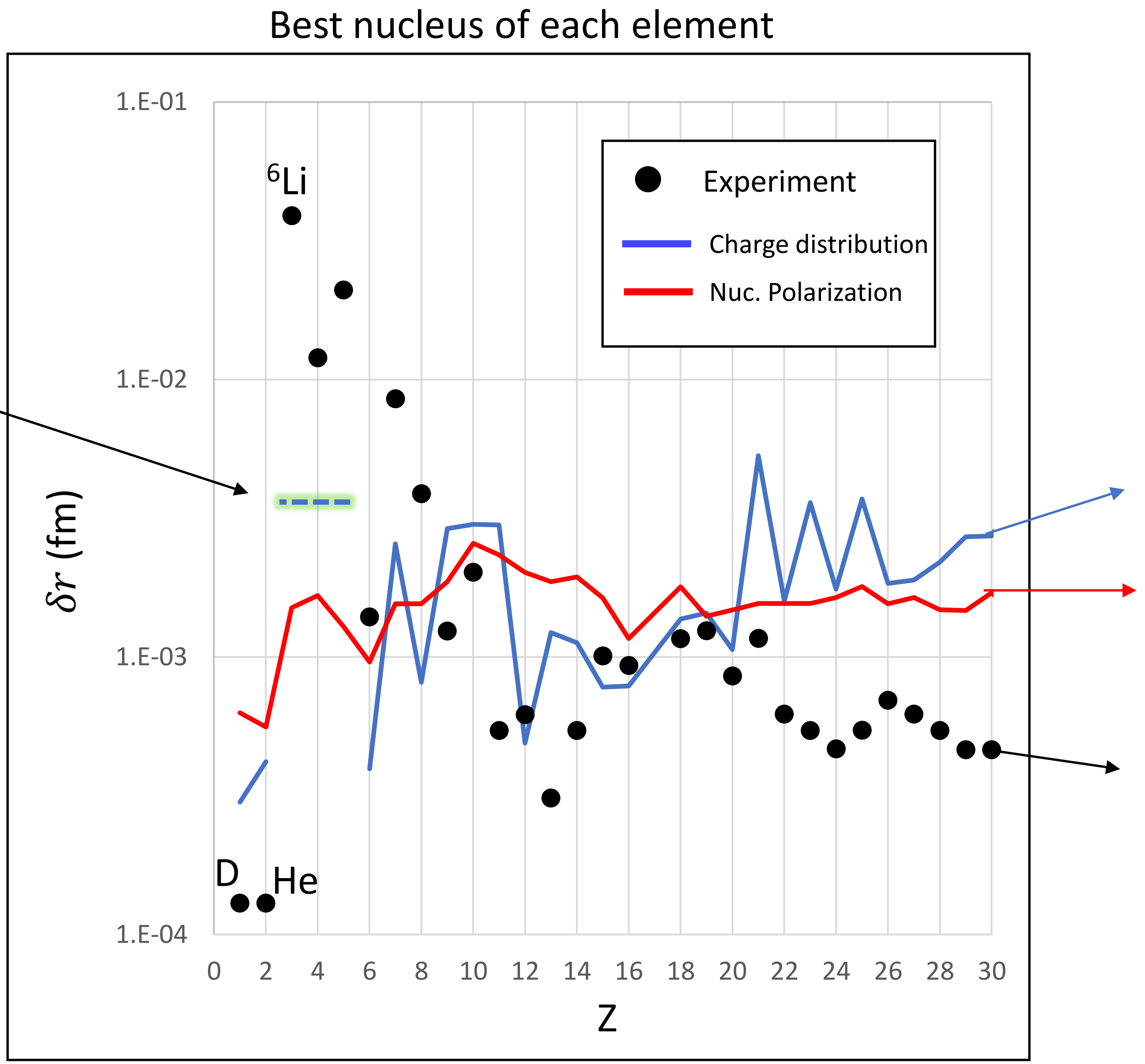
Consequence of v-factor analysis – Blue Line



Consequence of v-factor analysis – Blue Line

How about Li, Be and B?

