Radiative corrections: 3π channel

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Radiative corrections and Monte Carlo tools for Strong 2020

University of Zurich, 5/6/2023

Hoferichter, Hoid, BK, Schuh, in progress

Motivation: radiative corrections for $\pi^+\pi^-\pi^0$

- second largest exclusive channel next to $\pi^+\pi^-$:

Channel	KNT18	DHMZ17	Difference
Data based channels ($\sqrt{s} \leq 1.8$ GeV)			
$\pi^+\pi^-$	503.74 ± 1.96	506.70 ± 2.58	-2.96
$\pi^+\pi^-\pi^0$	47.70 ± 0.89	46.20 ± 1.45	1.50
$\pi^+\pi^-\pi^+\pi^-$	13.99 ± 0.19	13.68 ± 0.31	0.31
$\pi^+\pi^-\pi^0\pi^0$	18.15 ± 0.74	18.03 ± 0.54	0.12
K^+K^-	23.00 ± 0.22	23.06 ± 0.41	-0.06
$K_S^0K_L^0$	13.04 ± 0.19	12.82 ± 0.24	0.22
Total	693.3 ± 2.5	693.1 ± 3.4	0.2

A. Keshavarzi, Mainz 2018

$\longrightarrow$ cross-checked dispersively — Hoferichter, Hoid, BK 2019

- (infrared-finite) $\pi^+\pi^-\gamma$ contribution:

$$a_\mu^{\pi^+\pi^-\gamma}\big|_{\leq 0.95\,\text{GeV}} = 4.34(4) \times 10^{-10}$$ Moussallam 2013

$\longrightarrow$ expect $a_\mu^{\pi^+\pi^-\pi^0\gamma} \sim 0.4 \times 10^{-10}$

The *anomalous* process $\gamma^* \to 3\pi$

- $\gamma^*(q) \to \pi^+\pi^-\pi^0$: odd intrinsic parity

$$\langle 0|j_\mu(0)|\pi^+(p_+)\pi^-(p_-)\pi^0(p_0)\rangle = -\epsilon_{\mu\nu\rho\sigma}\, p_+^\nu p_-^\rho p_0^\sigma\, \mathcal{F}(s,t,u;q^2)$$

s,t,u: pion–pion invariant masses, $s+t+u = q^2 + 3M_\pi^2$

- normalisation fixed from Wess–Zumino–Witten anomaly:

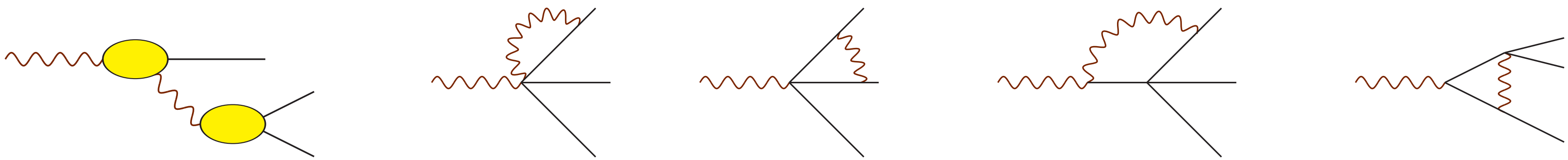
$$\mathcal{F}(0,0,0;0) = F_{3\pi} = \frac{1}{4\pi^2 F_\pi^3}$$

$\longrightarrow$ not part of scalar QED

- coupling of negative mass dimension $\longrightarrow$ nonrenormalisable!

