Dispersive Approach to Massive Two-Loop Amplitudes

Andrea Gurgone

5th Workstop / Thinkstart:

Radiative corrections and Monte Carlo tools for Strong 2020

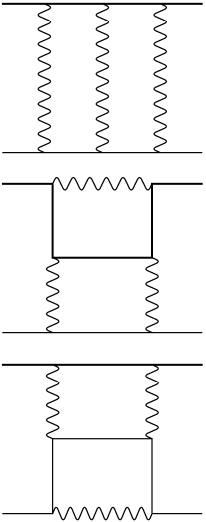
Zurich – 6 June 2023

UNIVERSITÀ DI PAVIA

INFN PAVIA

Introduction

- Goal: Evaluation of **two-loop box** amplitudes with **full mass** dependency.
- We need a compromise between an (insane) analytical calculation and an (inefficient) complete numerical computation.
- General idea: Reduce one of the two sub-loops in a self-energy by introducing two **Feynman parameters**, so that the corresponding two-point function can be re-written in a propagator-like form by using a **dispersion relation**.
- This leads to a standard one-loop amplitude with a further 3-dimensional integral (2 Feynman parameters + 1 dispersion parameter) to be evaluated numerically.
- Currently considered for $\mu e \to \mu e$ at NNLO in MESMER, but it can be also applied to $ee \to ee$, $ee \to \mu\mu$ and $ee \to \gamma\gamma$.
- No numerical result yet, but let's explain how the method works.



References

1. M. Awramik, M. Czakon and A. Freitas, *Electroweak two-loop corrections to the effective weak mixing angle*, JHEP 11 (2006) 048 [hep-ph/0608099]

2. Q. Song and A. Freitas, *On the evaluation of two-loop electroweak box diagrams for $e^+e^- \to HZ$ production*, JHEP 04 (2021) 179 [2101.00308]

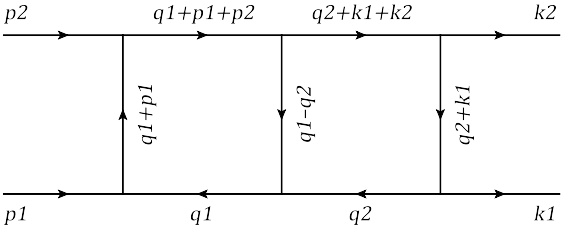
3. A. Freitas and Q. Song, *Two-Loop Electroweak Corrections with Fermion Loops to $e^+e^- \to ZH$*, Phys. Rev. Lett. 130 (2023) 031801 [2209.07612]

4. A. Aleksejevs, *Dispersion Approach in Two-Loop Calculations*, Phys. Rev. D 98 (2018) 036021 [1804.08914]

5. A. Aleksejevs, *Crossed Topology in Two-Loop Dispersive Approach*, [1809.05592]

Benchmark diagram

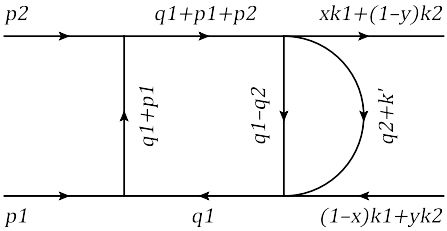
In this talk: scalar two-box diagram $p_1\, p_2 \to k_1\, k_2$ with all masses equal to m.



$$\mathcal{I} = \int d^4q_1\, d^4q_2\, \frac{1}{q_1^2 - m^2 + i\epsilon}\, \frac{1}{(q_1+p_1)^2 - m^2 + i\epsilon}\, \frac{1}{(q_1+p_1+p_2)^2 - m^2 + i\epsilon}$$
$$\times\, \frac{1}{(q_1-q_2)^2 - m^2 + i\epsilon}\, \frac{1}{q_2^2 - m^2 + i\epsilon}\, \frac{1}{(q_2+k_1)^2 - m^2 + i\epsilon}\, \frac{1}{(q_2+k_1+k_2)^2 - m^2 + i\epsilon}$$

Step 1: Feynman parametrisation

We apply the Feynman parametrisation to the three propagators depending on q_2 but not q_1 to reduce the q_2 sub-loop into a self-energy sub-diagram $\longrightarrow$ x, y parameters:



