Radiative corrections to $e^+e^- \to \pi^+\pi^-$ (and $\pi^+\pi^- \to \pi^+\pi^-$)

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Outline

Introduction

Illustration of the approach: the forward-backward asymmetry in $e^+e^- \to \pi^+\pi^-$

Dispersive approach to radiative corrections to $\pi\pi$ scattering

Dispersive approach to FSR in $e^+e^- \to \pi^+\pi^-$

Conclusions and outlook

Work done in collaboration with
Martina Cottini, Martin Hoferichter, Joachim Monnard and Jacobo Ruiz de Elvira

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HVP contribution to $(g-2)_\mu$

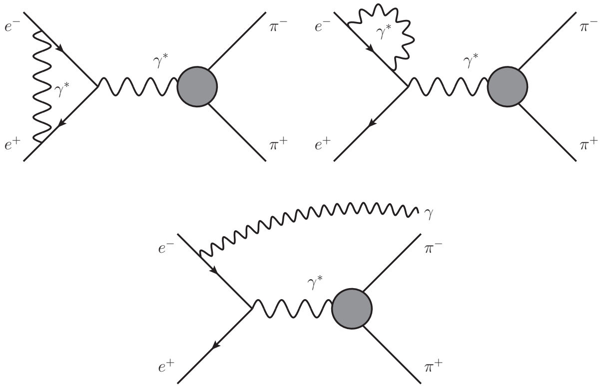
Contribution	Value $\times 10^{11}$
QED	116 584 718.931(104)
Electroweak	153.6(1.0)
HVP (e^+e^-, LO + NLO + NNLO)	6845(40)
HLbL (phenomenology + lattice + NLO)	92(18)
Total SM Value	116 591 810(43)
Experiment	116 592 061(41)
Difference: $\Delta a_\mu := a_\mu^{\text{exp}} - a_\mu^{\text{SM}}$	251(59)

HVP dominant source of theory uncertainty
rel. size $\sim 0.6\% \Rightarrow$ RC in $e^+e^- \to \pi^+\pi^-$ must be under control

RC evaluation based on models so far
A dispersive approach could lead to model-independent results

Radiative corrections to $e^+e^- \to \pi^+\pi^-$

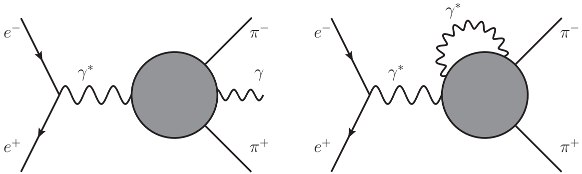
Initial State Radiation:



can be calculated in QED in terms of $F_\pi^V(s)$

Radiative corrections to $e^+e^- \to \pi^+\pi^-$

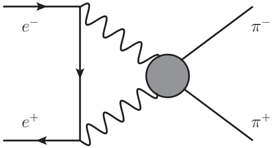
Final State Radiation:



requires hadronic matrix elements beyond $F_\pi^V(s)$
known in ChPT to one loop Kubis, Meißner (01)

Radiative corrections to $e^+e^- \to \pi^+\pi^-$

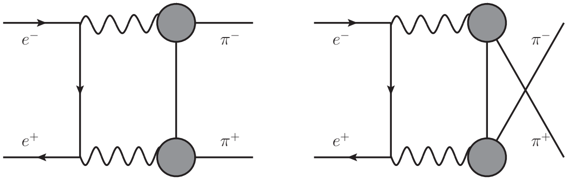
Interference terms:



also require hadronic matrix elements beyond $F_\pi^V(s)$

Radiative corrections to $e^+e^- \to \pi^+\pi^-$

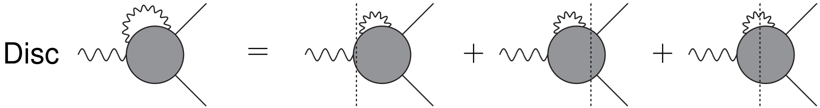
Interference terms:



also require hadronic matrix elements beyond $F_\pi^V(s)$
other than in the 1π-exchange approximation;

do not contribute to the total cross section because C-odd
but to the forward-backward asymmetry

Dispersive approach to FSR



Neglecting intermediate states beyond 2π, unitarity reads

$$\frac{\text{Disc} F_\pi^{V,\alpha}(s)}{2i} = \frac{(2\pi)^4}{2} \int d\Phi_2 F_\pi^V(s) \times T_{\pi\pi}^{\alpha *}(s,t) + \frac{(2\pi)^4}{2} \int d\Phi_2 F_\pi^{V,\alpha}(s) \times T_{\pi\pi}^{*}(s,t) + \frac{(2\pi)^4}{2} \int d\Phi_3 F_\pi^{V,\gamma}(s,t) T_{\pi\pi}^{\gamma *}(s,\{t_i\})$$

$\Rightarrow$ need $T_{\pi\pi}^{\alpha}$ as well as $T_{\pi\pi}^{\gamma}$ and $F_\pi^{V,\gamma}$ as input

The DR for $F_\pi^{V,\alpha}(s)$ takes the form of an integral equation

Outline

Forward-backward asymmetry

$$\frac{d\sigma_0}{dz} = \frac{\pi\alpha^2\beta^3}{4s}(1-z^2)\left|F_\pi^V(s)\right|^2, \qquad \beta = \sqrt{1-\frac{4M_\pi^2}{s}}, \qquad z = \cos\theta$$

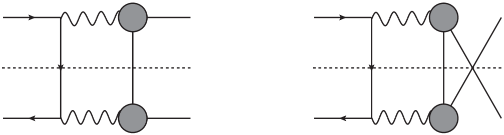
$$A_{\text{FB}}(z) = \frac{\frac{d\sigma}{dz}(z) - \frac{d\sigma}{dz}(-z)}{\frac{d\sigma}{dz}(z) + \frac{d\sigma}{dz}(-z)}$$

$$\left.\frac{d\sigma}{dz}\right|_{C\text{-odd}} = \frac{d\sigma_0}{dz}\left[\delta_{\text{soft}}(m_\gamma^2,\Delta) + \delta_{\text{virt}}(m_\gamma^2)\right] + \left.\frac{d\sigma}{dz}\right|_{\text{hard}}(\Delta)$$

$$\delta_{\text{soft}} = \frac{2\alpha}{\pi}\left\{\log\frac{m_\gamma^2}{4\Delta^2}\log\frac{1+\beta z}{1-\beta z} + \log(1-\beta^2)\log\frac{1+\beta z}{1-\beta z} + \ldots\right\}$$