Radiative corrections $\gamma\pi^{\pm} \to \pi^{\pm}\pi^{0}$

- can be calculated in chiral perturbation theory (with virtual photons):



Ametller, Knecht, Talavera 2001

$\longrightarrow$ photon *t*-channel pole kinematically enhanced; irrelevant for $\gamma^* \to 3\pi$ kinematics

$\longrightarrow$ requires (unknown) counterterms

Ananthanarayan, Moussallam 2002

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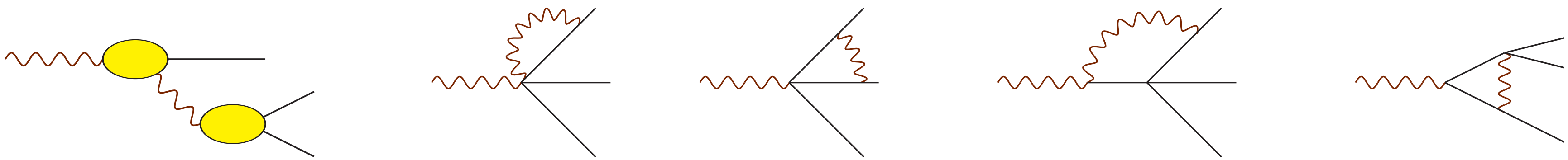

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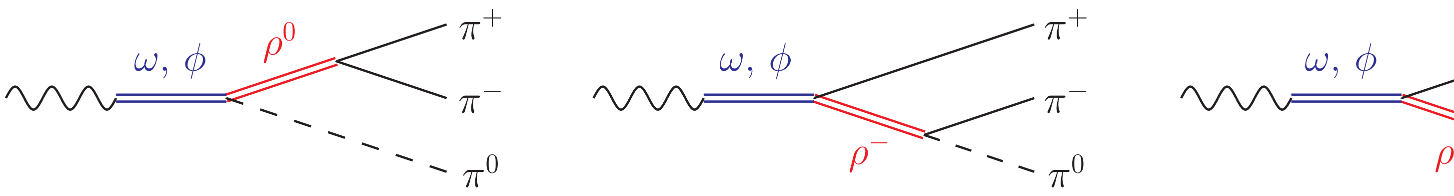
- misses dominating resonance dynamics away from threshold:



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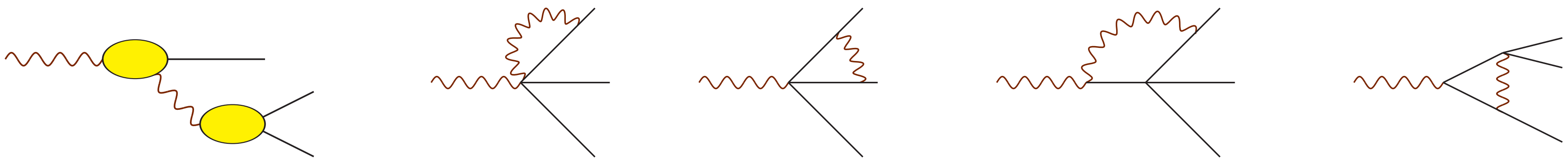

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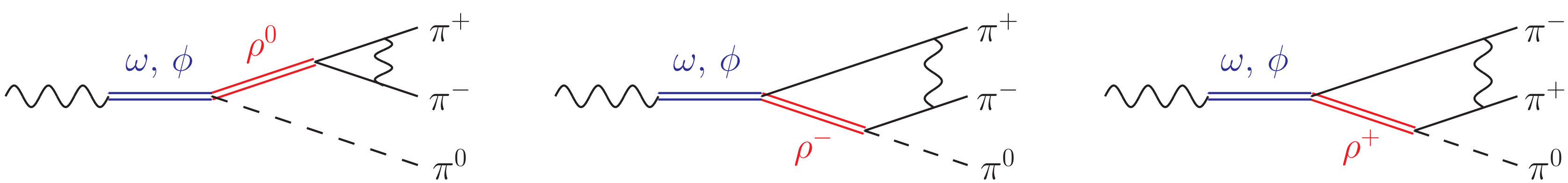
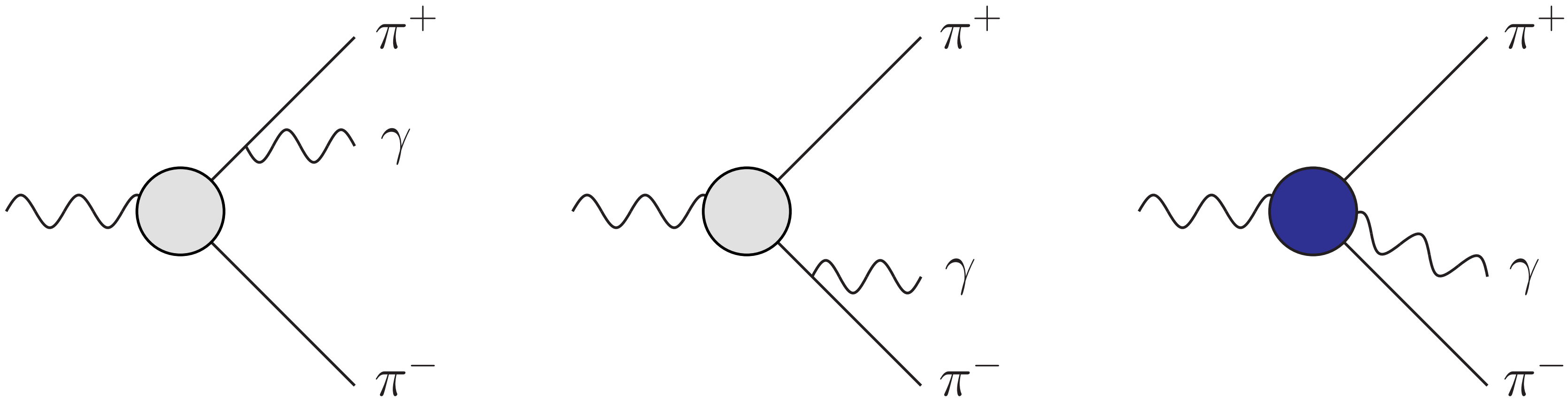