Auxiliary momentum and mass:

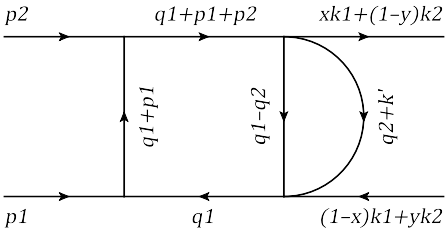
$$k' \equiv (1-x)k_1 + yk_2$$

$$m'^2 \equiv m^2 - xy(k_1+k_2)^2 - (1-x-y)(xk_1^2 + yk_2^2)$$

$$\frac{1}{q_2^2 - m^2 + i\epsilon}\,\frac{1}{(q_2+k_1)^2 - m^2 + i\epsilon}\,\frac{1}{(q_2+k_1+k_2)^2 - m^2 + i\epsilon} = \int_0^1 dx \int_0^{1-x} dy\, \frac{2}{[(q_2+k')^2 - m'^2 + i\epsilon]^3}$$

Step 2: Two-point scalar function

We can now write the q_2 sub-loop in terms of the (derivative of the) two-point scalar function B_0:



External effective momenta:

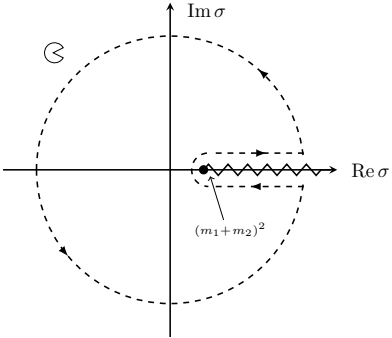
$$k_1' \equiv x k_1 + (1-y) k_2$$

$$k_2' \equiv (1-x) k_1 + y k_2$$

$$\begin{aligned}\mathcal{I}_2 &= \int dx\, dy \int d^4 q_2 \, \frac{1}{(q_1 - q_2)^2 - m^2 + i\epsilon} \frac{2}{\left[(q_2 + k')^2 - m'^2 + i\epsilon\right]^3} \\ &= \int dx\, dy \, \frac{\partial^2}{\partial \left(m'^2\right)^2} \, B_0 \left((q_1 + k')^2, m'^2, m^2 \right)\end{aligned}$$

Dispersion relation for $m'^2 > 0$

If $m'^2 \in \mathbb{R}$ we can write the dispersion relation for B_0 using the usual pac-man contour:



$$B_0\left((q_1+k')^2;\, m'^2,\, m^2\right) =$$

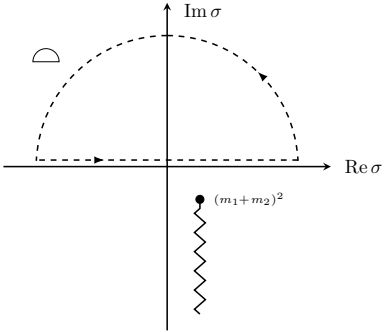
$$= \frac{1}{2\pi i}\oint_{\circlearrowleft} d\sigma\, \frac{B_0\left(\sigma;\, m'^2,\, m^2\right)}{\sigma-(q_1+k')^2-i\epsilon} =$$

$$= \frac{1}{2\pi i}\int_{(m_1+m_2)^2}^{+\infty} d\sigma\, \frac{\mathrm{Disc}\, B_0\left(\sigma;\, m'^2,\, m^2\right)}{\sigma-(q_1+k')^2-i\epsilon}$$

$$\mathrm{Disc}\, B_0\left(\sigma\right) = \lim_{\eta\to 0}\left[B_0\left(\sigma+i\eta\right)-B_0\left(\sigma-i\eta\right)\right] = 2i\,\mathrm{Im}\, B_0\left(\sigma\right) \equiv 2\pi i\,\Delta B_0\left(\sigma\right) = \frac{2\pi i}{\sigma}\sqrt{\lambda\left(\sigma,\, m'^2,\, m^2\right)}$$