Calculation of $\delta_{\rm virt}$ in the 1π-exchange approximation

- cut the diagrams in the t (or u) channel



- represent the subamplitude $e^+e^- \to \pi^+\pi^-$ dispersively

$$\frac{F_\pi^V(s)}{s} = \frac{1}{s - m_\gamma^2} - \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{{\rm Im} F_\pi^V(s')}{s'} \frac{1}{s - s'}$$

- which leads to

GC, Hoferichter, Monnard, Ruiz de Elvira (22)

$$\delta_{\rm virt} = \bar\delta_{\rm virt}(m_\gamma^2, m_\gamma^2) - \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{{\rm Im} F_\pi^V(s')}{s'}\left[\bar\delta_{\rm virt}(s', m_\gamma^2) + \bar\delta_{\rm virt}(m_\gamma^2, s')\right]$$
$$+ \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{{\rm Im} F_\pi^V(s')}{s'} \frac{1}{\pi}\int_{4M_\pi^2}^\infty ds'' \frac{{\rm Im} F_\pi^V(s'')}{s''} \bar\delta_{\rm virt}(s', s''),$$

Calculation of $\delta_{\rm virt}$ in the 1π-exchange approximation

GC, Hoferichter, Monnard, Ruiz de Elvira (22)

$$\bar{\delta}_{\rm virt} = -\frac{\mathrm{Re} F_\pi^V(s)}{2\beta^2 s(1-z^2)|F_\pi^V(s)|^2}\frac{\alpha}{\pi}$$

$$\times \mathrm{Re}\Big[4t(M_\pi^2-t)\Big(C_0(m_e^2,t,M_\pi^2,s',m_e^2,M_\pi^2)+C_0(m_e^2,t,M_\pi^2,s'',m_e^2,M_\pi^2)\Big)$$

$$-4t\Big(sC_0(m_e^2,s,m_e^2,m_e^2,s',s'')-tC_0(M_\pi^2,s,M_\pi^2,M_\pi^2,s',s'')\Big)$$

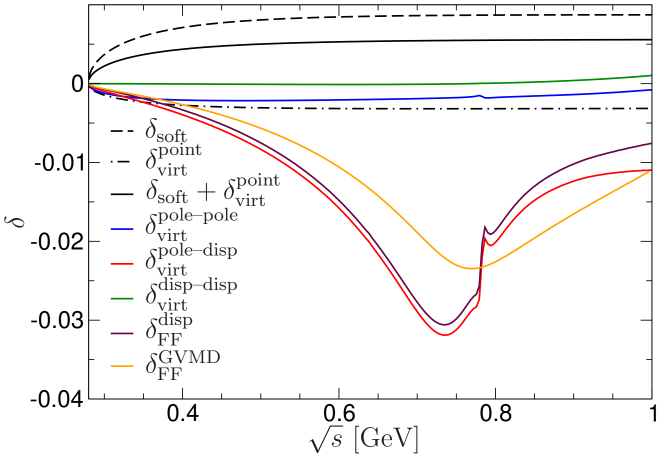
$$+4(M_\pi^2-t)\Big((M_\pi^2-t)^2+M_\pi^4+t(s'+s''-u)\Big)$$

$$\times D_0(m_e^2,m_e^2,M_\pi^2,M_\pi^2,s,t,s',m_e^2,s'',M_\pi^2)-(t\leftrightarrow u)\Big]$$

$$+(\mathrm{Re}\to\ \mathrm{Im})$$

Numerical analysis

GC, Hoferichter, Monnard, Ruiz de Elvira (22)



GVMD describes well preliminary CMD3 data Ignatov, Lee (22)

Numerical analysis

GC, Hoferichter, Monnard, Ruiz de Elvira (22)

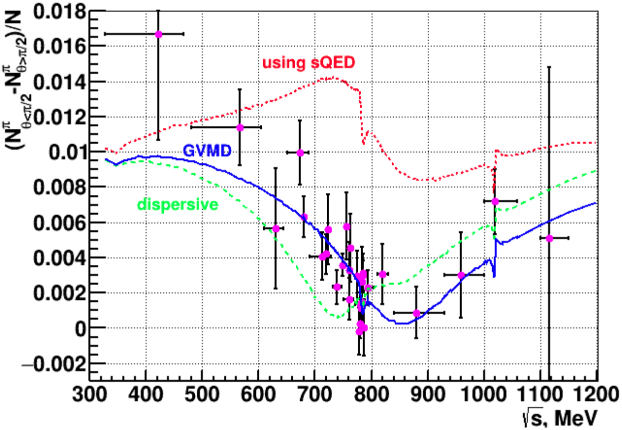


Figure courtesy of F. Ignatov

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$\pi\pi$ scattering amplitude in the isospin limit

Phenomenological representation of $A(s,t,u)$ below ~ 1 GeV

$$A(s,t,u) = A(s,t,u)_{SP} + A(s,t,u)_d$$

where A_{SP} is the unitarity contribution of S and P waves

$$A(s,t,u)_{SP} = \frac{32\pi}{3}\left\{W^0(s) - W^2(s) + \frac{9}{2}(s-u)W^1(t) + \frac{3}{2}W^2(t) + (t \leftrightarrow u)\right\}$$

and (with $\sqrt{s_2} \sim 2$ GeV)

$$\begin{aligned}
W^0(s) &= \frac{a_0^0\, s}{4M_\pi^2} + \frac{s(s-4M_\pi^2)}{\pi}\int_{4M_\pi^2}^{s_2} ds' \,\frac{\mathrm{Im}\, t_0^0(s')}{s'(s'-4M_\pi^2)(s'-s)} \\
W^1(s) &= \frac{s}{\pi}\int_{4M_\pi^2}^{s_2} ds' \,\frac{\mathrm{Im}\, t_1^1(s')}{s'(s'-4M_\pi^2)(s'-s)} \\
W^2(s) &= \frac{a_0^2\, s}{4M_\pi^2} + \frac{s(s-4M_\pi^2)}{\pi}\int_{4M_\pi^2}^{s_2} ds' \,\frac{\mathrm{Im}\, t_0^2(s')}{s'(s'-4M_\pi^2)(s'-s)}
\end{aligned}$$