$\longrightarrow$ radiative corrections will not easily factorise

$2\pi\gamma$: infrared enhanced contributions

Moussallam 2013



- decomposition Born (incl. virtual $\longrightarrow$ IR-finite $\eta_{2\pi}$!) + rest:

$$a_\mu^{\pi^+\pi^-\gamma} = a_\mu^{\text{Born}} + \hat{a}_\mu^{\pi^+\pi^-\gamma}$$

$$a_\mu^{\text{Born}}|_{\leq 0.95\,\text{GeV}} = 4.19 \times 10^{-10}$$

$$\hat{a}_\mu^{\pi^+\pi^-\gamma}|_{\leq 0.95\,\text{GeV}} = 0.15(4) \times 10^{-10}$$

$\longrightarrow$ "infrared enhanced" Born terms dominate by far!

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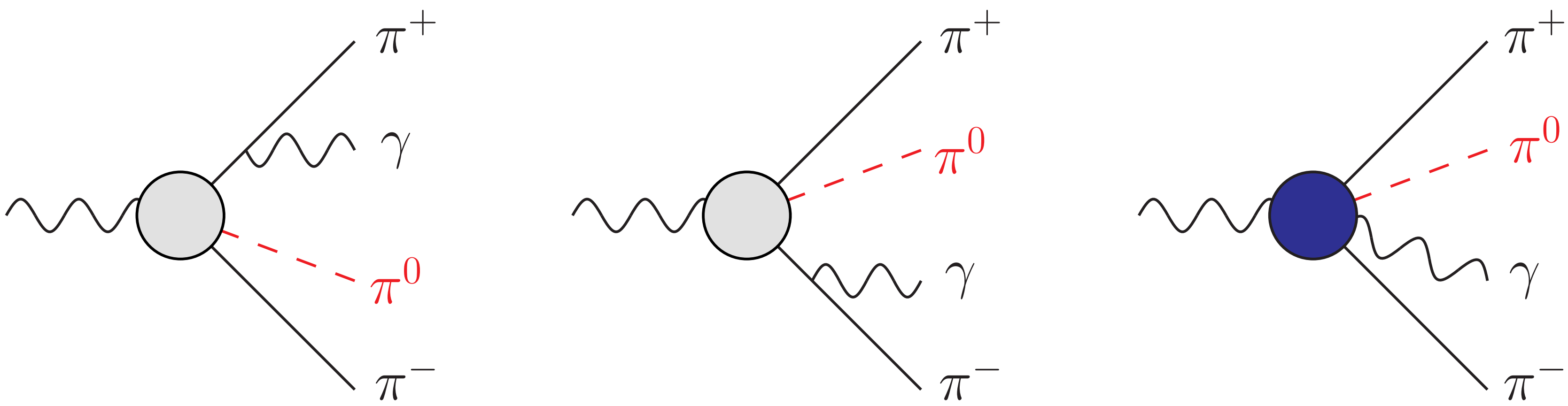

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$\longrightarrow$ assume this hierarchy for $3\pi\gamma$, too!