“Worst case scenario”:
With ancient scattering,
or extrapolated v-factors



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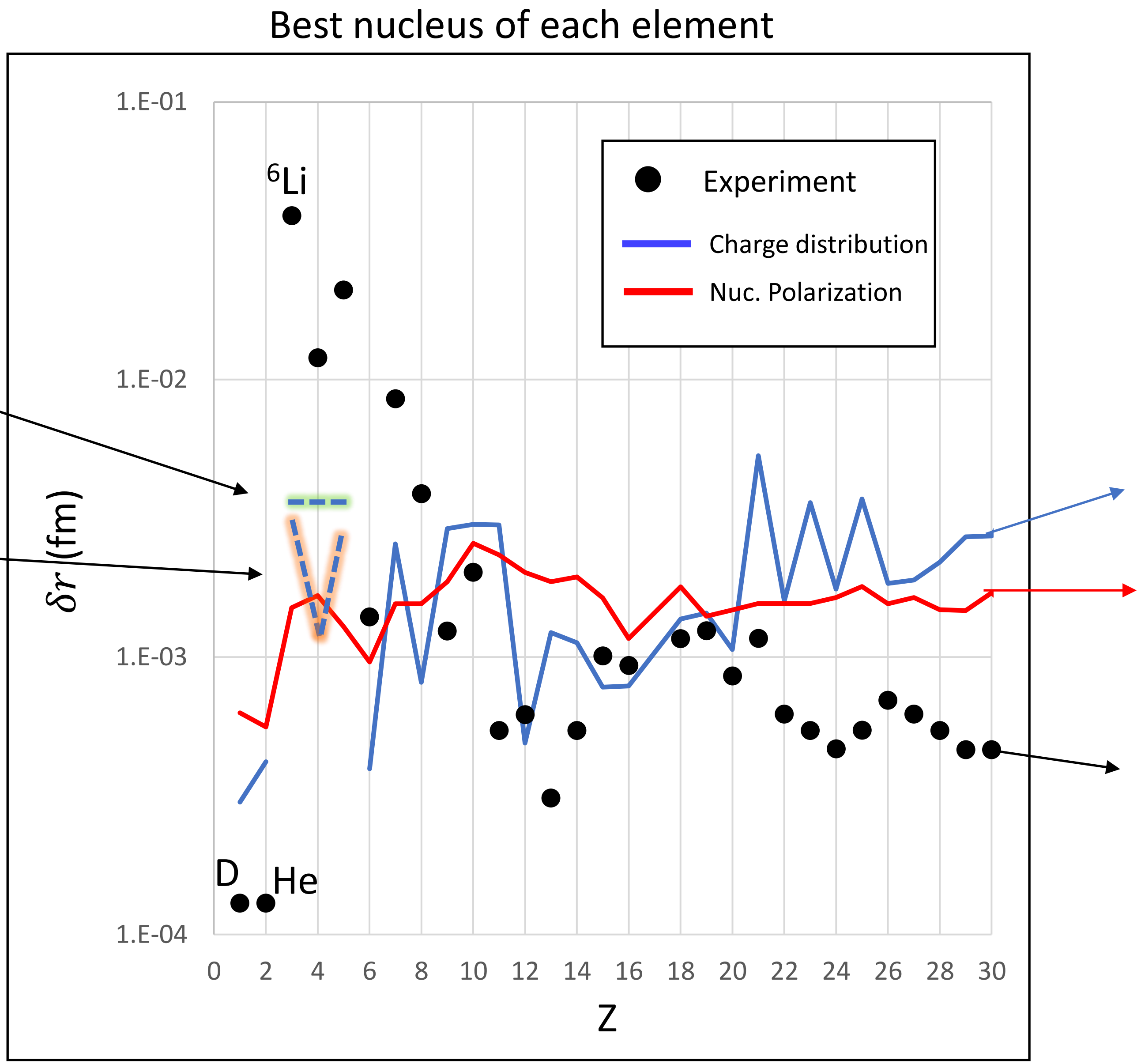
Dipole vs. Gaussian parametrization
Of charge distribution (Krutov et. al. 2016)