$\pi\pi$ scattering amplitude in the isospin limit

Phenomenological representation of $A(s,t,u)$ below $\sim$ 1 GeV

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$$A(s,t,u)_{SP} = \frac{32\pi}{3}\left\{W^0(s) - W^2(s) + \frac{9}{2}(s-u)W^1(t) + \frac{3}{2}W^2(t) + (t \leftrightarrow u)\right\}$$

where $t^I_\ell(s)$ are partial wave projections of isospin amplitudes

$$\begin{aligned}
T^0(s,t,u) &= 3A(s,t,u) + A(t,u,s) + A(u,s,t)\\
T^1(s,t,u) &= A(t,u,s) - A(u,s,t)\\
T^2(s,t,u) &= A(t,u,s) + A(u,s,t)
\end{aligned}$$

$\pi\pi$ scattering amplitude in the isospin limit

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$$A(s,t,u)_{SP} = \frac{32\pi}{3}\left\{W^0(s) - W^2(s) + \frac{9}{2}(s-u)W^1(t) + \frac{3}{2}W^2(t) + (t \leftrightarrow u)\right\}$$

$A_d =$ effect of higher waves and higher energies

For $\sqrt{s} < 1$ GeV small and smooth contribution $\Rightarrow$ polynomial

$\pi\pi$ scattering amplitude away from the isospin limit

We need to consider three different effects: GC, Gasser, Rusetsky (09)

1. strong isospin breaking: effects proportional to $(m_u - m_d)$
2. effects proportional to $M_{\pi^+} - M_{\pi^0}$
3. effects due to photon exchanges

Each of them can be considered separately from the other two

At low energy effects 1. are small $\sim O((m_u - m_d)^2)$

At higher energies they generate π^0-η as well as ρ-ω mixing

These can be (and are) described phenomenologically
(and π^0-η mixing is not relevant for $F_\pi^V(s)$)

The rest of the talk concerns the other two

Roy equations away from the isospin limit

First we need to switch from the isospin to the charge basis

$$\begin{aligned}
T^c(s,t,u) &= \frac{1}{3}T^0(s,t,u) + \frac{1}{2}T^1(s,t,u) + \frac{1}{6}T^2(s,t,u) \\
T^x(s,t,u) &= \frac{1}{3}T^0(s,t,u) - \frac{1}{3}T^2(s,t,u) \\
T^n(s,t,u) &= \frac{1}{3}T^0(s,t,u) + \frac{2}{3}T^2(s,t,u)
\end{aligned}$$

where

$$T^c := T(\pi^+\pi^- \to \pi^+\pi^-),\ T^x := T(\pi^+\pi^- \to \pi^0\pi^0),\ T^n := T(\pi^0\pi^0 \to \pi^0\pi^0)$$

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where

$$T^c := T(\pi^+\pi^- \to \pi^+\pi^-),\ T^x := T(\pi^+\pi^- \to \pi^0\pi^0),\ T^n := T(\pi^0\pi^0 \to \pi^0\pi^0)$$

and with crossed channels

$$\begin{aligned} T^{++}(s,t,u) := T(\pi^+\pi^+ \to \pi^+\pi^+) &= T^c(t,u,s) \\ T^{+}(s,t,u) := T(\pi^+\pi^0 \to \pi^+\pi^0) &= T^x(t,u,s)\ . \end{aligned}$$

Roy equations away from the isospin limit

First we need to switch from the isospin to the charge basis

$$
\begin{aligned}
T^c(s,t,u) &= \frac{1}{3}T^0(s,t,u) + \frac{1}{2}T^1(s,t,u) + \frac{1}{6}T^2(s,t,u) \\
T^x(s,t,u) &= \frac{1}{3}T^0(s,t,u) - \frac{1}{3}T^2(s,t,u) \\
T^n(s,t,u) &= \frac{1}{3}T^0(s,t,u) + \frac{2}{3}T^2(s,t,u)
\end{aligned}
$$

Then adapt (elastic) unitarity relations

$$
\begin{aligned}
\mathrm{Im}\, T_S(s) &= T_S(s)\rho(s)T_S^*(s)\,, \quad \text{with} \quad T_S = \begin{pmatrix} t_{n,S}(s) & -t_{x,S}(s) \\ -t_{x,S}(s) & t_{c,S}(s) \end{pmatrix}, \\
\rho(s) &= \begin{pmatrix} \sigma_0(s)\theta(s-4M_{\pi^0}^2) & 0 \\ 0 & 2\sigma(s)\theta(s-4M_\pi^2) \end{pmatrix}
\end{aligned}
$$

where

$$
\sigma_0(s) = \sqrt{1-4M_{\pi^0}^2/s}, \quad \sigma(s) = \sqrt{1-4M_\pi^2/s}
$$

Roy equations away from the isospin limit

First we need to switch from the isospin to the charge basis

$$
\begin{aligned}
T^c(s,t,u) &= \frac{1}{3}T^0(s,t,u) + \frac{1}{2}T^1(s,t,u) + \frac{1}{6}T^2(s,t,u) \\
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T^n(s,t,u) &= \frac{1}{3}T^0(s,t,u) + \frac{2}{3}T^2(s,t,u)
\end{aligned}
$$

Then adapt (elastic) unitarity relations

$$
\begin{aligned}
\mathrm{Im} t_{n,S}(s) &= \sigma_0(s)|t_{n,S}(s)|^2 + 2\sigma(s)|t_{x,S}(s)|^2 \\
\mathrm{Im} t_{x,S}(s) &= \sigma_0(s)t_{n,S}(s)t^*_{x,S}(s) + 2\sigma(s)t_{x,S}(s)t^*_{c,S}(s) \\
\mathrm{Im} t_{c,S}(s) &= \sigma_0(s)|t_{x,S}(s)|^2 + 2\sigma(s)|t_{c,S}(s)|^2 \ .
\end{aligned}
$$

where

$$
\sigma_0(s) = \sqrt{1 - 4M^2_{\pi^0}/s}, \quad \sigma(s) = \sqrt{1 - 4M^2_{\pi}/s}
$$