Amplitude representation $\gamma^* \to 3\pi$

$$\langle 0|j_\mu(0)|\pi^+(p_+)\pi^-(p_-)\pi^0(p_0)\rangle = -\epsilon_{\mu\nu\rho\sigma}\, p_+^\nu p_-^\rho p_0^\sigma\, \mathcal{F}(s,t,u;q^2)$$

- “reconstruction theorem”: neglect discontinuities in F-waves. . .

  $\longrightarrow$ decomposition into “single-variable” functions (at fixed q^2)

$$\mathcal{F}(s,t,u;q^2) = \mathcal{F}(s,q^2) + \mathcal{F}(t,q^2) + \mathcal{F}(u,q^2)$$

- (s-channel) P-wave projection: $f_1(s,q^2) = \mathcal{F}(s,q^2) + \hat{\mathcal{F}}(s,q^2)$

  $\hat{\mathcal{F}}(s,q^2)$: contribution from crossed channels $\mathcal{F}(t/u,q^2)$

  $\longrightarrow$ dispersive Khuri–Treiman representation of $\mathcal{F}(s,q^2)$

  + parameterisation of q^2 dependence fitted to data

Hoferichter, Hoid, BK 2019 + . . . [cf. spares]

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- strategy: transfer $\eta_{2\pi}(q^2)$ from $\gamma^* \to \pi^+\pi^-$
  to $\eta_{2\pi}(s)$ in $\gamma^*\pi^0 \to \pi^+\pi^-$ P-wave
- subtlety: partial waves diverge at $s_{\text{PT}} = (\sqrt{q^2} - M_\pi)^2$
  $\longrightarrow$ need to apply fudge factor to F-waves and higher

Radiative corrections in 3π

- fudge factor option 1: constant correction for $f_3(s,q^2)+\ldots$

$$\sigma_{3\pi(\gamma)}(q^2) \propto \int_{s_-}^{s_+} \mathrm{d}s \int_{t_-}^{t_+} \mathrm{d}t\, K(s,t;q^2) \Bigg| \underbrace{\big[\mathcal{F}(s,q^2)+\hat{\mathcal{F}}(s,q^2)\big]}_{f_1(s,q^2)} \sqrt{1+\frac{\alpha}{\pi}\eta_{2\pi}(s)}$$

$$+ \underbrace{\big[\mathcal{F}(t,q^2)+\mathcal{F}(u,q^2)-\hat{\mathcal{F}}(s,q^2)\big]}_{f_3(s,q^2)+\ldots} \sqrt{1+\frac{\alpha}{\pi}\eta_{2\pi}(s_{\mathrm{PT}})}\Bigg|^2$$