Dispersion relation for $m'^2 < 0$

If $m'^2 \in \mathbb{C}$ we have to change the contour accordingly (no branch point on the real axis):



$$B_0\left((q_1+k')^2;\, m'^2,\, m^2\right) =$$

$$= \frac{1}{2\pi i}\oint_{\frown} d\sigma\, \frac{B_0\left(\sigma;\, m'^2,\, m^2\right)}{\sigma - (q_1+k')^2 - i\epsilon}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{+\infty} d\sigma\, \frac{B_0\left(\sigma;\, m'^2,\, m^2\right)}{\sigma - (q_1+k')^2 - i\epsilon}$$

Depending on the values of $(x,\, y)$ the mass m' can be either real or complex $\longrightarrow \int dxdy$ must be splitted in a real and complex region where to apply the correct dispersion relation (but let's focus only on the real case).

Step 3: Dispersion relation

We can re-write the two-point function using the appropriate dispersion relation $\longrightarrow$ σ parameter:

$$\mathcal{I}_2 = \int dx\, dy\, \frac{\partial^2}{\partial\,(m'^2)^2}\, B_0\left((q_1+k')^2, m'^2, m^2\right) = \int dx\, dy\, \frac{\partial^2}{\partial\,(m'^2)^2} \int_{\sigma_0}^{\infty} d\sigma\, \frac{\Delta B_0\left(\sigma;\, m'^2, m^2\right)}{\sigma - \tilde{q}_1^2}$$

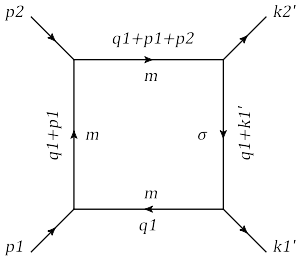
where $\tilde{q}_1 \equiv q_1 + k' + i\epsilon$ and $\sigma_0 \equiv (m+m')^2$

Since $\Delta B_0\left(\sigma_0;\, m'^2, m^2\right) = 0$ we can use the Leibniz rule to move the m'^2 derivative inside the integral, allowing us to re-write the integrand so that the lower bound $\sigma \to \sigma_0$ is not divergent

$$\mathcal{I}_2 = \int dx\, dy \left\{ \int_{\sigma_0}^{\infty} d\sigma\, \partial^2_{m'}\, \Delta B_0\left(\sigma;\, m'^2, m^2\right) \left(\frac{1}{\sigma - \tilde{q}_1^2} - \frac{\sigma_0}{\sigma\,(\sigma_0 - \tilde{q}_1^2)} \right) \right.$$
$$\left. + \frac{\sigma_0}{\sigma_0 - \tilde{q}_1^2}\, \partial^2_{m'}\, B_0\left(0;\, m'^2, m^2\right) \right\}$$

Step 4: One-loop integral

The q_2 sub-loop has been reduced to an effective propagator with momentum $q_1 + k_1'$ and mass σ, reducing the full amplitude to a one-loop integral that can be computed with standard techniques.



$$\mathcal{I} = \int d^4 q_1 \, \frac{1}{q_1^2 - m^2} \, \frac{1}{(q_1 + p_1)^2 - m^2} \, \frac{1}{(q_1 + p_1 + p_2)^2 - m^2} \times \mathcal{I}_2$$

$$= - \int dx \, dy \left\{ \int_{\sigma_0}^{\infty} d\sigma \, \partial_{m'}^2 \, \Delta B_0 \left(\sigma; \, m'^2, m^2\right) \left[D_0 \left(\sigma\right) - \frac{\sigma_0}{\sigma} D_0 \left(\sigma_0\right) \right] \right.$$

$$\left. + \, \sigma_0 \, \partial_{m'}^2 \, \Delta B_0 \left(0; \, m'^2, m^2\right) D_0 \left(\sigma_0\right) \right\}$$

where $D_0 \left(p_1^2, \, p_2^2, \, k_2'^2, \, k_1'^2, \, s, \, t'; \, m^2, \, m^2, \, m^2, \, \sigma\right) \equiv D_0 \left(\sigma\right)$ with $t' \equiv (p_1 - k_1')^2$ is the 4-point scalar function.