Roy equations away from the isospin limit

This leads to the following Roy eqs.

$$T^n_{SP}(s,t,u) = 32\pi \left(W^{00}_{n,S}(s) + W^{+-}_{n,S}(s) + (s \leftrightarrow t) + (s \leftrightarrow u) \right)$$

$$\begin{aligned} W^{00}_{n,S}(s) &= \frac{a^{00}_n s}{4M^2_{\pi^0}} + \frac{s(s-4M^2_{\pi^0})}{\pi} \int_{4M^2_{\pi^0}}^{s_2} ds' \frac{\mathrm{Im} t^{00}_{n,S}(s')}{s'(s'-4M^2_{\pi^0})(s'-s)} \\ W^{+-}_{n,S}(s) &= \frac{s(s-4M^2_{\pi^0})}{\pi} \int_{4M^2_{\pi}}^{s_2} ds' \frac{\mathrm{Im} t^{+-}_{n,S}(s')}{s'(s'-4M^2_{\pi^0})(s'-s)} \ , \end{aligned}$$

Roy equations away from the isospin limit

This leads to the following Roy eqs.

$$
\begin{aligned}
T^{++}_{SP}(s,t,u) &= 32\pi \Big[W^{++}_{S}(s) + W^{00}_{c,S}(t) + W^{+-}_{c,S}(t) + W^{00}_{c,S}(u) + W^{+-}_{c,S}(u) \\
&\quad + (s-u) W^{+-}_{c,P}(t) + (s-t) W^{+-}_{c,P}(u) \Big] \\
W^{++}_{S}(s) &= \frac{a^{++}\, s}{4M_\pi^2} + \frac{s(s-4M_\pi^2)}{\pi} \int_{4M_\pi^2}^{s_2} ds' \frac{\mathrm{Im}\, t^{++}_{S}(s')}{s'(s'-4M_\pi^2)(s'-s)} \\
W^{+-}_{c,S}(s) &= \frac{a^{+-}_{c}\, s}{4M_\pi^2} + \frac{s(s-4M_\pi^2)}{\pi} \int_{4M_\pi^2}^{s_2} ds' \frac{\mathrm{Im}\, t^{+-}_{c,S}(s')}{s'(s'-4M_\pi^2)(s'-s)} \\
W^{00}_{c,S}(s) &= \frac{s(s-4M_\pi^2)}{\pi} \int_{4M_{\pi^0}^2}^{s_2} ds' \frac{\mathrm{Im}\, t^{00}_{c,S}(s')}{s'(s'-4M_\pi^2)(s'-s)} \\
W^{+-}_{c,P}(s) &= \frac{s}{\pi} \int_{4M_\pi^2}^{s_2} ds' \frac{3\,\mathrm{Im}\, t^{+-}_{c,P}(s')}{s'(s'-4M_\pi^2)(s'-s)} \, .
\end{aligned}
$$

Via crossing this provides also a representation for T^c

Roy equations away from the isospin limit

This leads to the following Roy eqs.

$$T^x_{SP}(s,t,u) = 32\pi\Big[W^{+-}_{x,S}(s) + W^{00}_{x,S}(s) + W^{+0}_S(t) + W^{+0}_S(u) + \left(t(s-u)+\Delta^2_\pi\right) W^{+0}_P(t) + \left(u(s-t)+\Delta^2_\pi\right) W^{+0}_P(u)\Big]$$

$$W^{+-}_{x,S}(s) = \frac{a^{+-}_x\, s}{4M^2_\pi} + \frac{s(s-4M^2_\pi)}{\pi}\int_{4M^2_\pi}^{s_2} ds' \frac{\mathrm{Im}t^{+-}_{x,S}(s')}{s'(s'-4M^2_\pi)(s'-s)}$$

$$W^{00}_{x,S}(s) = \frac{s(s-4M^2_\pi)}{\pi}\int_{4M^2_{\pi^0}}^{s_2} ds' \frac{\mathrm{Im}t^{00}_{x,S}(s')}{s'(s'-4M^2_\pi)(s'-s)}$$

$$W^{+0}_S(s) = \frac{a^{+0}_c\, s}{4\bar{M}^2_\pi} + \frac{s(s-4\bar{M}^2_\pi)}{\pi}\int_{4\bar{M}^2_\pi}^{s_2} ds' \frac{\mathrm{Im}t^{+0}_S(s')}{s'(s'-4\bar{M}^2_\pi)(s'-s)}$$

$$W^{+0}_P(s) = \frac{1}{\pi}\int_{4\bar{M}^2_\pi}^{s_2} ds' \frac{3\mathrm{Im}t^{+0}_P(s')}{\lambda(s', M^2_\pi, M^2_{\pi^0})(s'-s)} ,$$

$$\Delta_\pi := M^2_\pi - M^2_{\pi^0} \quad \bar{M}_\pi := (M_\pi + M_{\pi^0})/2$$

Roy eqs. and $M_\pi^2 - M_{\pi^0}^2$ effects

- Input for Roy eqs.:
  $\text{Im}t_\ell^I(s)$ above $\sqrt{s_1} \sim 1.15$ GeV and scattering lengths
- numerical solutions $\Rightarrow$ partial waves for $4M_\pi^2 \le s \le s_1$
- assume: input above s_1 does not change for $\Delta_\pi \neq 0$
- starting point: solutions in the isospin limit;
  evaluating the dispersive integrals with $\Delta_\pi \neq 0$
  $\Rightarrow$ calculation of the desired effects
- the effect on $F_\pi^V(s)$ is small (the $\pi^0\pi^0$ only appears in the t-channel of the $\pi\pi$ amplitude in the unitarity relation)

Scattering lengths for $M_\pi \neq M_{\pi^0}$

GC and M. Cottini, extracted from Knecht-Urech (98) and Knecht-Nehme (02)

$$a_n^{00} = \frac{M_{\pi^0}^2}{32\pi F_\pi^2}\left\{1 + \xi_0\left[4\bar{\ell}_1 + 8\bar{\ell}_2 - \frac{3}{2}\bar{\ell}_3 + 2\bar{\ell}_4 - \frac{23}{2} - \frac{9-11\delta_\pi}{(1-\delta_\pi)}L_\pi \right.\right.$$
$$\left.\left. + 9j_0\left(4M_{\pi^0}^2\right)\right] + \xi\delta_\pi\left(\frac{\bar{k}_{31}}{9} - \frac{10}{9}\bar{k}_2 - \bar{k}_4\right)\right\}$$

$$a^{++} = -\frac{M_\pi^2}{16\pi F_\pi^2}\left\{1 - \delta_\pi - \xi\left[\frac{4}{3}\left(\bar{\ell}_1 + 2\bar{\ell}_2\right) - \frac{1}{2}\left(\bar{\ell}_3 + 4\bar{\ell}_4\right)(1-\delta_\pi)^2\right.\right.$$
$$\left.\left. + \frac{1}{2}\left(1 + 3\delta_\pi + \frac{88}{9}\delta_\pi^2\right) - \delta_\pi(1-\delta_\pi)\left(\frac{\bar{k}_{31}}{9} - 4\bar{k}_{32} + \frac{62}{9}\bar{k}_2 + 5\bar{k}_4\right)\right]\right\}$$

where $\delta_\pi = 1 - M_{\pi^0}^2/M_\pi^2$, $\xi_0 = M_{\pi^0}^2/(16\pi^2F_\pi^2) = \xi(1-\delta_\pi)$,
$L_\pi := -\ln(1-\delta_\pi) = \delta_\pi + \mathcal{O}(\delta_\pi^2)$

Scattering lengths for $M_\pi \neq M_{\pi^0}$

GC and M. Cottini, extracted from Knecht-Urech (98) and Knecht-Nehme (02)

$$
\begin{aligned}
a_x^{+-} = -\frac{M_\pi^2}{32\pi F_\pi^2\,(2-\delta_\pi)}\Bigg\{ 2\,(3-\delta_\pi) + \frac{\xi}{3}\Bigg[& \frac{33+158\delta_\pi-29\delta_\pi^2-36\delta_\pi^3}{3} \\
& + 8\bar{\ell}_1\left(1+\delta_\pi-\delta_\pi^2\right) + 4\bar{\ell}_2\,(2-\delta_\pi)^2 - 3\bar{\ell}_3\,(1-\delta_\pi)^2 \\
& + 12\bar{\ell}_4\left(3-4\delta_\pi+\delta_\pi^2\right) + 2\left(2-11\delta_\pi-18\delta_\pi^2+9\delta_\pi^3\right)\lambda_\pi \\
& + \frac{3}{2}\left(6-7\delta_\pi+\delta_\pi^3\right) j_0\left(4M_\pi^2\right) + \frac{4}{3}\delta_\pi\left(16-21\delta_\pi+2\delta_\pi^2\right)\bar{j}_{+0}^{(1)} + \frac{3}{2}\delta_\pi^4\bar{j}_{+0}^{(2)} \\
& + \delta_\pi\left(\frac{2}{3}\bar{k}_{31}(3-\delta_\pi) + 12\bar{k}_{32}(1-\delta_\pi) + \frac{4}{3}\bar{k}_2(3+5\delta_\pi) + 12\bar{k}_4\right)\Bigg]\Bigg\}
\end{aligned}
$$

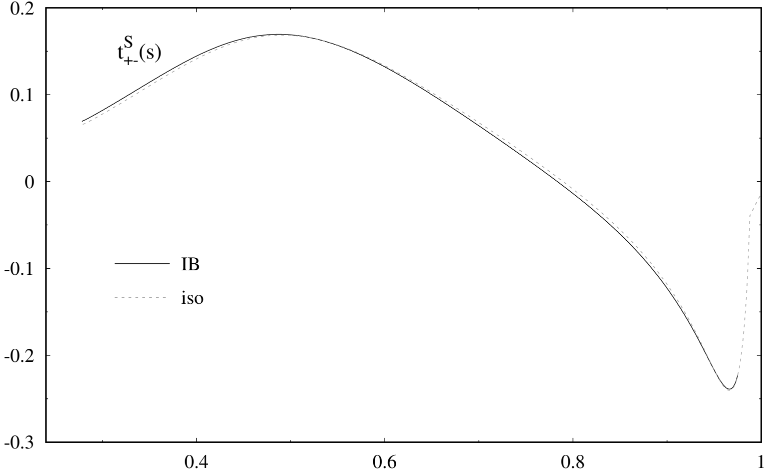
where $\lambda_\pi := L_\pi/\delta_\pi = 1 + \mathcal{O}(\delta_\pi)$

Scattering lengths for $M_\pi \neq M_{\pi^0}$

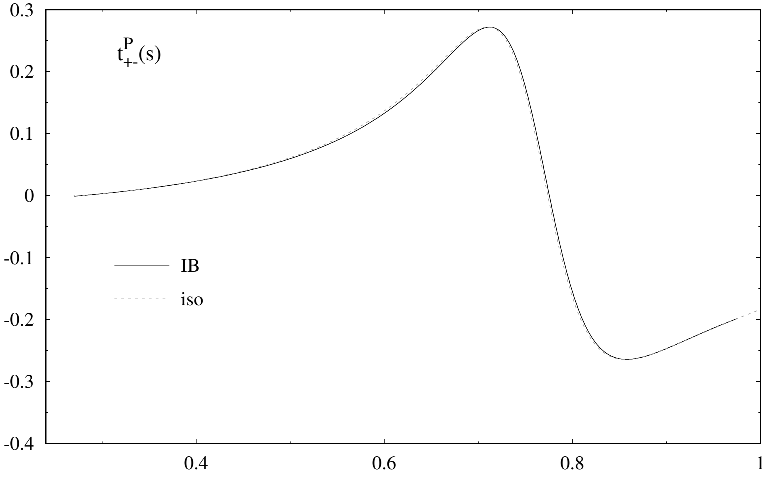
GC and M. Cottini, extracted from Knecht-Urech (98) and Knecht-Nehme (02)

$$
\begin{aligned}
a_c^{+0} = \frac{M_\pi^2}{32\pi F_\pi^2 \eta} \Bigg\{ & \frac{1-\delta_\pi}{1-\delta_\pi/2} - \xi \Bigg[\frac{1}{3} \Bigg(\frac{\left(17 + 10\delta_\pi - 23\delta_\pi^2\right) + 8\eta\,(2-\delta_\pi)}{3} \\
& + 8\,(1-\delta_\pi)\,\bar{\ell}_1 - 4\,(2-\delta_\pi)\,(2-\delta_\pi-4\eta)\,\bar{\ell}_2 - 3\,(1-\delta_\pi)^2\left(\bar{\ell}_3 + 4\bar{\ell}_4\right) \Bigg) \\
& + \frac{2}{3}\left[6 - 8\eta\,(2-\delta_\pi) - \delta_\pi^2\left(7 - 9\delta_\pi + 6\delta_\pi^2\right)\right]\lambda_\pi \\
& + (2-\delta_\pi)\eta \left((1-2\eta^{-1})^2 + \frac{\eta^2\delta_\pi^4}{64} \right) \bar{j}_{+0}\left(4\bar{M}_\pi^2\right) \\
& - \frac{\eta^2\delta_\pi^2}{4}\,(2-\delta_\pi)\left(1-2\eta^{-1}\right)\bar{\bar{j}}_{+0}\left(4\bar{M}_\pi^2\right) + \frac{\delta_\pi^4}{8}\left(4 - \eta\,(2-\delta_\pi)\right) j_{+0}^{(2)} \\
& - \left((2-\delta_\pi)\left(\frac{16}{9\eta^2} + \frac{\delta_\pi^4\eta^2}{16} \right) - \frac{4}{9}\left(8 - 12\delta_\pi + 3\delta_\pi^2 - 4\delta_\pi^3\right) \right) j_{+0}^{(1)} \\
& - \delta_\pi\,(1-\delta_\pi)\left(\frac{2}{9}\bar{k}_{31} - 4\bar{k}_{32} + \frac{52}{9}\bar{k}_2 + 4\bar{k}_4 \right) \Bigg] \frac{1}{2-\delta_\pi} \Bigg\}
\end{aligned}
$$

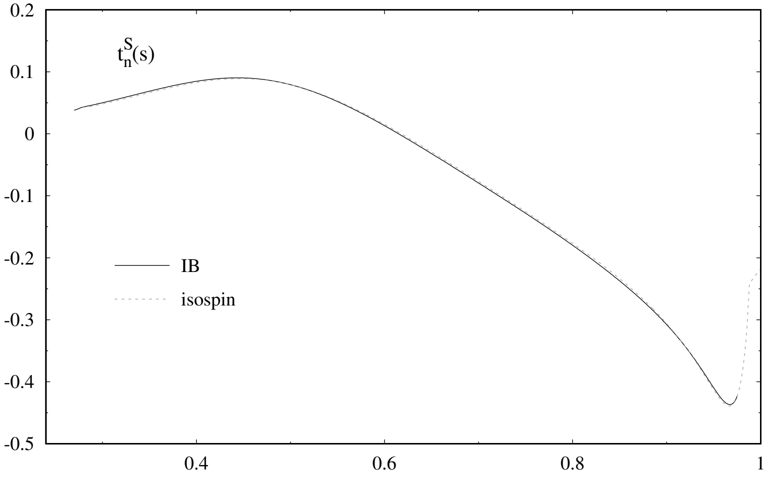
Results (preliminary)



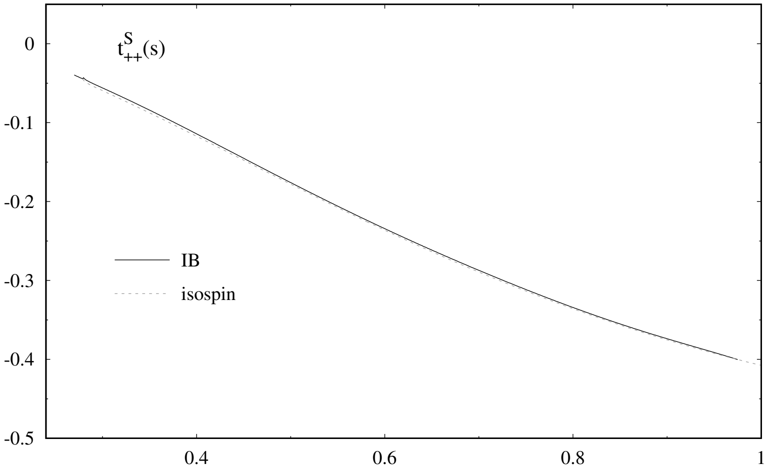
Results (preliminary)



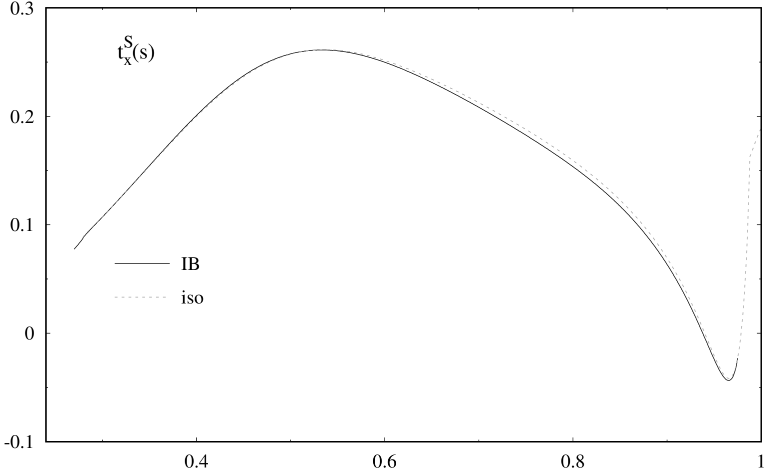
Results (preliminary)



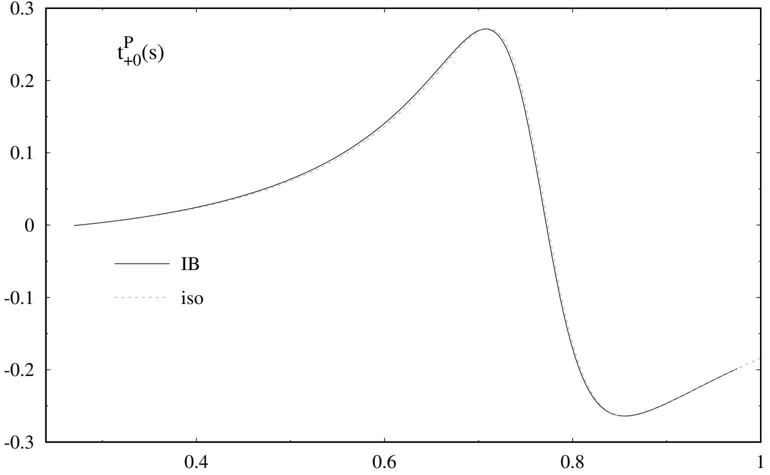
Results (preliminary)



