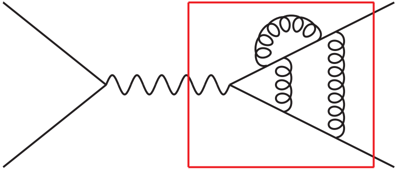
Karlsruhe Institute of Technology

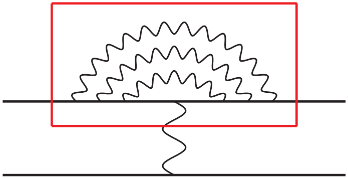
Towards N^{3}LO: three-loop form factors

5th Workstop / Thinkstart: Radiative corrections and Monte Carlo tools for Strong 2020 | June 5 - 9, 2023

Fabian Lange
in collaboration with Matteo Fael, Kay Schönwald, Matthias Steinhauser | June 6, 2023

Motivation





- Form factors are basic building blocks for many physical observables:
  - $t\bar{t}$ production at hadron and e^+e^- colliders
  - Higgs production and decay
  - Low-energy e^+e^- collisions and μe scattering
  - ...
- Form factors exhibit an universal infrared behavior which is interesting to study

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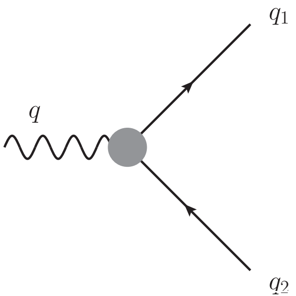
The process

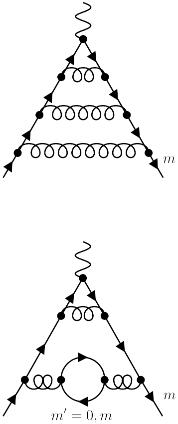


$$X(q) \to Q(q_1) + \bar{Q}(q_2)$$

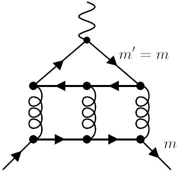
$$q_1^2 = q_2^2 = m^2, \qquad q^2 = s = \hat{s} \cdot m^2$$

vector :	$j_\mu^{\mathrm{v}} = \overline{\psi}\gamma_\mu\psi,$	$\Gamma_\mu^{\mathrm{v}} = F_1^{\mathrm{v}}(s)\gamma_\mu - \frac{\mathrm{i}}{2m}F_2^{\mathrm{v}}(s)\sigma_{\mu\nu}q^\nu$
axial-vector :	$j_\mu^{\mathrm{a}} = \overline{\psi}\gamma_\mu\gamma_5\psi,$	$\Gamma_\mu^{\mathrm{a}} = F_1^{\mathrm{a}}(s)\gamma_\mu\gamma_5 - \frac{1}{2m}F_2^{\mathrm{a}}(s)q_\mu\gamma_5$
scalar :	$j^{\mathrm{s}} = m\overline{\psi}\psi,$	$\Gamma^{\mathrm{s}} = mF^{s}(s)$
pseudo-scalar :	$j^{\mathrm{p}} = \mathrm{i}m\overline{\psi}\gamma_5\psi,$	$\Gamma^{\mathrm{p}} = \mathrm{i}mF^{\mathrm{p}}(s)\gamma_5$

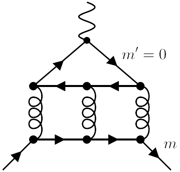
Types of contributions



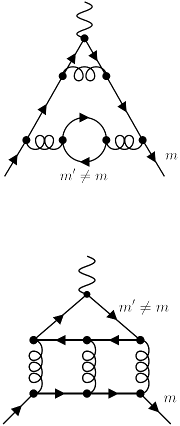
nonsinglet



n_{h} singlet



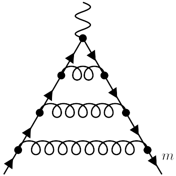
n_{l} singlet



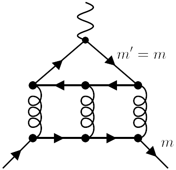
contributions with $m' \neq m$

Status of massive QCD corrections

nonsinglet:



singlet:



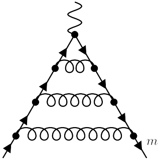
$F_i^{(2)}$ (**NNLO**):

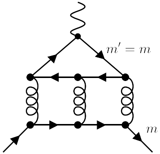
- fermionic contributions [Hoang, Teubner 1997]
- QED [Bonciani, Mastrolia, Remiddi 2003]
- complete (except contributions with $m' \neq m$) [Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi 2004 - 2005]

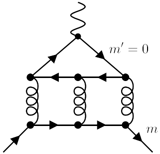
$F_i^{(3)}$ (**NNNLO**):

- nonsinglet large N_c [Henn, Smirnov, Smirnov, Steinhauser 2016; Lee, Smirnov, Smirnov, Steinhauser 2018; Ablinger, Blümlein, Marquard, Rana, Schneider 2 $\times$ 2018; Lee, Smirnov, Smirnov, Steinhauser 2018]
- nonsinglet n_l [Lee, Smirnov, Smirnov, Steinhauser 2018; Ablinger, Blümlein, Marquard, Rana, Schneider 2 $\times$ 2018]
- nonsinglet n_h (partially) [Blümlein, Marquard, Rana, Schneider 2019]
- complete (except contributions with $m' \neq m$) [Fael, **FL**, Schönwald, Steinhauser 2 $\times$ 2022 + 2023]

Setup







	nonsinglet	n_h-singlet	n_l-singlet
diagrams	271	66	66
families	34	17	13
integrals	302671	106883	127980
masters	422	316	158

- Generate diagrams with `qgraf` [Nogueira 1991]
- Map to predefined integral families with `q2e/exp` [Harlander, Seidensticker, Steinhauser 1998; Seidensticker 1999]
- `FORM` [Vermaseren 2000; Kuipers, Ueda, Vermaseren, Vollinga 2013; Ruijl, Ueda, Vermaseren 2017] for Lorentz, Dirac, and color algebra [van Ritbergen, Schellekens, Vermaseren 1998]
- Reduction to master integrals with `Kira` [Maierhöfer, Usovitsch, Uwer 2017; Klappert, **FL**, Maierhöfer, Usovitsch 2020] and `Fermat` [Lewis]
  - Construct good basis where denominators factorize in ϵ and $\hat{s}$ with `ImproveMasters.m` [Smirnov, Smirnov 2020]
- Establish differential equations in $\hat{s}$ with `LiteRed` [Lee 2012 + 2013]

Algorithm to solve master integrals (I)

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$$\frac{\partial}{\partial \hat{s}} M_n = A_{nm}(\epsilon, \hat{s}) M_m$$

- Compute expansion around $\hat{s} = 0$ by:
  - Inserting an ansatz for the master integrals into the differential equation:

$$\boxed{M_n(\epsilon, \hat{s} = 0) = \sum_{i=-3}^{\infty} \sum_{j=0}^{j_{\max}} c_{ij}^{(n)} \, \epsilon^i \, \hat{s}^j}$$

  - Compare coefficients in ϵ and $\hat{s}$ to establish linear system of equations for $c_{ij}^{(n)}$:

$$c_{12}^{(1)} \epsilon \hat{s}^2 + \dots = 52 c_{33}^{(1)} \epsilon \hat{s}^2 + \dots + 127 c_{14}^{(4)} \epsilon \hat{s}^2 + \dots$$

  - Solve system in terms of small number of boundary constants using `Kira` with `FireFly` [Klappert, **FL** 2019; Klappert, Klein, **FL** 2020] :

$$c_{12}^{(1)} = 52 c_{33}^{(1)} + 127 c_{14}^{(4)}$$

- Compute boundary values to fix remaining constants

Algorithm to solve master integrals (II)

$$\frac{\partial}{\partial \hat{s}} M_n = A_{nm}(\epsilon, \hat{s}) M_m$$

- Repeat for $\hat{s} = \hat{s}_1$:
  - Insert an ansatz around $\hat{s} = \hat{s}_1$ into the differential equation:

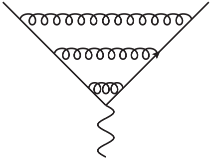
$$M_n(\epsilon, \hat{s} = \hat{s}_1) = \sum_{i=-3}^{\infty} \sum_{j=0}^{j_{\max}} c_{ij}^{(n)} \, \epsilon^i \, (\hat{s} - \hat{s}_1)^j$$

  - Compare coefficients in ϵ and $\hat{s}$ and solve system:

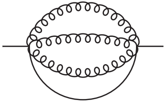
$$c_{12}^{(1)} \epsilon (\hat{s} - \hat{s}_1)^2 + \cdots = 12 c_{44}^{(1)} \epsilon (\hat{s} - \hat{s}_1)^2 + \cdots - 23 c_{04}^{(4)} \epsilon (\hat{s} - \hat{s}_1)^2 + \dots \qquad \Rightarrow \qquad c_{12}^{(1)} = 12 c_{44}^{(1)} - 23 c_{04}^{(4)}$$

  - Match this new expansion to previous expansion around $\hat{s} = 0$ numerically in between, e.g. at $\hat{s}_1/2$, to fix the boundary constants
- Repeat

Calculation of boundary conditions: nonsinglet

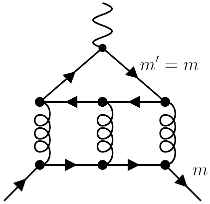


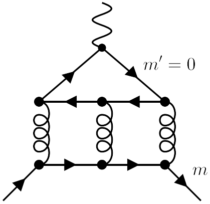
$\xRightarrow{s \to 0}$



- For $s = 0$ the nonsinglet master integrals reduce to 3-loop on-shell propagators:
  - Well studied in the literature [Laporta, Remiddi 1996; Melnikov, van Ritbergen 1999; Lee, Smirnov 2010]
  - Some higher-order terms were missing for our calculation
  - Using the dimensional-recurrence relations from [Lee, Smirnov 2010] we calculated them with `SummerTime.m` [Lee, Mingulov 2015] and `PSLQ` [Ferguson, Bailey, Arno 1999]

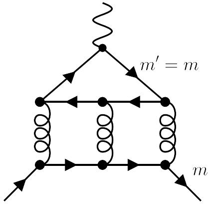
Calculation of boundary conditions: singlet

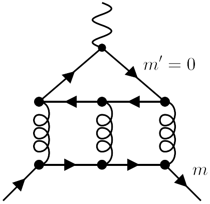




- n_h singlet boundary conditions require asymptotic expansion due to massless cuts:
  - Reveal regions with `ASY.m` [Smirnov, Pak 2010; Jantzen, Smirnov, Smirnov 2012]
  - Naive region same as for nonsinglet
  - Remaining regions can be integrated directly or with `HyperInt` [Panzer 2014]

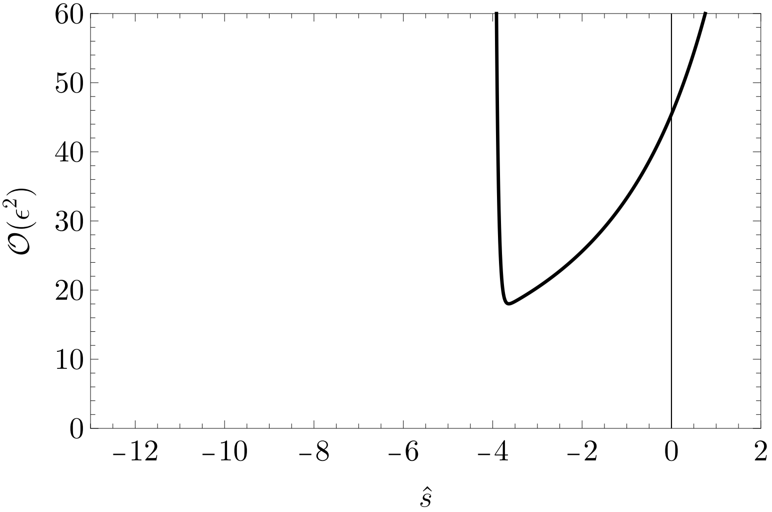
Calculation of boundary conditions: singlet

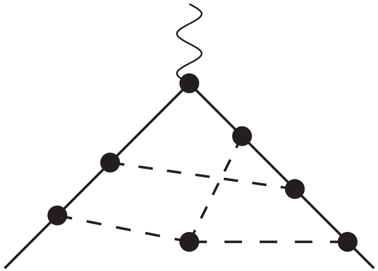




- n_h singlet boundary conditions require asymptotic expansion due to massless cuts:
  - Reveal regions with `ASY.m` [Smirnov, Pak 2010; Jantzen, Smirnov, Smirnov 2012]
  - Naive region same as for nonsinglet
  - Remaining regions can be integrated directly or with `HyperInt` [Panzer 2014]
- n_l singlet boundary conditions:
  - Even more massless cuts and direct integration for some regions too complicated
  - Instead use `AMFlow` [Liu, Ma 2022] to compute them numerically at $\hat{s} = -1$ with 86 digits

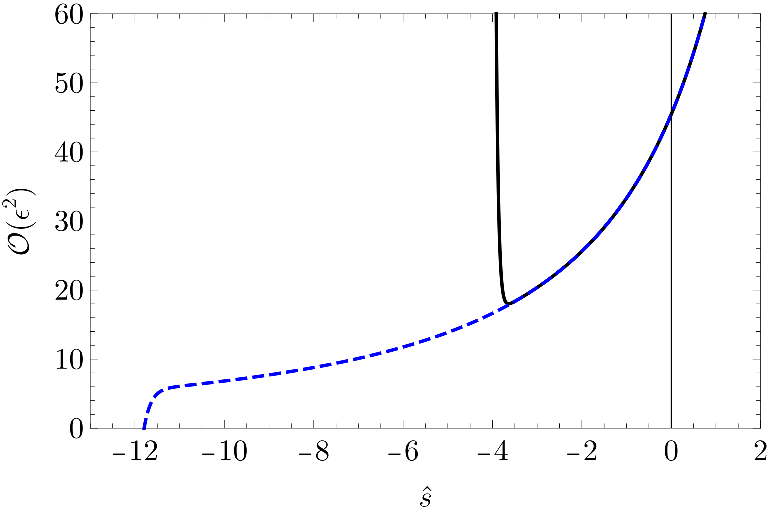
Example

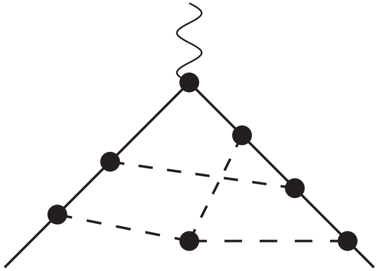




- Expansion around $\hat{s} = 0$

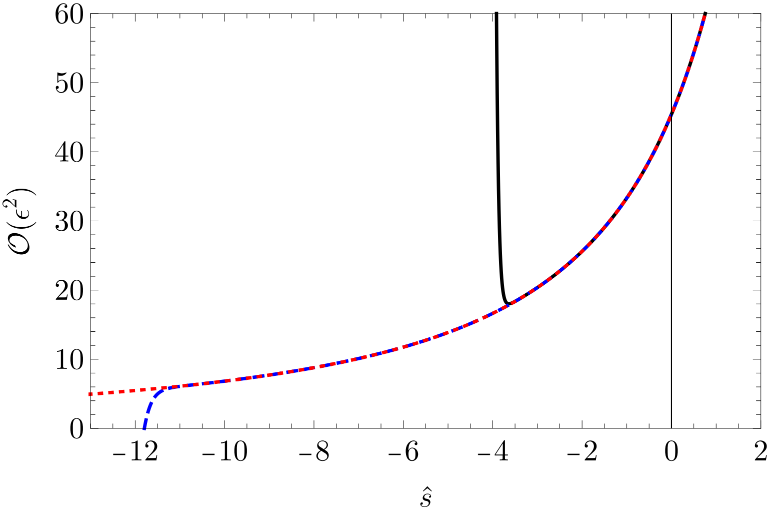
Example

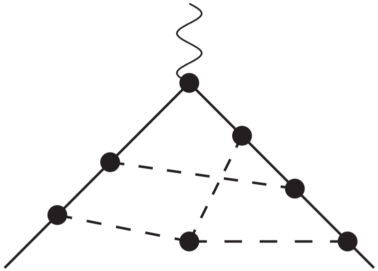




- Expansion around $\hat{s} = 0$
- Expansion around $\hat{s} = -4$, matched at $\hat{s} = -2$

Example





- Expansion around $\hat{s} = 0$
- Expansion around $\hat{s} = -4$, matched at $\hat{s} = -2$
- Expansion around $\hat{s} = -8$, matched at $\hat{s} = -6$

Results – analytic expansion around $\hat{s} = 0$

$$F_1^{\mathrm{v,f,(3)}}(\hat{s}=0) = \Bigg\{ C_F^3\Big(-15a_4 - \frac{17\pi^2\zeta_3}{24} - \frac{18367\zeta_3}{1728} + \frac{25\zeta_5}{8} - \frac{5l_2^4}{8} - \frac{19}{40}\pi^2 l_2^2 + \frac{4957\pi^2 l_2}{720} + \frac{3037\pi^4}{25920} - \frac{24463\pi^2}{7776} + \frac{13135}{20736}\Big) + C_A C_F^2\Big(\frac{19a_4}{2} - \frac{\pi^2\zeta_3}{9} + \frac{17725\zeta_3}{3456} - \frac{55\zeta_5}{32} + \frac{19l_2^4}{48} - \frac{97}{720}\pi^2 l_2^2 + \frac{29\pi^2 l_2}{240} - \frac{347\pi^4}{17280} - \frac{4829\pi^2}{10368} + \frac{707}{288}\Big) + C_A^2 C_F\Big(-a_4 + \frac{7\pi^2\zeta_3}{96} + \frac{4045\zeta_3}{5184} - \frac{5\zeta_5}{64} - \frac{l_2^4}{24} + \frac{67}{360}\pi^2 l_2^2 - \frac{5131\pi^2 l_2}{2880} + \frac{67\pi^4}{8640} + \frac{172285\pi^2}{186624} - \frac{7876}{2187}\Big)\Bigg\}\hat{s} + \text{fermionic corrections} + \mathcal{O}(\hat{s}^2)$$

- $l_2 = \ln(2)$, $a_4 = \mathrm{Li}_4(1/2)$ and $C_A = 3$, $C_F = 4/3$ for QCD
- Expansions for all currents are available up to $\mathcal{O}(\hat{s}^{67})$

Results – high-energy limit

$$
\begin{aligned}
F_1^{\mathrm{v,f},(3)}\Big|_{s\to-\infty} &= 4.7318C_F^3 - 20.762C_F^2C_A + 8.3501C_FC_A^2 + \Big[3.4586C_F^3 - 4.0082C_F^2C_A - 6.3561C_FC_A^2\Big]l_s \\
&+ \Big[1.4025C_F^3 + 0.51078C_F^2C_A - 2.2488C_FC_A^2\Big]l_s^2 + \Big[0.062184C_F^3 + 0.90267C_F^2C_A - 0.42778C_FC_A^2\Big]l_s^3 \\
&+ \Big[-0.075860C_F^3 + 0.20814C_F^2C_A - 0.035011C_FC_A^2\Big]l_s^4 + \Big[-0.023438C_F^3 + 0.019097C_F^2C_A\Big]l_s^5 \\
&+ \Big[-0.0026042C_F^3\Big]l_s^6 - \Big\{-92.918C_F^3 + 123.65C_F^2C_A - 47.821C_FC_A^2 + \Big[-10.381C_F^3 + 2.3223C_F^2C_A \\
&+ 17.305C_FC_A^2\Big]l_s + \Big[4.9856C_F^3 - 19.097C_F^2C_A + 8.0183C_FC_A^2\Big]l_s^2 + \Big[3.0499C_F^3 - 6.8519C_F^2C_A + 1.9149C_FC_A^2\Big]l_s^3 \\
&+ \Big[0.67172C_F^3 - 0.91213C_F^2C_A + 0.24069C_FC_A^2\Big]l_s^4 + \Big[0.13229C_F^3 - 0.051389C_F^2C_A + 0.0043403C_FC_A^2\Big]l_s^5 \\
&+ \Big[0.0041667C_F^3 - 0.0010417C_F^2C_A - 0.00052083C_FC_A^2\Big]l_s^6\Big\}\frac{m^2}{s} + \mathcal{O}\left(\frac{m^4}{s^2}\right) + \text{fermionic contributions}
\end{aligned}
$$

Results – high-energy limit

$$
\begin{aligned}
F_1^{\mathrm{v,f},(3)}\Big|_{s\to-\infty} &= 4.7318C_F^3 - 20.762C_F^2C_A + 8.3501C_FC_A^2 + \Big[3.4586C_F^3 - 4.0082C_F^2C_A - 6.3561C_FC_A^2\Big]l_s \\
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&+ \boxed{\Big[-0.0026042C_F^3\Big]l_s^6} - \Big\{ -92.918C_F^3 + 123.65C_F^2C_A - 47.821C_FC_A^2 + \Big[-10.381C_F^3 + 2.3223C_F^2C_A \\
&+ 17.305C_FC_A^2\Big]l_s + \Big[4.9856C_F^3 - 19.097C_F^2C_A + 8.0183C_FC_A^2\Big]l_s^2 + \Big[3.0499C_F^3 - 6.8519C_F^2C_A + 1.9149C_FC_A^2\Big]l_s^3 \\
&+ \Big[0.67172C_F^3 - 0.91213C_F^2C_A + 0.24069C_FC_A^2\Big]l_s^4 + \Big[0.13229C_F^3 - 0.051389C_F^2C_A + 0.0043403C_FC_A^2\Big]l_s^5 \\
&+ \boxed{\Big[0.0041667C_F^3 - 0.0010417C_F^2C_A - 0.00052083C_FC_A^2\Big]l_s^6}\Big\}\frac{m^2}{s} + \mathcal{O}\left(\frac{m^4}{s^2}\right) + \text{fermionic contributions}
\end{aligned}
$$

- Dedicated calculation of leading logarithms [Liu, Penin, Zerf 2017] :

$$
F_1^{\mathrm{v,f},(3)} = -\frac{C_F^3}{384}l_s^6 - \frac{m^2}{s}\left(\frac{C_F^3}{240} - \frac{C_F^2C_A}{960} - \frac{C_FC_A^2}{1920}\right)l_s^6 + \ldots, \quad \text{with } l_s = \ln\left(\frac{m^2}{-s}\right)
$$

- We reproduce these terms with high precision

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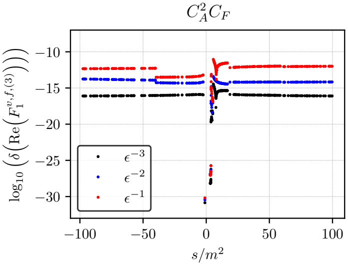
Results – pole cancellation

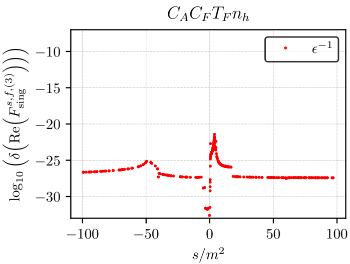
- We use the pole cancellation to estimate the precision
- To estimate the number of significant digits we use

$$\log_{10}\left(\left|\frac{\text{expansion} - \text{analytic CT}}{\text{analytic CT}}\right|\right)$$

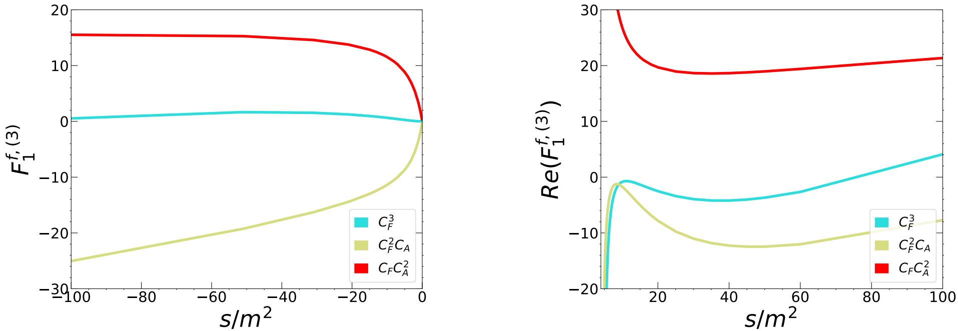
$\Rightarrow$ We estimate at least 8 correct digits for the finite terms for QCD and 10 correct digits for QED

- Most regions for most color factors and especially singlet contributions much more precise





Results – some plots: nonsinglet



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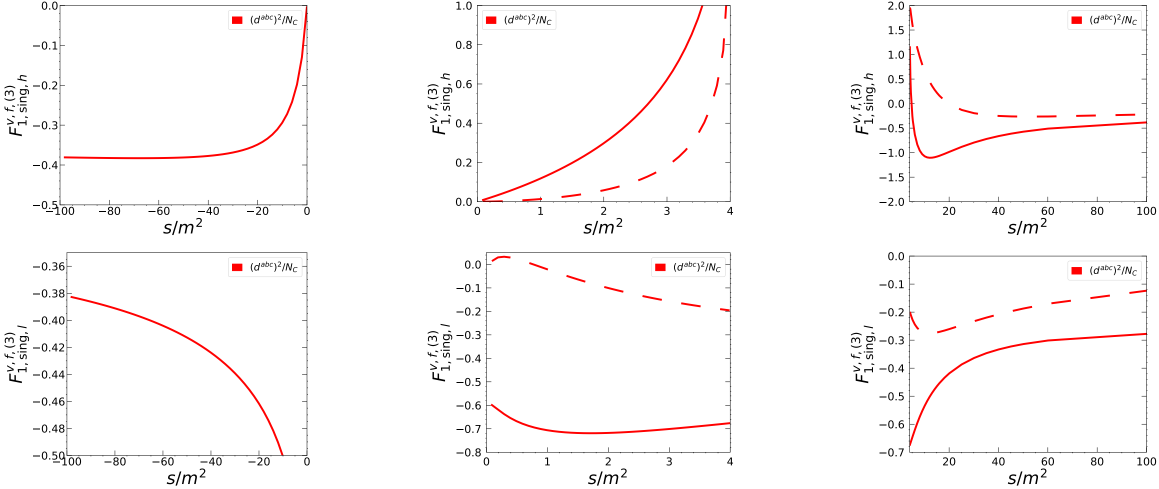
Results – some plots: singlet



Public implementation – `Mathematica`

- Bare as well as both ultraviolet and infrared finite form factors implemented as grids for `Mathematica`: `formfactors3l`

```
Get["FormFactors3l.m"]

In[]  := FormFactorBareNonSing[veF1, 0, -1]
Out[] := 77.0506 cA^2 cR+95.0634 cA cR^2+0.467466 cR^3
         -21.9243 cA cR I2R nh-11.5582 cR^2 I2R nh+0.751403 cR I2R^2 nh^2
         -62.6063 cA cR I2R nl-45.5408 cR^2 I2R nl+9.35837 cR I2R^2 nh nl
         +11.8102 cR I2R^2 nl^2

In[]  := FormFactorRenNonSing[veF1, -1]
Out[] := 3.10714 cA^2 cR-3.23413 cA cR^2+0.0144347 cR^3
         +0.0435081 cA cR I2R nh-0.0640418 cR^2 I2R nh
         -0.0107609 cR I2R^2 nh^2-2.59041 cA cR I2R nl
         +1.02032 cR^2 I2R nl+0.000282528 cR I2R^2 nh nl
         +0.494057 cR I2R^2 nl^2
```

Public implementation – Fortran

- Ultraviolet renormalized, infrared unsubtracted form factors implemented as grids in `Fortran` library `ff3l`
- Specialization to QED by adding suffix `_qed` to function calls
- Ready for Monte-Carlo tools

```
program example1
  use ff3l
  implicit none

  double complex :: f1v
  double precision :: s = 10
  integer :: eporder

  do eporder = -3,0
    f1v = ff3l_veF1(s,eporder)
    print *,"F1( s = ",s,", ep = ",eporder," ) = ", f1v
  enddo
end program example1
```

Summary

- Calculated massive form factors at NNLO in QCD
- Applied a semianalytic method by constructing series expansions and matching numerically
- Reproduce known results from the literature:
  - large-N_c limit, n_l, and partial n_h contributions
  - static, high-energy, and threshold expansions
- Checked chiral Ward identity for singlet contributions
- Estimate precision to at least 8 significant digits over the whole real axis for QCD and 10 significant digits for QED
- Results available as grids for both `Mathematica` and `Fortran`

Outlook: contributions with a second mass $m' \neq m$ (extremely preliminary)

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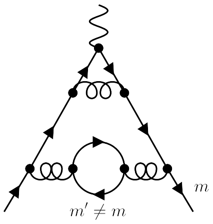
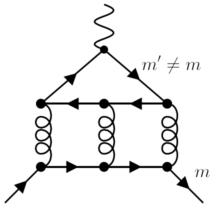
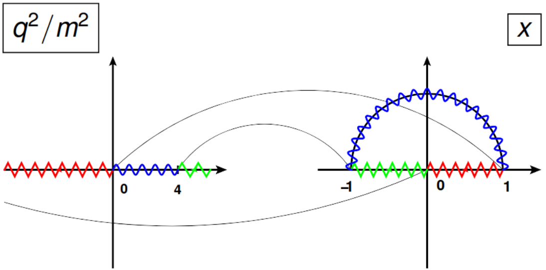




- NNLO:
  - 1 bubble diagram
  - Can be expressed through dispersion integral over vacuum polarization as for the hadronic corrections [Fael 2018]
  - Leptonic vacuum polarization known through four loops in QED [Sturm 2013] and top contribution to three loops in QCD [Chetyrkin, Kühn, Steinhauser 1995 + 1996]
- NNNLO:
  - 42 nonsinglet diagrams, some of them with topologies beyond bubbles
  - 152 master integrals
  - Boundaries at $s = 0$ available, but already complicated functions of ratio $\frac{m'}{m}$ [Fael, Schönwald, Steinhauser 2020]
  - Then symbolic expansions in $\frac{m'}{m}$ and $\frac{s}{m}$ and match numerically?
  - Or solve numerically in relevant range $\frac{s}{m}$ for fixed $\frac{m'}{m}$ like [Boughezal, Czakon, Schutzmeier 2007; Czakon, Niggetiedt 2020] ?

Why numerical?

$$\boxed{q^2 = s = -\frac{(1-x)^2}{x}}$$



- Large-N_c and n_l contributions at NNNLO can be written as iterated integrals over letters

$$\frac{1}{x}, \ \frac{1}{1+x}, \ \frac{1}{1-x}, \ \frac{1}{1-x+x^2}, \ \frac{x}{1-x+x^2}$$

- n_h terms already contain structures beyond iterated integrals (elliptic integrals)

$\Rightarrow$ No ready-to-use tools available for analytic solution

$\Rightarrow$ Instead: Full solution through analytic series expansions and numerical matching

Series expansions

- Different ansätze for different points:

regular point: $$M_n(\epsilon, \hat{s} = \hat{s}_0) = \sum_{i=-3}^{\infty} \sum_{j=0}^{j_{\max}} c_{ij}^{(n)} \, \epsilon^i \, (\hat{s} - \hat{s}_0)^j$$

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$s = \pm\infty$ (high-energy limit): $$M_n(\epsilon, \hat{s} \to \pm\infty) = \sum_{i=-3}^{\infty} \sum_{j=-s_{\min}}^{j_{\max}} \sum_{k=0}^{i+6} c_{ijk}^{(n)} \, \epsilon^i \, \hat{s}^{-j} \, \ln^k(\hat{s})$$

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$s = 4m^2$ (2-particle threshold):
$$M_n(\epsilon, \hat{s} = 4) = \sum_{i=-3}^{\infty} \sum_{j=-s_{\min}}^{j_{\max}} \sum_{k=0}^{i+3} c_{ijk}^{(n)} \, \epsilon^i \, \left[\sqrt{4 - \hat{s}}\right]^j \ln^k\left(\sqrt{4 - \hat{s}}\right)$$

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$s = 16m^2$ (4-particle threshold): $M_n(\epsilon, \hat{s} = 16) = \sum_{i=-3}^{\infty} \sum_{j=-s_{\min}}^{j_{\max}} \sum_{k=0}^{i+3} c_{ijk}^{(n)}\, \epsilon^i\, \left[\sqrt{16 - \hat{s}}\right]^j \ln^k\left(\sqrt{16 - \hat{s}}\right)$

Series expansions

- Different ansätze for different points:

regular point: $$M_n(\epsilon, \hat{s} = \hat{s}_0) = \sum_{i=-3}^{\infty} \sum_{j=0}^{j_{\max}} c_{ij}^{(n)} \, \epsilon^i \, (\hat{s} - \hat{s}_0)^j$$

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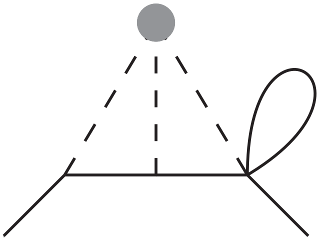
- We construct expansions up to $j_{\max} = 50$ around

$$\hat{s} = \{-\infty, -32, -28, -24, -16, -12, -8, -4, 0, 1, 2, 5/2, 3, 7/2, 4, \\ 9/2, 5, 6, 7, 8, 10, 12, 14, 15, 16, 17, 19, 22, 28, 40\}$$

and similar for the n_h-singlet contributions

Calculation of boundary conditions – n_h-singlets



- The singlet diagrams can have massless cuts, therefore the limit $\hat{s} \to 0$ demands an asymptotic expansion.
- We reveal regions with `ASY.m` [Smirnov, Pak 2010; Jantzen, Smirnov, Smirnov 2012] ($y = \sqrt{-\hat{s}}$):
  - ✓ $y^{-0\epsilon}$: taylor expansion of the integrand, same as for the non-singlet
  - ✓ $y^{-2\epsilon}$: integrals can be performed for general ϵ in terms of Γ functions
  - ✓ $y^{-4\epsilon}$: one integral was calculated using `HyperInt` [Panzer 2014]

$\Rightarrow$ We obtain analytic boundary conditions in the limit $\hat{s} \to 0$.

Calculation of boundary conditions – n_l-singlets

- The singlet diagrams can have massless cuts, therefore the limit $\hat{s} \to 0$ demands an asymptotic expansion.
- We reveal regions with `ASY.m` [Smirnov, Pak 2010; Jantzen, Smirnov, Smirnov 2012] ($y = \sqrt{-\hat{s}}$):
  - ✓ $y^{-0\epsilon}$: taylor expansion of the integrand, same as for the non-singlet
  - ✓ $y^{-2\epsilon}$: integrals can be performed for general ϵ in terms of Γ functions
  - ✓ $y^{-4\epsilon}$: integrals can be performed with `HyperInt` and Mellin-Barnes methods
  - ✗ $y^{-6\epsilon}$: direct integration for some integrals quite involved

$\Rightarrow$ For the n_l-singlets we changed strategy and calculated the masters at $\hat{s} = -1$ with `AMFLow` [Liu, Ma 2022] and matched from there.

Moebius Transformations

- The radius of convergence is at most the distance to the closest singularity.
- We can extend the radius of convergence by changing to a new expansion variable.
- If we want to expand around the point x_k with the closest singularities at x_{k-1} and x_{k+1}, we can use:

$$y_k = \frac{(x - x_k)(x_{k+1} - x_{k-1})}{(x - x_{k+1})(x_{k-1} - x_k) + (x - x_{k-1})(x_{k+1} - x_k)}$$

- The variable change maps $\{x_{k-1}, x_k, x_{k+1}\} \to \{-1, 0, 1\}$.

Renormalization and infrared structure

UV renormalization

- $\overline{\mathrm{MS}}$ renormalization of α_s
- On-shell renormalization of mass Z_m^{OS}, wave function Z_2^{OS}, and (if needed) currents [Chetyrkin, Steinhauser 1999; Melnikov, van Ritbergen 2000]
- Much more involved renormalization for the axial and pseudoscalar singlet contributions

IR subtraction

- Structure of infrared poles given by cusp anomalous dimension Γ_{cusp} [Grozin, Henn, Korchemski, Marquard 2014]
- Define finite form factors $F = Z_{\text{IR}} F^{\text{finite}}$ with UV-renormalized form factor F and

$$Z_{\text{IR}} = 1 - \frac{\alpha_s}{\pi}\frac{1}{2\epsilon}\Gamma_{\text{cusp}}^{(1)} - \left(\frac{\alpha_s}{\pi}\right)^2\left(\frac{\cdots}{\epsilon^2} + \frac{1}{4\epsilon}\Gamma_{\text{cusp}}^{(2)}\right) - \left(\frac{\alpha_s}{\pi}\right)^3\left(\frac{\cdots}{\epsilon^3} + \frac{\cdots}{\epsilon^2} + \frac{1}{6\epsilon}\Gamma_{\text{cusp}}^{(3)}\right)$$

- $\Gamma_{\text{cusp}} = \Gamma_{\text{cusp}}(x)$ depends on kinematics
- Γ_{cusp} universal for all currents

Treatment of γ_5

- For non-singlet diagrams always an even number of γ_5 matrices appear on a fermion line.
  $\Rightarrow$ Use anti-commuting γ_5.
- In the singlet diagrams odd numbers of γ_5 appear on a fermion line.
  $\Rightarrow$ Use Larin's prescription [Larin 1992] :

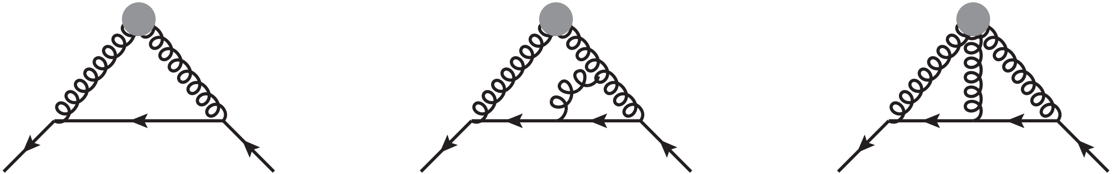
$$\boxed{\gamma_\mu \gamma_5 \to \frac{1}{3!} \epsilon_{\mu\nu\rho\sigma} \gamma^\nu \gamma^\rho \gamma^\sigma},$$

  where the contraction of two ϵ tensors is done in $d = 4 - 2\epsilon$ dimensions.

✓ Finite (multiplicative) renormalization constants for all currents are known.

- Only the sum of singlet and non-singlet diagrams renormalizes multiplicative, so the non-singlet has to be calculated in the Larin scheme as well (we use this as a cross-check).

Chiral Ward identity



- The non-renormalization of the Adler-Bell-Jackiw (ABJ) anomaly implies:

$$\boxed{\left(\partial^\mu j^a_\mu\right)_{\mathrm{R}} = 2\left(j^p\right)_{\mathrm{R}} + \frac{\alpha_s}{4\pi} T_F \left(G\tilde{G}\right)_{\mathrm{R}}},$$

  with the pseudoscalar gluonic operator $G\tilde{G} = \epsilon_{\mu\nu\rho\sigma} G^{a,\mu\nu} G^{a,\rho\sigma}$
- This relation can be used to check the correct treatment of γ_5.
- For the form factors this leads to the identity:

$$\boxed{F^{a,f}_{\mathrm{sing},1} + \frac{s}{4m^2} F^{a,f}_{\mathrm{sing},2} = F^{p,f}_{\mathrm{sing}} + \frac{\alpha_s}{4\pi} T_F F^f_{G\tilde{G}}}$$

- We calculated the form factor associated to $G\tilde{G}$ up to $\mathcal{O}(\alpha_s^2)$ for this check.

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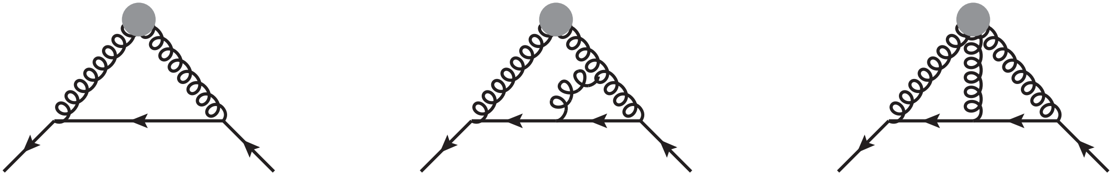
Chiral Ward identity



- The new topologies introduce 3 (1), 24 (15) master integrals (new wrt. the form factor calculation).
- We calculate the masters by the algorithm outlined in [Ablinger, Blümlein, Marquard, Rana, Schneider 2018] :
  1. Uncouple coupled blocks of the differential equation into a higher order one with `OreSys` [Gerhold 2002] and `Sigma` [Schneider 2007] .
  2. Solve the higher order differential equations via the factorization of the differential operator with `HarmonicSums` [Ablinger 2011-] .
  3. The boundary conditions can be found by direct integration in the asymptotic limit $\hat{s} \to 0$.
- We can express the result up to $\mathcal{O}(\alpha_s^2)$ in terms of harmonic polylogarithms. [Remiddi, Vermseren 1999]

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Results – threshold expansion around $s = 4m^2$

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- Close to threshold we can construct cross-sections and decay rates like

$$\sigma(e^+e^- \to Q\bar{Q}) = \sigma_0 \beta \underbrace{\left(\left| F_1^v + F_2^v \right|^2 + \frac{\left|(1-\beta^2)F_1^v + F_2^v\right|^2}{2(1-\beta^2)} \right)}_{=3/2\,\Delta^v}$$

with the quark velocity $\beta = \sqrt{1 - 4m^2/s}$

- Real radiation suppressed by β^3

$\Rightarrow$ Direct phenomenological relevance

- We find (with $l_{2\beta} = \ln(2\beta)$)

$$\begin{aligned}\Delta^{v,(3)} &= C_F^3 \left[-\frac{32.470}{\beta^2} + \frac{1}{\beta}\left(14.998 - 32.470 l_{2\beta}\right) \right] + C_A^2 C_F \frac{1}{\beta}\left[16.586 l_{2\beta}^2 - 22.572 l_{2\beta} + 42.936 \right] \\ &+ C_A C_F^2 \left[\frac{1}{\beta^2}\left(-29.764 l_{2\beta} - 7.7703\right) + \frac{1}{\beta}\left(-12.516 l_{2\beta} - 11.435\right) \right] \\ &+ \mathcal{O}(\beta^0) + \text{fermionic contributions}\end{aligned}$$

- Agrees with dedicated calculation [Kiyo, Maier, Maierhöfer, Marquard 2009]