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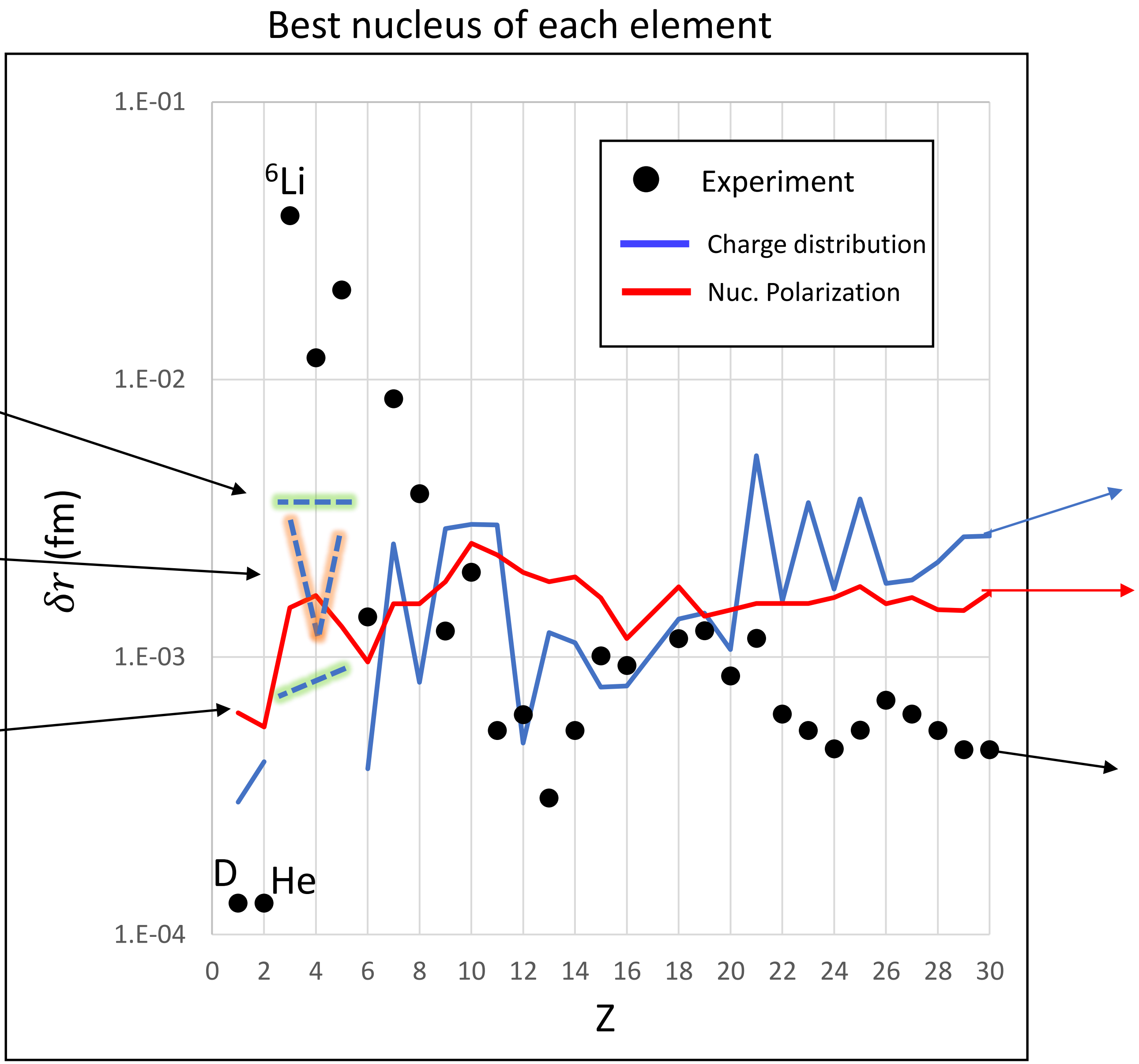
“Best case scenario”:
Get SOG/FB analysis of 6Li?
New scattering efforts at JLAB (ongoing)