- fudge factor option 2: same factor for complete amplitude

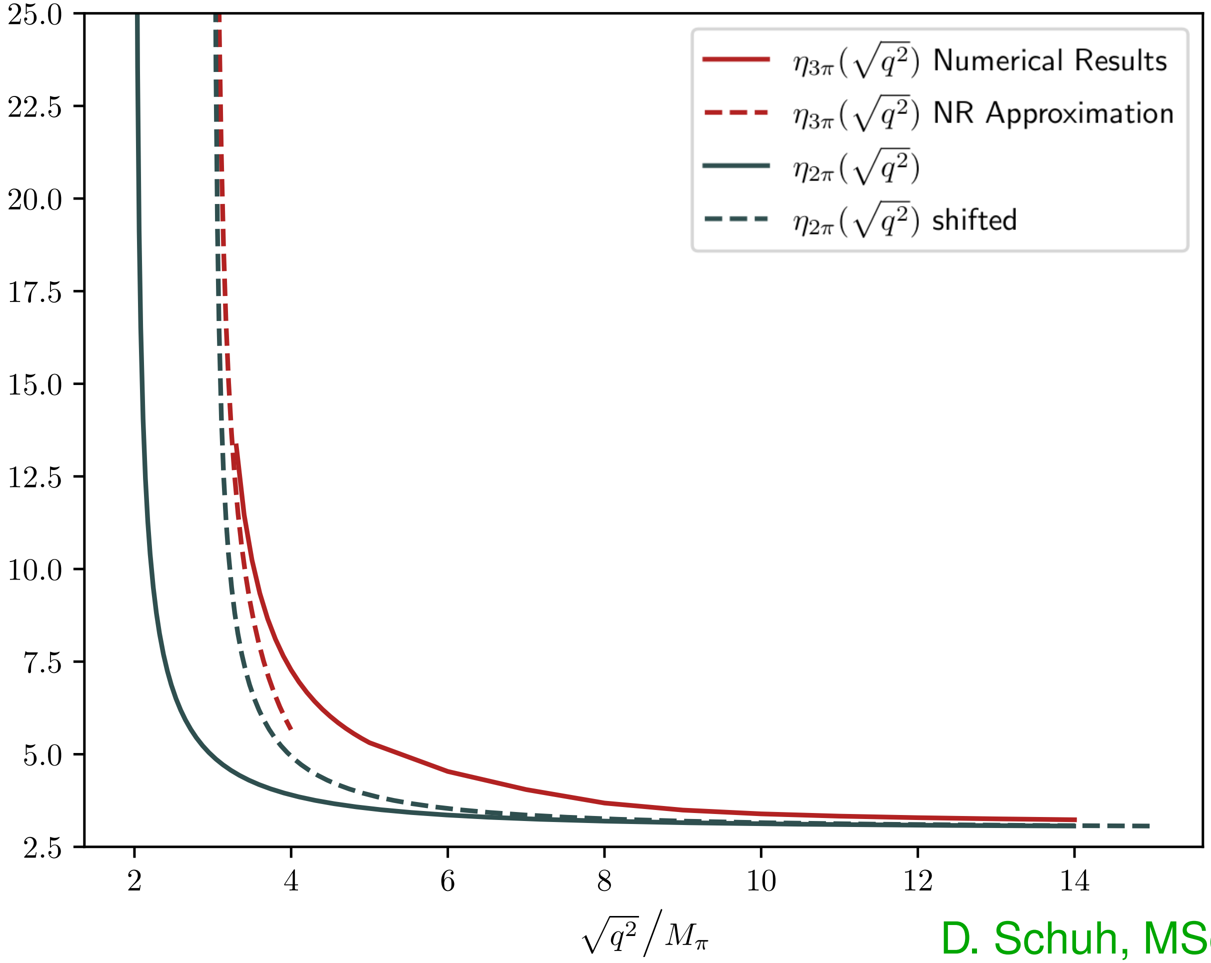
$$\sigma_{3\pi(\gamma)}(q^2) \propto \int_{s_-}^{s_+} \mathrm{d}s \int_{t_-}^{t_+} \mathrm{d}t\, K(s,t;q^2) \big|\mathcal{F}(s,t,u,q^2)\big|^2 \left(1+\frac{\alpha}{\pi}\eta_{2\pi}(s)\right)$$

$\longrightarrow$ difference is negligible

- define $\eta_{3\pi}(q^2)$ from ratio

$$\frac{\sigma_{3\pi(\gamma)}(q^2)}{\sigma^0_{3\pi}(q^2)} \equiv 1+\frac{\alpha}{\pi}\eta_{3\pi}(q^2)$$

Radiative corrections in 3π: results



D. Schuh, MSc thesis 2023

- near-threshold behaviour of $\eta_{3\pi}(q^2)$ cross-checked by analytic nonrelativistic expansion
- comparison to (shifted) $\eta_{2\pi}(q^2)$: nontrivial Dalitz plot effects