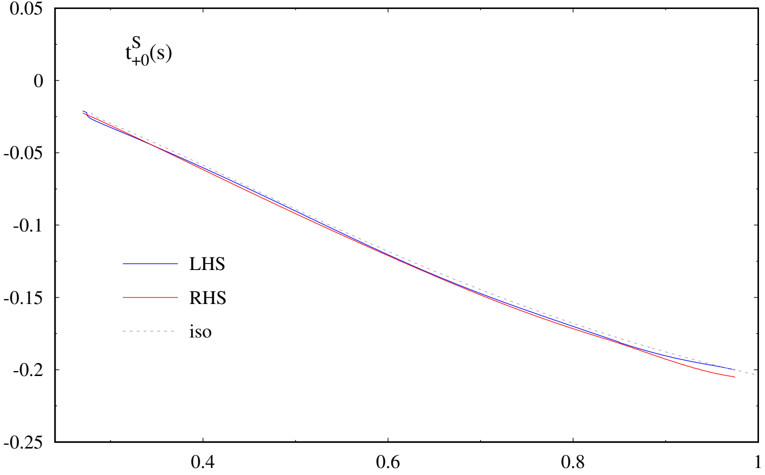
Results (preliminary)



Results (preliminary)



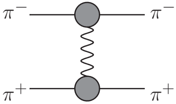
Results (preliminary)



Roy eqs. and photon-exchange effects

Photon-exchange diagrams are $O(\alpha)$ effects not included in the Roy eqs.

$$T_B(t, s, u) :=$$

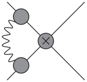
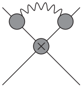
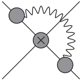


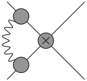
$$= 4\pi\alpha \frac{s-u}{t} F_\pi^V(t)^2$$

$$T_B^c(s, t, u) = T_B(t, s, u) + T_B(s, t, u)$$

- Adding such a contribution to the T^c amplitude upsets the unitarity relations for all amplitudes
- we are interested in corrections only up to $O(\alpha)$
  $\Rightarrow$ set up an iterative scheme

Roy eqs. and photon-exchange effects: 1. iteration

$T_D^c(s,t,u) :=$  $+$  $+$  $+$ flipped diags.

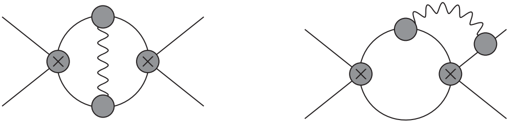
$T_D^x(s,t,u) :=$ 

"Triangle diagrams" $\Rightarrow$ topology of box diagrams
$\Rightarrow$ double-spectral representation

Starting point for further iterations:

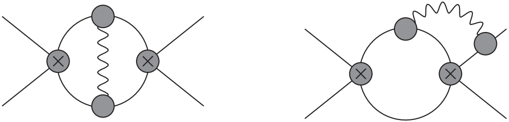
$$\begin{aligned}
T_1^c(s,t,u) &= T_0^c(s,t,u) + T_B^c(s,t,u) + T_D^c(s,t,u) \\
T_1^x(s,t,u) &= T_0^x(s,t,u) + T_D^x(s,t,u) \\
T_1^n(s,t,u) &= T_0^n(s,t,u)
\end{aligned}$$

Roy eqs. and photon-exchange effects: 2. iteration



Diagrams have to be cut in all possible ways:
$\Rightarrow$ contributions from subamplitudes with real photons

Roy eqs. and photon-exchange effects: 2. iteration



Diagrams have to be cut in all possible ways:
⇒ contributions from subamplitudes with real photons

Expression after further iterations:

$$
\begin{aligned}
T_1^c(s,t,u) &= T_0^c(s,t,u) + T_B^c(s,t,u) + T_D^c(s,t,u) + \sum_{k=2} R_k^c(s,t,u) \\
T_1^x(s,t,u) &= T_0^x(s,t,u) + T_D^x(s,t,u) + \sum_{k=2} R_k^x(s,t,u) \\
T_1^n(s,t,u) &= T_0^n(s,t,u) + \sum_{k=2} R_k^n(s,t,u)
\end{aligned}
$$

Roy eqs. and photon-exchange effects: comments

- starting from the 2. iteration the evaluation of the R^i_{k+1} is done as follows:
  1. project the R^i_k amplitudes onto partial waves
  2. insert these into the unitarity relations combined with the projections of T^i_0
  3. add the contribution of subdiagrams with real photons
  4. solve the corresponding dispersion relation
- subtraction constants can be fixed by matching to ChPT
- iteration number k corresponds to chiral $O(p^{2k})$
- χPT $\pi\pi \to \pi\pi$ w/ RC at one loop Knecht, Urech (98), Knecht, Nehme (02)
  $\Rightarrow$ subtraction constants for all R^i_k, $k \geq 2$ can be set to zero

Outline

Introduction

Illustration of the approach: the forward-backward asymmetry in $e^+e^- \to \pi^+\pi^-$

Dispersive approach to radiative corrections to $\pi\pi$ scattering

Dispersive approach to FSR in $e^+e^- \to \pi^+\pi^-$

Conclusions and outlook

Dispersive treatment of FSR in $e^+e^- \to \pi^+\pi^-$

$$\frac{\mathrm{Disc}F_\pi^{V,\alpha}(s)}{2i} = \frac{(2\pi)^4}{2}\int d\Phi_2 F_\pi^V(s) \times T_{\pi\pi}^{\alpha *}(s,t) + \frac{(2\pi)^4}{2}\int d\Phi_2 F_\pi^{V,\alpha}(s) \times T_{\pi\pi}^{*}(s,t) + \frac{(2\pi)^4}{2}\int d\Phi_3 F_\pi^{V,\gamma}(s,t) T_{\pi\pi}^{\gamma *}(s,\{t_i\})$$

Long digression $\Rightarrow$ $T_{\pi\pi}^{\alpha}$

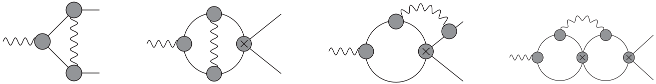
Approximation: only 2π intermediate states for $F_\pi^{V,\gamma}$ and $T_{\pi\pi}^{\gamma}$:

All subamplitudes known $\Rightarrow$ $F_\pi^{V,\gamma}$ and $T_{\pi\pi}^{\gamma}$ ✓

Evaluation of $F_\pi^{V,\alpha}$

Having evaluated all the following diagrams

J. Monnard, PhD thesis 2021



Evaluation of $F_\pi^{V,\alpha}$

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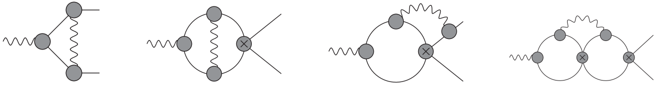
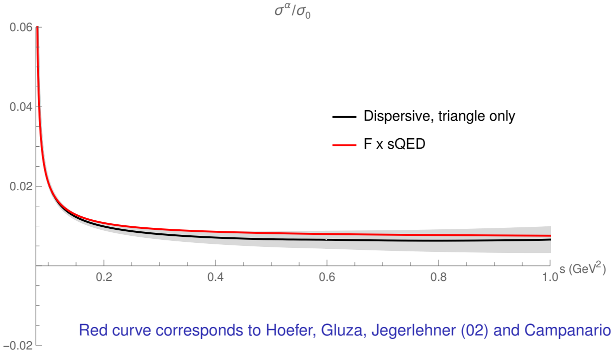
J. Monnard, PhD thesis 2021



the results for $\sigma(e^+e^- \to \pi^+\pi^-(\gamma))$ look as follows:

Preliminary!

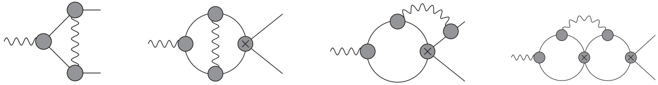


Red curve corresponds to Hoefer, Gluza, Jegerlehner (02) and Campanario et al. (19) (?)

Evaluation of $F_\pi^{V,\alpha}$

Having evaluated all the following diagrams

J. Monnard, PhD thesis 2021



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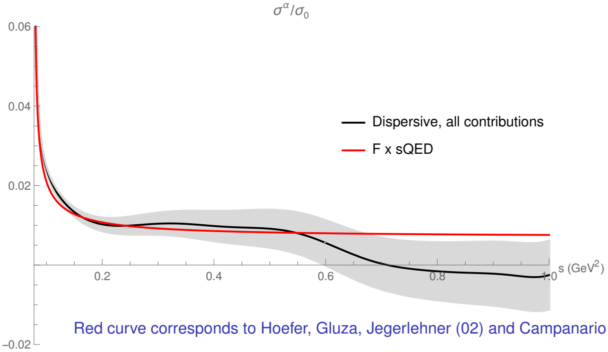
Preliminary!



Red curve corresponds to Hoefer, Gluza, Jegerlehner (02) and Campanario et al. (19) (?)

Impact on a_μ^{HVP}

Ideally: use calculated RC in the data analysis (future?).

Quick estimate of the impact: thanks to M. Hoferichter and P. Stoffer

1. remove RC from the measured $\sigma(e^+e^- \to \pi^+\pi^-(\gamma))$
2. fit with the dispersive representation for $F_\pi^V(s)$
3. insert back the RC

The impact on a_μ^{HVP} is evaluated by comparing to the result obtained by removing RC with $\eta(s)$ calculated in sQED

$$10^{11}\Delta a_\mu^{\mathrm{HVP}} = \begin{cases} 10.2 \pm 0.5 \pm 5 & \text{FsQED} \\ 10.5 \pm 0.5 \pm (?) & \text{triangle} \\ 13.2 \pm 0.5 & \text{full} \end{cases}$$

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Conclusions and outlook

Conclusions and outlook

- We have developed a **dispersive formalism** for evaluating RC to the $\pi\pi$ scattering amplitude and $F_\pi^V(s)$

  work in progress GC, J. Monnard, J. Ruiz de Elvira

- approximation: include only up to 2π intermediate states $\Rightarrow$ finite system of equations (numerical solutions)
- preliminary evaluation of the corrections to $F_\pi^V(s)$ and a_μ^{HVP} shows **no unexpectedly large effects** J. Monnard, PhD thesis, 2021
- **other than** in the forward-backward asymmetry

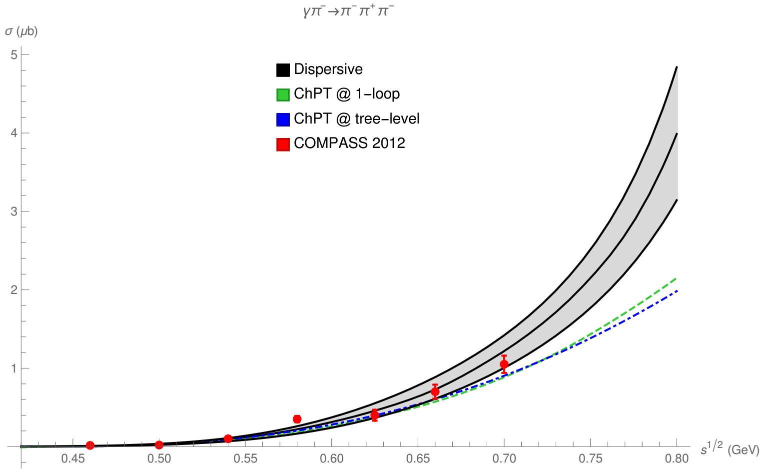
  (if compared to naive sQED) Ignatov, Lee (22), GC, Hoferichter, Monnard, Ruiz de Elvira (22)

- final goal: ready-to-use code which can be implemented in MC and used in data analysis
- we plan to apply the same approach to $\tau \to \pi\pi\nu_\tau$

  M. Cottini and S. Holz, work in progress

Backup Slides

$\gamma\pi \to 3\pi$



$\gamma\pi \to 3\pi$