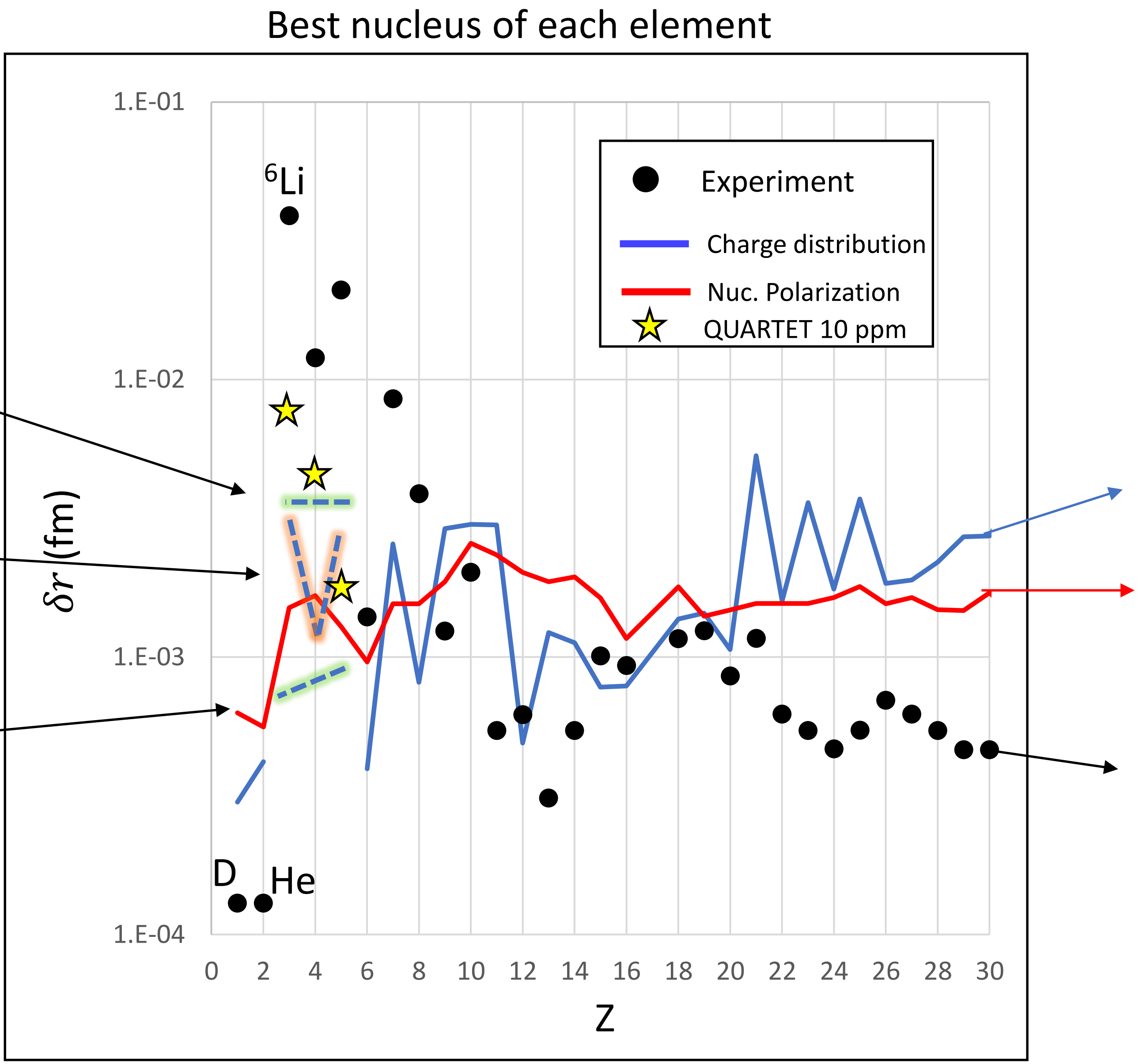
Consequence of v-factor analysis – Blue Line

How about Li, Be and B?

“Worst case scenario”:
With ancient scattering,
or extrapolated v-factors

Dipole vs. Gaussian parametrization
Of charge distribution (Krutov et. al. 2016)

“Best case scenario”:
Get SOG/FB analysis of 6Li?
New scattering efforts at JLAB (ongoing)



Modern analysis (Indelicato Recipe):

1. Introduced into code in 2015
2. Modern way to combine electron scattering and muonic measurements
3. Applied (only?) for mu^{12}C

LKB

Method of calculation for muonic atoms

- Use many points inside the nucleus (several thousands)
- Calculate with Vacuum polarization (Uehling) in perturbation or in the Dirac equation (all orders)
- Use many different values of the parameters (RMS radius, Fermi t parameter)
- Do the calculation for different models (SOG, FB, 2 parameters Fermi, Gauss, Exponential) and compare Barrett's radii
- Find the pair of Fermi parameters that reproduces the order 4 and 6 Barrett moments for a given experimental charge distribution
- Fit as a function of R using Friar's functional shape
- Combine all fitted contributions into a an equation E(R) and solve E(R)=Experimental value to get R.