Summary / Result

- radiative corrections to $\gamma^* \to 3\pi$ not calculable in scalar QED
- ChPT insufficient: resonance-rich in relevant energy range
- infrared enhanced corr.: $\eta_{2\pi}$ applied to $\pi^+\pi^-$ invariant mass
  $\eta_{3\pi}$ correction from numerical Dalitz plot integration
- estimate $a_\mu^{3\pi\gamma}$:

$$a_\mu^{2\pi}|_{\leq 1\,\text{GeV}} = 495.0(2.6) \times 10^{-10} \qquad a_\mu^{3\pi}|_{\leq 1.8\,\text{GeV}} = 46.2(8) \times 10^{-10}$$

$$a_\mu^{2\pi\gamma}|_{\leq 0.95\,\text{GeV}} = 4.34(4) \times 10^{-10} \qquad a_\mu^{3\pi\gamma}|_{\leq 1.8\,\text{GeV}} = 0.47(1) \times 10^{-10}$$

Colangelo, Hoferichter, Stoffer 2018; Hoferichter, Hoid, BK 2019
Moussallam 2013; Schuh 2023

$\longrightarrow$ expected effect in the 1% range

- contributes to broader analysis of isospin breaking in 3π channel

in progress

Spares

Dispersive representation $\gamma^* \to 3\pi$

Unitarity relation for $\mathcal{F}(s, q^2)$:

$$\text{disc}\,\mathcal{F}(s,q^2) = 2i\{\underbrace{\mathcal{F}(s,q^2)}_{\text{right-hand cut}} + \underbrace{\hat{\mathcal{F}}(s,q^2)}_{\text{left-hand cut}}\} \times \theta(s - 4\,M_\pi^2) \times \sin\,\delta_1^1(s)\,e^{-i\delta_1^1(s)}$$

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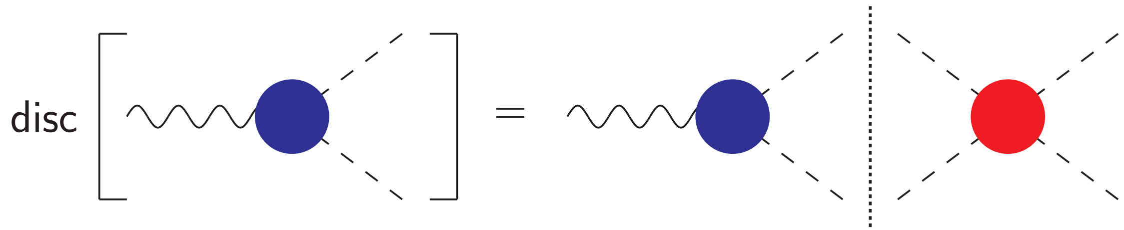
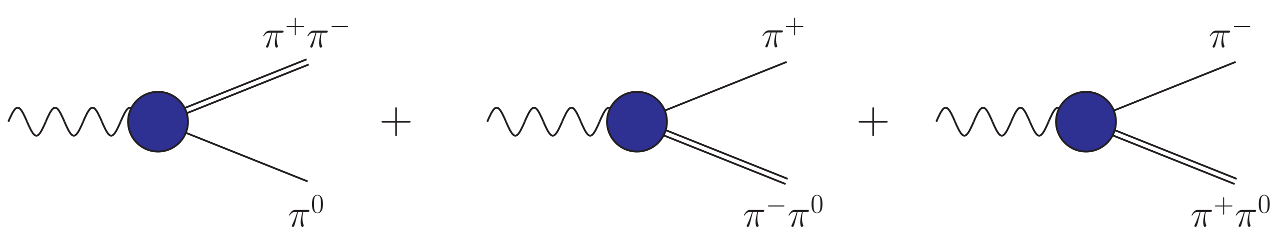


- right-hand cut only $\longrightarrow$ Omnès problem

$$\mathcal{F}(s,q^2) = a(q^2)\,\Omega(s)\;, \qquad \Omega(s) = \exp\left\{ \frac{s}{\pi}\int_{4M_\pi^2}^{\infty} \frac{ds'}{s'} \frac{\delta_1^1(s')}{s'-s} \right\}$$

$\longrightarrow$ amplitude given in terms of pion vector form factor

$$\mathcal{F}(s,t,u;q^2) =$$



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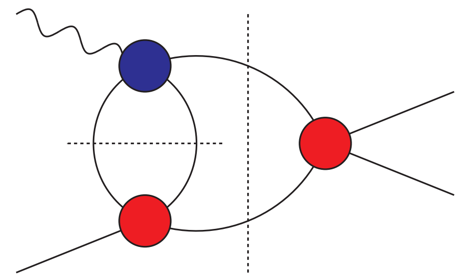
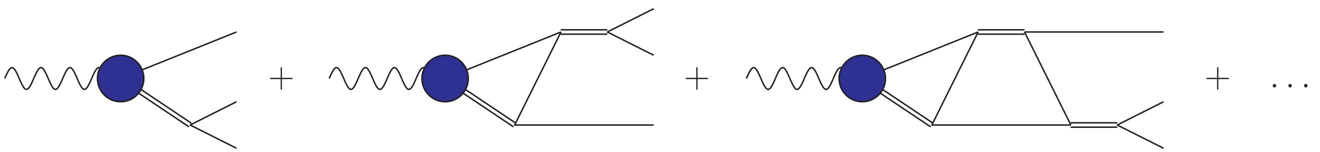


- inhomogeneities $\hat{\mathcal{F}}(s,q^2)$: angular averages over the $\mathcal{F}(t)$, $\mathcal{F}(u)$

$$\mathcal{F}(s,q^2) = a(q^2)\,\Omega(s)\left\{1 + \frac{s^2}{\pi}\int_{4M_\pi^2}^\infty \frac{ds'}{s'^2}\frac{\sin\delta_1^1(s')\hat{\mathcal{F}}(s',q^2)}{|\Omega(s')|(s'-s)}\right\}$$

$$\hat{\mathcal{F}}(s,q^2) = \frac{3}{2}\int_{-1}^{1} dz\,(1-z^2)\mathcal{F}\big(t(s,z),q^2\big)$$

$\mathcal{F}(s,q^2) =$ 

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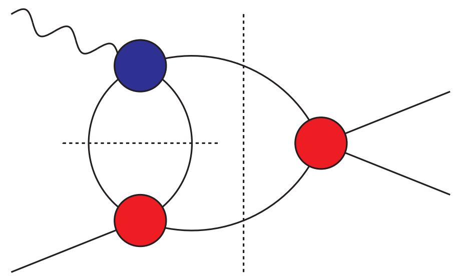

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- crossed-channel scatt. between s-, t-, u-channel (left-hand cuts)

Dispersive representation $\gamma^* \to 3\pi$

- parameterisation of subtraction function $a(q^2)$

  $\longrightarrow$ to be fitted to $e^+e^- \to 3\pi$ cross section data:

$$a(q^2) = \frac{F_{3\pi}}{3} + \frac{q^2}{\pi}\int_{\rm thr}^{\infty} ds' \frac{\operatorname{Im}\mathcal{A}(s')}{s'(s'-q^2)} + C_n(q^2)$$

- $\mathcal{A}(q^2)$ includes resonance poles:

$$\mathcal{A}(q^2) = \sum_V \frac{c_V}{M_V^2 - q^2 - i\sqrt{q^2}\Gamma_V(q^2)} \qquad V = \omega, \phi, \omega', \omega''$$

  c_V real

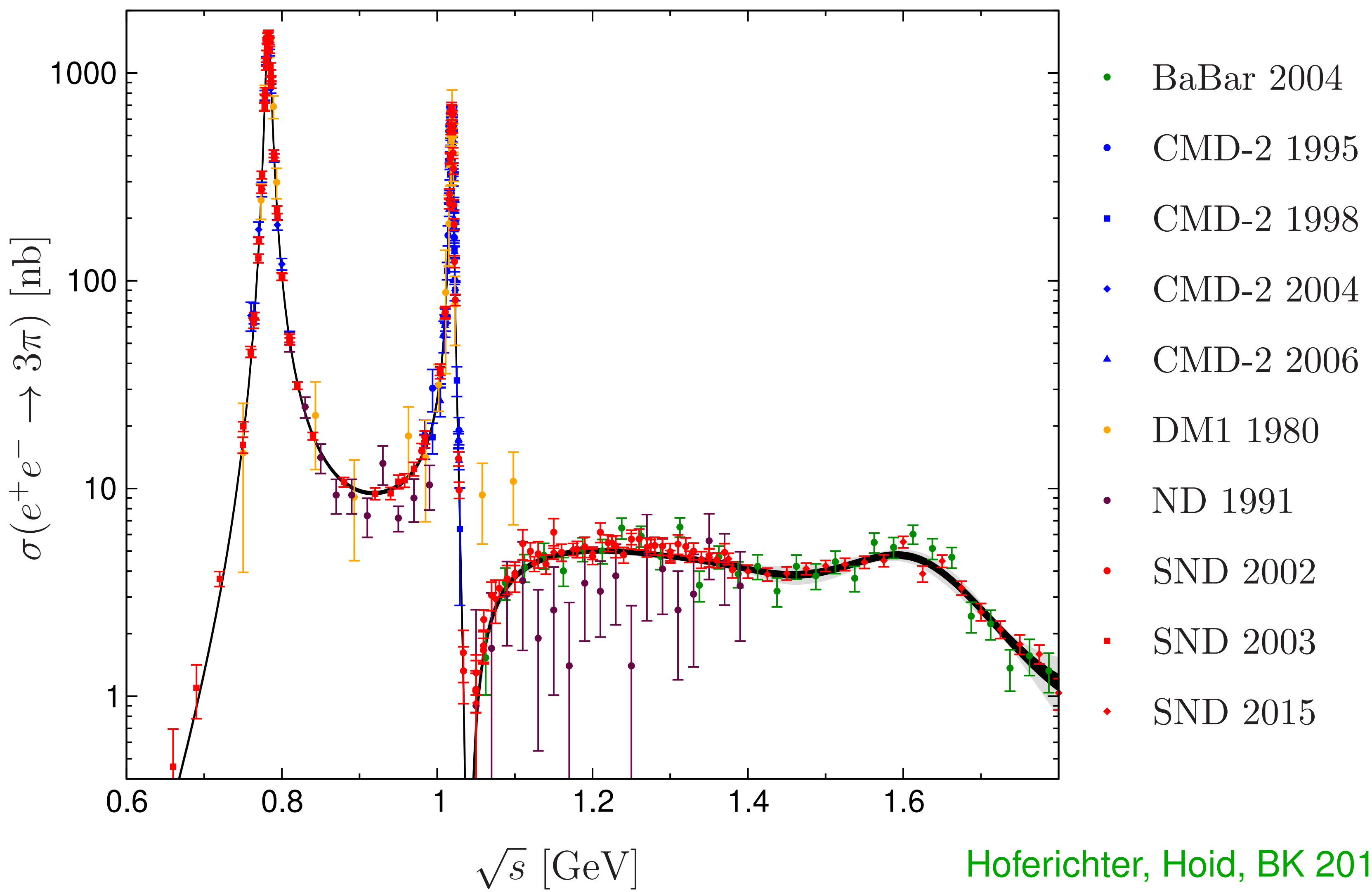
- conformal polynomial (inelasticities); S-wave cusp eliminated:

$$C_n(q^2) = \sum_{i=1}^{n} c_i \Big(z(q^2)^i - z(0)^i \Big), \qquad z(q^2) = \frac{\sqrt{s_{\text{inel}} - s_1} - \sqrt{s_{\text{inel}} - q^2}}{\sqrt{s_{\text{inel}} - s_1} + \sqrt{s_{\text{inel}} - q^2}}$$

- exact implementation of $\gamma^* \to 3\pi$ anomaly:

$$\frac{F_{3\pi}}{3} = \frac{1}{\pi}\int_{s_{\text{thr}}}^{\infty} ds' \frac{\operatorname{Im} a(s')}{s'}$$

Fit results $e^+e^- \to 3\pi$ data up to 1.8 GeV



Hoferichter, Hoid, BK 2019

- black / gray bands represent fit and total uncertainties
- vacuum polarisation removed from the cross section