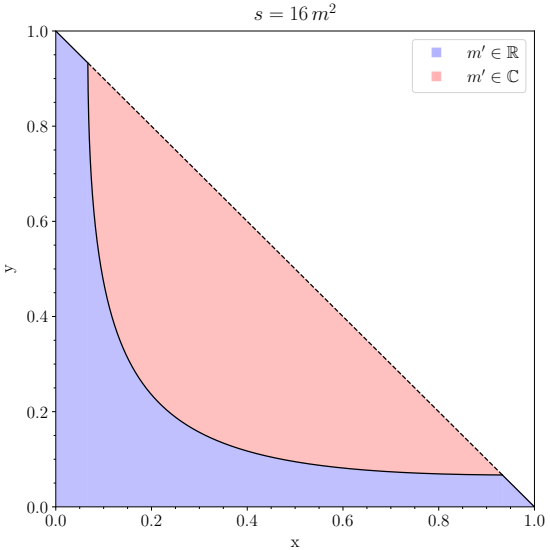
The remaining 3-dimensional integral can be evaluated numerically with an improved efficiency.

Outlook

- We have done the math, so the next step is to obtain stable numerical results for the scalar case and compare them with other methods (WIP).
- We are considering this method for $\mu e \to \mu e$ scattering at NNLO in MESMER, namely for the two-box and box-triangle diagrams.
- The idea can also be applied to other leptonic processes such as $ee \to ee$, $ee \to \mu\mu$ and $ee \to \gamma\gamma$, both for VV at NNLO and RVV at N^{3}LO.
- A comparison with massification can be important to evaluate its uncertainty.

Backup

Real and complex regions of m'



$$m'^2 = [1 - xyz - (1-x-y)(x+y)]\, m^2, \;\; z = s/m$$

When $m'^2 > 0$? $(0 \le x \le 1,\; 0 \le y \le 1-x,\; z > 4)$

$$\begin{cases} 0 \le y \le 1 - x & \text{for} \quad 0 \le x \le x_a \\ 0 \le y \le \frac{1}{2}A(x) - \frac{1}{2}B(x) & \text{for} \quad x_a \le x \le x_b \\ 0 \le y \le 1 - x & \text{for} \quad x_b \le x \le 1 \end{cases}$$

$$A(x) = 1 - 2x + xz$$

$$B(x) = \sqrt{z^2x^2 - 4zx^2 + 2xz - 3}$$

$$x_a = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4}{z}} \qquad x_b = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4}{z}}$$

Step 3b: Lower bound subtraction

$$\mathcal{I}_2 = \int dx\, dy\, \frac{\partial^2}{\partial\left(m'^2\right)^2} \int_{\sigma_0}^{\infty} d\sigma\, \frac{\Delta B_0\left(\sigma;\, m'^2, m^2\right)}{\sigma - \tilde{q}_1^2}$$

$$= \int dx\, dy\, \left\{ \int_{\sigma_0}^{\infty} d\sigma\, \frac{\partial_{m'}^2 \Delta B_0\left(\sigma;\, m'^2, m^2\right)}{\sigma - \tilde{q}_1^2} - \left[\frac{\partial_{m'} \Delta B_0\left(\sigma;\, m'^2, m^2\right)}{\sigma - \tilde{q}_1^2} \right]_{\sigma \to \sigma_0} \right\}$$

$$= \int dx\, dy\, \left\{ \int_{\sigma_0}^{\infty} d\sigma\, \partial_{m'}^2 \Delta B_0\left(\sigma;\, m'^2, m^2\right) \left[\frac{1}{\sigma - \tilde{q}_1^2} + \frac{\sigma_0}{\sigma} \frac{1}{\sigma_0 - \tilde{q}_1^2} - \frac{\sigma_0}{\sigma} \frac{1}{\sigma_0 - \tilde{q}_1^2} \right] \right\}$$

$$= \int dx\, dy\, \left\{ \int_{\sigma_0}^{\infty} d\sigma\, \partial_{m'}^2 \Delta B_0\left(\sigma;\, m'^2, m^2\right) \left(\frac{1}{\sigma - \tilde{q}_1^2} - \frac{\sigma_0}{\sigma\left(\sigma_0 - \tilde{q}_1^2\right)} \right) + \frac{\sigma_0}{\sigma_0 - \tilde{q}_1^2}\, \partial_{m'}^2 B_0\left(0;\, m'^2, m^2\right) \right\}$$