Indelicato Recipe for ^{12}C:

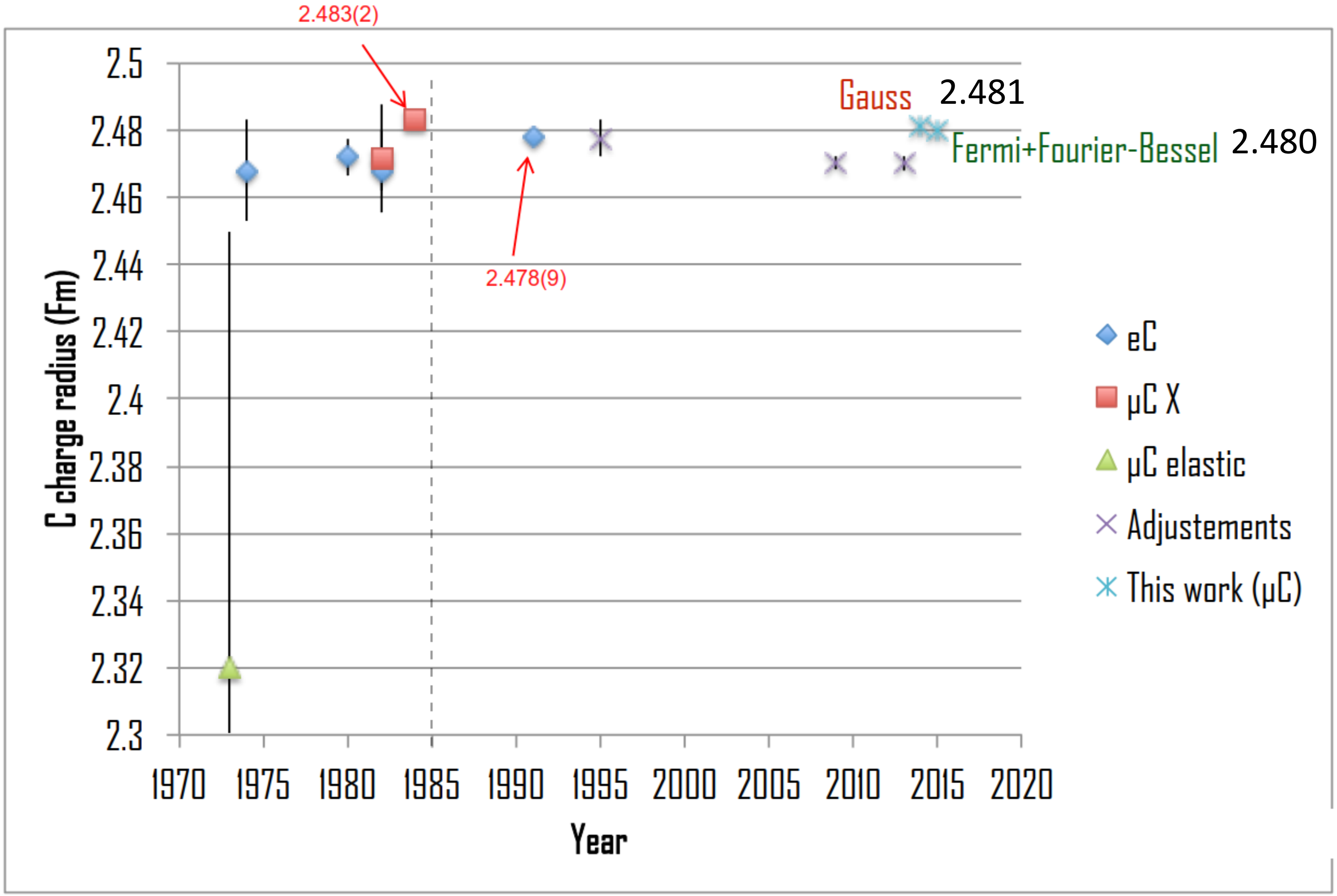
Convince Paul to publish (-:

Motivation:

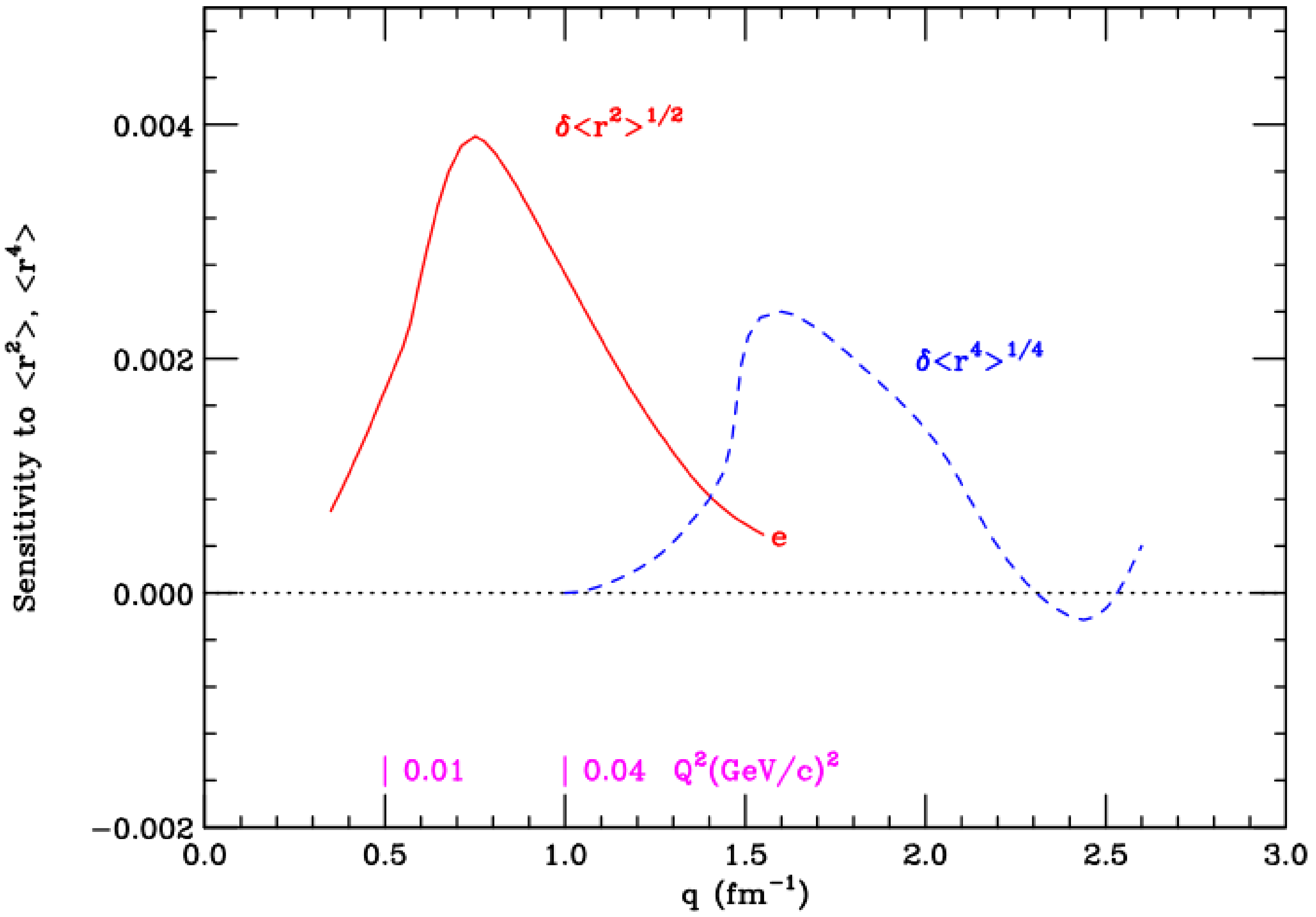
(communication with Wilfried Nörtershäuser)

He-like systems (carbon)	current Exp. uncertainty
$^3S_1 \rightarrow {}^3P_2$ transition	1.7 MHz
Δr/r	0.05 %*

*If atomic theory would match the experimental uncertainty



Summary:



0.004
0.002
0.000
−0.002
Sensitivity to $\langle r^2 \rangle$, $\langle r^4 \rangle$
$\delta\langle r^2 \rangle^{1/2}$
$\delta\langle r^4 \rangle^{1/4}$
e
| 0.01
| 0.04 $Q^2 (GeV/c)^2$
0.0
0.5
1.0
1.5
2.0
2.5
3.0
q (fm^{-1})

Sick on Li:

Analysis of world data for ^{6}Li (PRC84(11)024307)

use tail constraint as well

complication: p-tail or d-tail? (cluster structure of ^{6}Li)

SE_p=4.6MeV, SE_d=1.5MeV

as GFMC calculation (Pieper *et al.*) gives correct BE: use GFMC

Result

charge rms-radius = $2.589 \pm 0.039 fm$

comparatively large uncertainty due to poor low-q data

Modern analysis ^{12}C: