MODERN METHODS FOR THE COMPUTATION OF SCATTERING AMPLITUDES

Giuseppe De Laurentis

LTP Seminar

PAUL SCHERRER INSTITUT

Find these slides at gdelaurentis.github.io/slides/psi-ltp-seminar

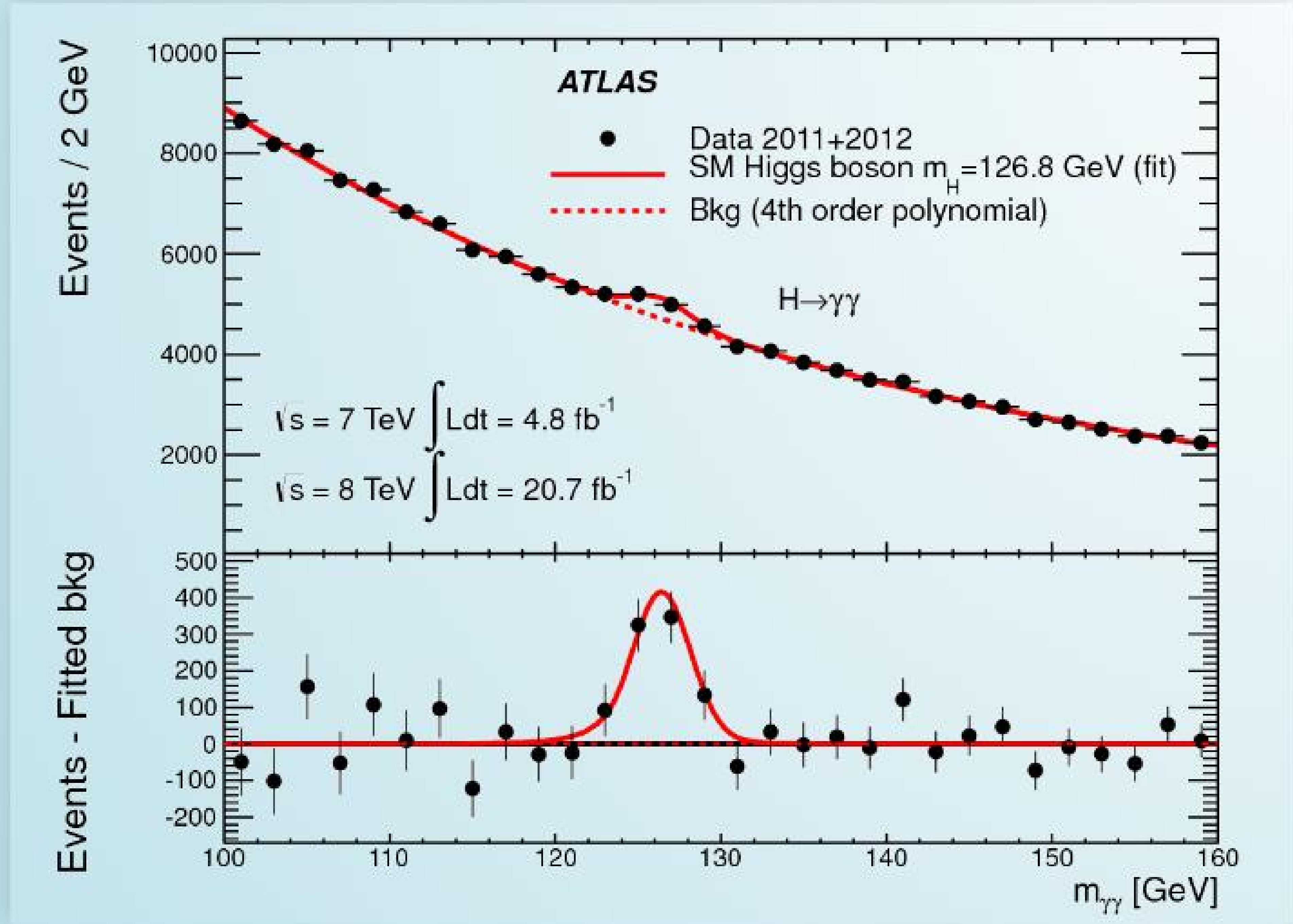
INTRODUCTION

Amplitudes and Cross Sections

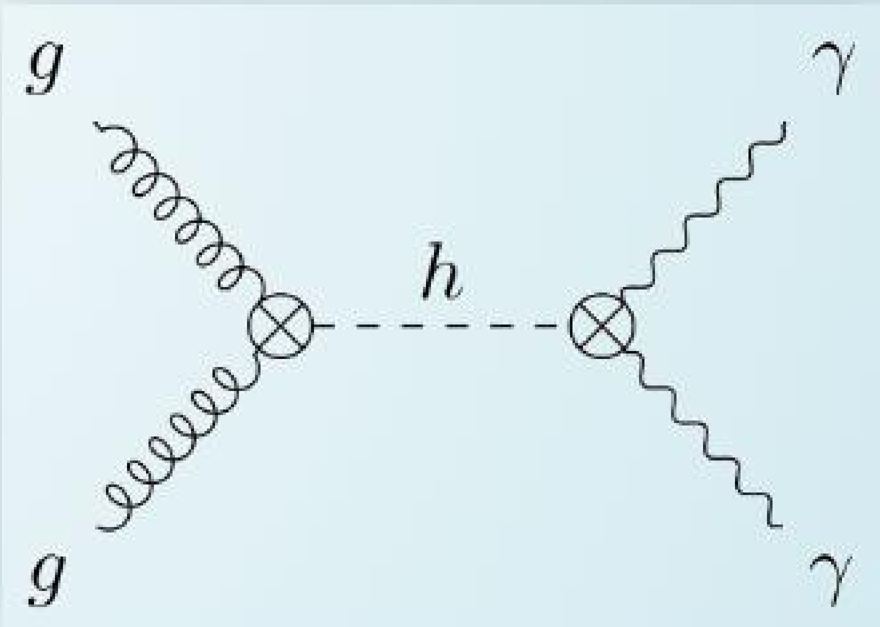
Amplitudes are a key element for computing cross sections. At hadron colliders, we have:

$$\sigma_{2\to n-2} = \sum_{a,b} \int dx_a dx_b f_{a/h_1}(x_a,\mu_F)\, f_{b/h_2}(x_b,\mu_F)\; \hat{\sigma}_{ab\to n-2}(x_a,x_b,\mu_F,\mu_R)$$

$$\hat{\sigma}_n = \frac{1}{2\hat{s}} \int d\text{LIPS}\; (2\pi)^4 \delta^4\Big(\sum_{i=1}^n p_i\Big)\; \overline{|\mathcal{A}(p_i,\lambda_i,a_i,\mu_F,\mu_R)|^2}$$



ATLAS Collaboration ('12)



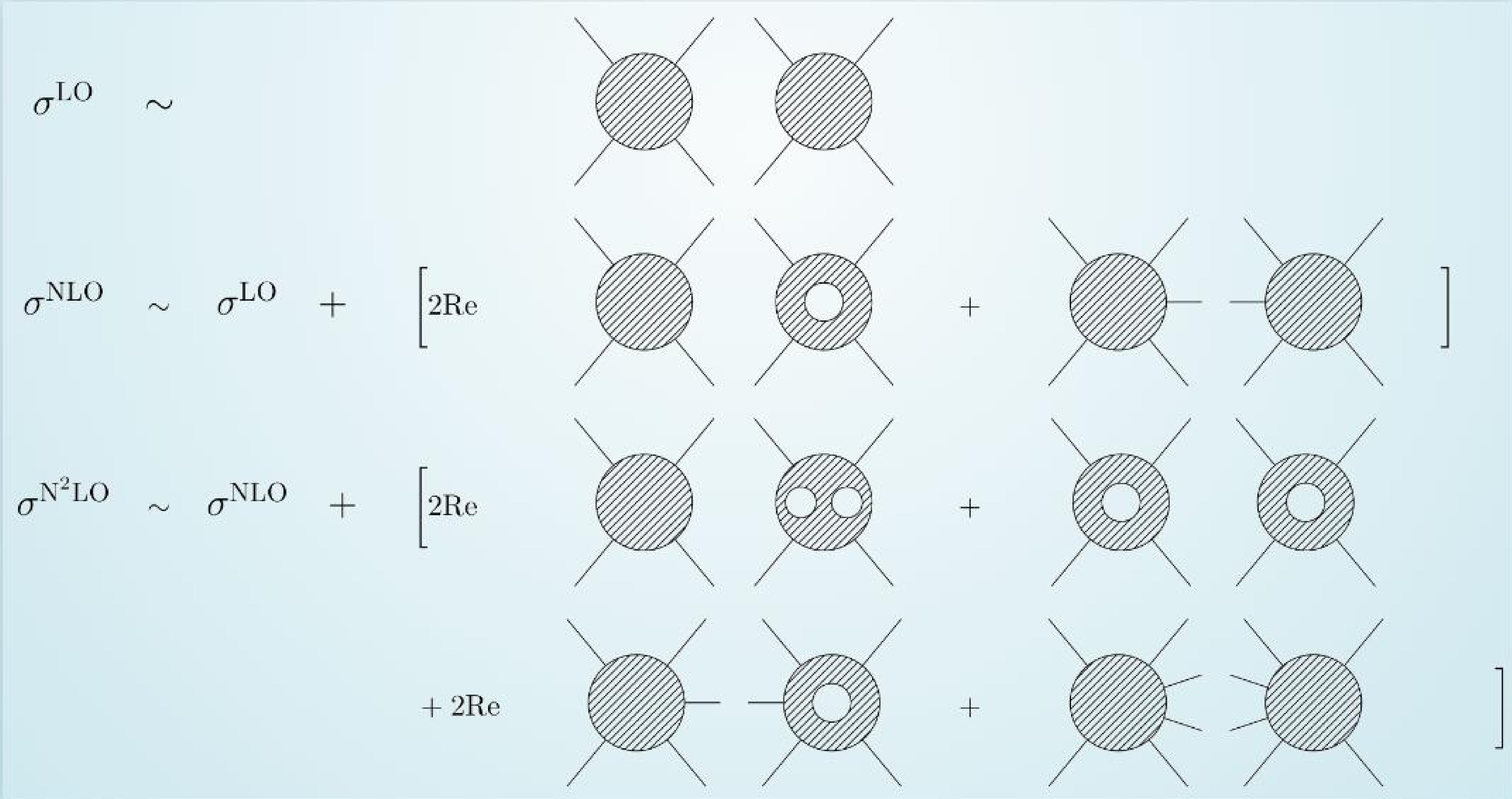
$$\mathcal{A}_{pp\to h\to\gamma\gamma} \sim \frac{1}{m_{\gamma\gamma}^2 - m_h^2 + i m_h \Gamma_h}$$

$\Rightarrow$ Breit–Wigner distribution

Perturbation Theory

$$\mathcal{A}_n/\alpha_s^k = \mathcal{A}_n^{\text{tree}} + \underbrace{\left(\frac{\alpha_s}{2\pi}\right)\mathcal{A}_n^{1-\text{loop}}}_{\sim 10\%} + \underbrace{\left(\frac{\alpha_s}{2\pi}\right)^2\mathcal{A}_n^{2-\text{loop}}}_{\sim 1\%}$$

Better predictions require both more loops and higher multiplicity.



Processes with additional soft or collinear radiation are indistinguishable from the Born.

State-of-the-Art

- Focus on all-gluon scattering, as a representative example.

$\mathcal{A}^{\ell-loops}_{n-gluons}$		multiplicity (n)				
		4	**5**	**6**	**7**	**8**
loops (ℓ)	**0**					
	1					
	2					
	3					

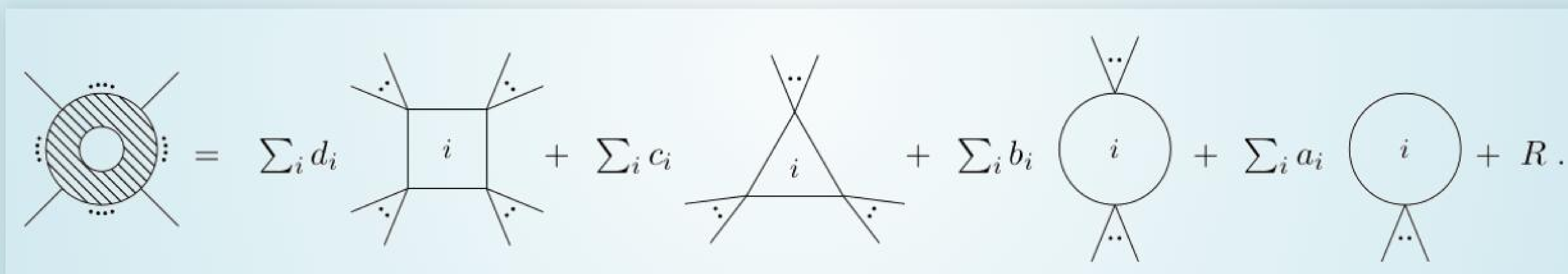
- Three-loop four-point (analytic) — Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi ('21)
- Two-loop five-point (analytic) — Abreu, Dormans, Febres Cordero, Ita, Page ('18) (Leading Color)
- One-loop six-point (analytic) — GDL, Maître ('19) (Previous results involve taking limits, sqrts, etc..)
- One-loop beyond six-point (solved, but only numerically) — BlackHat, Njet, Recola, OpenLoops, . . .
- Tree (solved) — Berends, Giele; Britto, Cachazo, Feng, Witten; Dixon, Henn, Plefka, Schuster ('10); . . .

THE STRUCTURE OF SCATTERING AMPLITUDES

Rational and Transcendental

Decomposition in terms of **master integrals** 't Hooft, Veltman; Bern, Dixon, Kosower; Ellis, Zanderighi

$$A_n^{1-\text{loop},D=4} = \sum_i d_i I^i_{Box} + \sum_i c_i I^i_{Triangle} + \sum_i b_i I^i_{Bubble} + \sum_i a_i I^i_{Tadpoles} + R$$

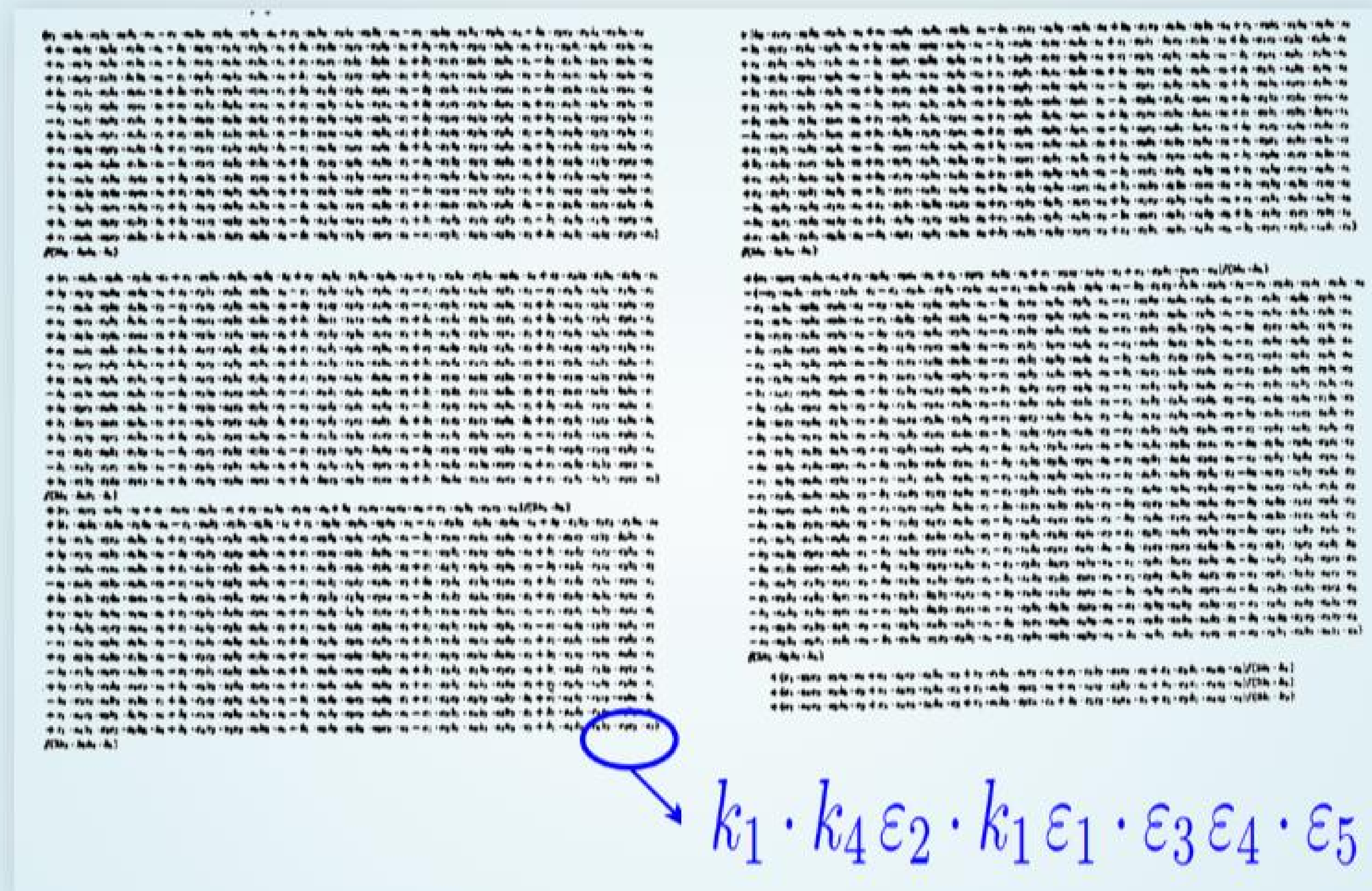


In general, in $D = 4 - 2\epsilon$, for a suitable choice of master integrals

$$A_n^{\ell-loop} = \sum_{i \in \text{masters}} \frac{c_i(\vec{p}, \vec{\lambda}, \epsilon)\, I_i(\vec{p}, \epsilon)}{\prod_j (\epsilon - a_{ij})}\,, \quad \text{with} \quad a_{ij} \in \mathbb{Q}$$

Feynman diagram by Feynman diagram

- Analytic computations can get very complicated very quickly. For example, for $A^{\text{tree}}_{5-\text{gluons}}$:



- This amplitude can be written as just

Parke, Taylor ('86), Berends, Giele ('88)

$$|A^{tree}(1_g^+ 2_g^+ 3_g^+ 4_g^- 5_g^-)|^2 = \frac{s_{45}^4}{s_{12}s_{23}s_{34}s_{45}s_{51}}$$

HOW DO WE COMPUTE SCATTERING AMPLITUDES EFFICIENTLY?

Multi-Loop Amplitudes from Trees

- Generalized unitarity relates products of tree amplitudes to loop amplitudes

$$\prod_{\text{trees}} A^{\text{tree}}(\vec{k}, \vec{\ell}|_{\text{cut}}) = \sum_{\substack{\text{topologies } \Gamma,\\ i \in M_\Gamma \cup S_\Gamma}} c_{i,\Gamma}(\vec{k}) \left(\frac{m_{i,\Gamma}(\vec{k}, \vec{\ell}|_{\text{cut}})}{\prod_{\text{props } j} \rho_j(\vec{k}, \vec{\ell}|_{\text{cut}})} \right)$$

C++ code

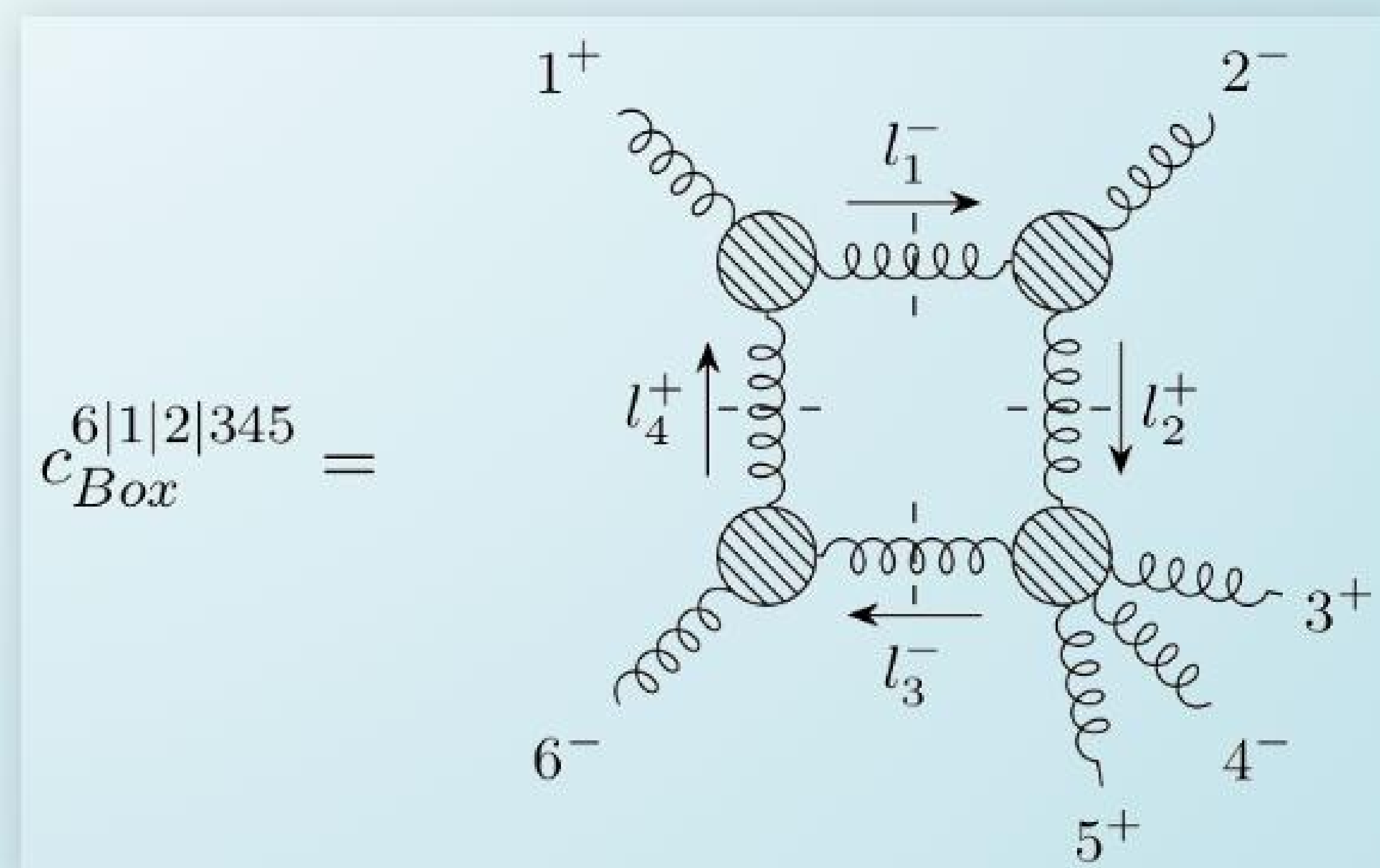
Caravel

Abreu, Dormans, Febres Cordero, Ita
Kraus, Page, Pascual, Ruf, Sotnikov ('20)

$$\left.\begin{array}{r} \underline{\text{Master integrals}}: \int d^D\vec{\ell}\, \frac{m_{i\in M_\Gamma}}{\prod_j \rho_j} \neq 0 \\ \underline{\text{Surface terms}}: \int d^D\vec{\ell}\, \frac{m_{i\in S_\Gamma}}{\prod_j \rho_j} = 0 \end{array}\right\}$$ Complex problem!

- The diagram on the right shows as example a one-loop box coefficient.

- In general, need to solve linear systems for the coefficients $c_{i,\Gamma}$.



Analytics from Numerics

Problem: direct analytic computation of the $c_{i,\Gamma}$ is not feasible.

- Floating-point evaluations ($\mathbb{R}$ or $\mathbb{C}$) would be sufficient for phenomenology.
  But they are so unstable, even this won't work.
- Could try rational inputs ($\mathbb{Q}$), but integers grow way too large at intermediate stages.
- Finite fields ($\mathbb{F}_p$) come to the rescue. von Manteuffel, Schabinger ('14), Peraro ('16)
  These are integers modulo a prime number p (no precision issue!):

  $$\mathbb{F}_p = \{0, 1, 2, \ldots, p-1\} \quad \text{with operations} \quad \{+, -, \times, \div\}$$

  The prime p needs to be large, to avoid accidental `DivisionByZero` .
- But we can't do phenomenology with $\mathbb{F}_p$!

Solution: sample $c_{i,\Gamma}$ in $\mathbb{F}_p \Rightarrow$ reconstruct analytic expression for $c_{i,\Gamma}$

Finite Fields

- Any rational number, other than multiples of $1/p$, has an equivalent in the finite field $\mathbb{F}_p$.
- For example, let's work with $p = 7$, i.e. with $\mathbb{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$:

$$\begin{aligned} & -1 \text{ is the additive inverse of } 1 \\ \Rightarrow \quad & -1 = 6 \text{ in } \mathbb{F}_7 \text{, because } 1 + 6 = 7 \,\%\, 7 = 0 \end{aligned}$$

$$\begin{aligned} & \tfrac{1}{3} \text{ is the multiplicative inverse of } 3 \\ \Rightarrow \quad & \tfrac{1}{3} = 5 \text{ in } \mathbb{F}_7 \text{, because } 3 \times 5 = 15 \,\%\, 7 = 1 \end{aligned}$$

The *Euclidean algorithm* allows to compute inverses without checking every entry.

- Numbers cannot grow out of control!

$$\frac{146118849457855618859788417553608602311208376528310383201 07}{185374227667620200647639434147201298352198123520 0} = 1251868773 \text{ in } \mathbb{F}_{2147483647}$$

2147483647 is $(2^{31} - 1)$ which is the largest possible value p working with 32-bits.

ANALYTIC RECONSTRUCTION

Common-Denominator Ansatz

$$c_{i,\Gamma}(\vec{x}) = \frac{\text{Num. poly}(\vec{x})}{\text{Denom. poly}(\vec{x})} = \frac{\text{Num. poly}(\vec{x})}{\prod_j W_j(\vec{x})}$$

- Interpolation in one variable (continued fraction) — Thiele (1909)

$$c_{i,\Gamma}(t) = c_{i,\Gamma}(t_0) + \cfrac{t-t_0}{\cfrac{t_0-t_1}{c_{i,\Gamma}(t_0)-c_{i,\Gamma}(t_1)} + \cfrac{t-t_2}{\dots + \cfrac{t-t_3}{\dots}}} = \frac{\text{Num. poly}(t)}{\text{Denom. poly}(t)}$$

Match denominator factors of $c_{i,\Gamma}(t)$ to $W_j(t) \Rightarrow$ obtain the denominator (this is the easy part).

- The numerator is **much** more complicated, in general — For spinors: GDL, Maître (2019)

$$\text{Num. poly}(\vec{x}) = \sum_{\vec{\alpha}} c_{\vec{\alpha}}\, x_1^{\alpha_1} \dots x_m^{\alpha_m}$$

- To solve the system must sample as many times as there are undertermined $c_{\vec{\alpha}}$'s.

Tools of the Trade

- In practice, using spinors $m = n(n-1)$ and there are constraints on $\vec{\alpha}$

Gröbner bases $\rightarrow$ constrain $\vec{\alpha}$

SINGULAR

Decker, Greuel, Pfister, Schönemann

Integer programming $\rightarrow$ enumerate sols. $\vec{\alpha}$

Google OR-Tools

Perron and Furnon (Google optimization team)

- Solving linear systems with CUDA in $\mathbb{C}$ or $\mathbb{F}_{p \leq 2^{31}-1}$ (currently private code)

System Size	Timing
8192	8 s
16384	51 s
32768	6m 30s

with RTX 2080ti 11GB
the absolute maximum is 52440 unknowns

(thanks gpu-Merlin!)

TAMING THE ALGEBRAIC COMPLEXITY

Problem: the least-common-denominator form is overly complex.
Its numerator can easily exceed 1 million monomials (e.g. 5-point 1-mass processes).

- For example, taking homogeneous expressions in 5 variables

$$c_{i,\Gamma}(x_1, \ldots, x_5) = \frac{\text{126 monomials of degree 5}}{x_1 x_2 x_3 x_4 x_5}$$

but say we knew that $x_1 x_2$ don't appear in the same denominator as the others, then

$$c_{i,\Gamma}(x_1, \ldots, x_5) = \frac{\text{15 monomials of degree 2}}{x_1 x_2} + \frac{\text{35 monomials of degree 3}}{x_3 x_4 x_5}$$

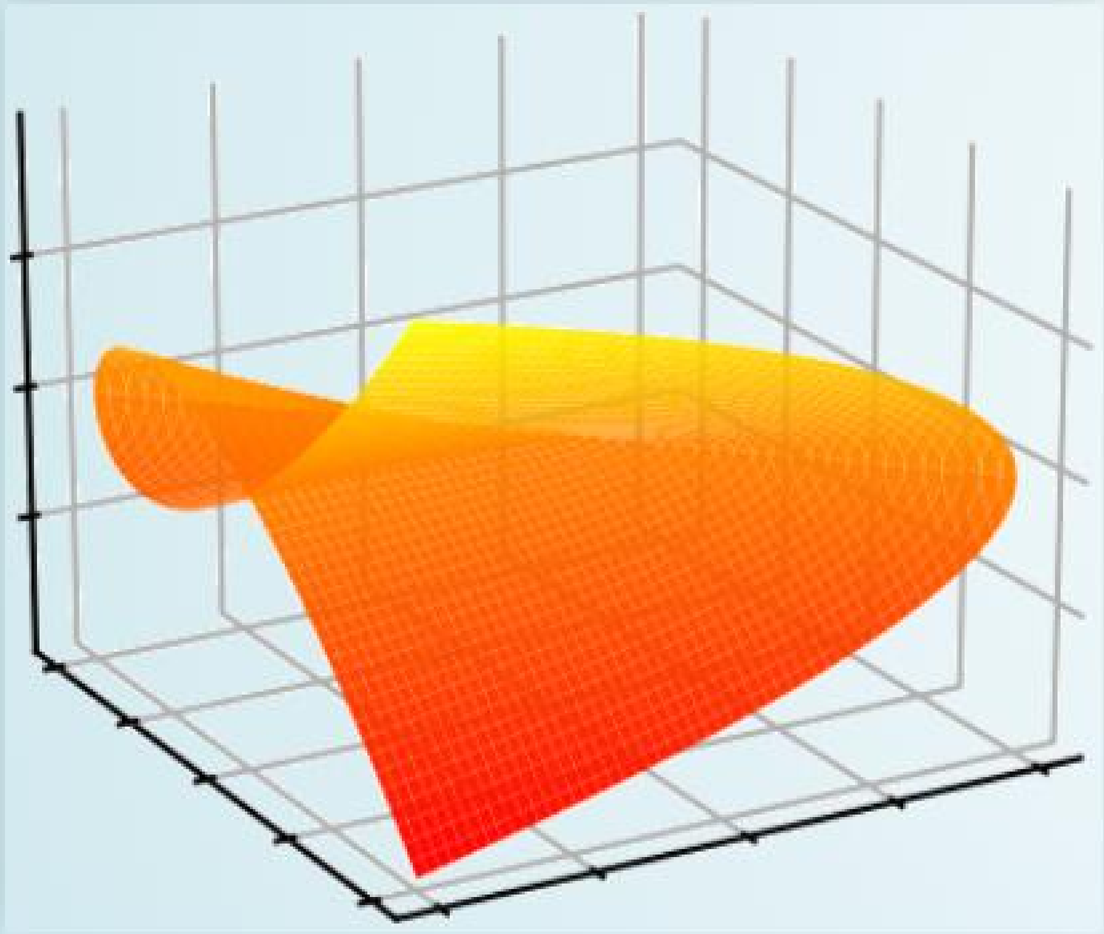
Goal: use partial-fraction decompositions,
but how to achieve this without an analytic expression?

THE GEOMETRY OF PHASE SPACE

based on: GDL, Page (JHEP 12 (2022) 140)

Least Common Denominator Redux

- Can't draw pictures in high (complex) dimensions, so let's consider the simplified case $\mathbb{R}[x, y, z]$.
- Denominator factors W_j correspond to *singular surfaces* .



$W_1 = (xy^2 + y^3 - z^2)$

Say we have:

$$W_1 = xy^2 + y^3 - z^2$$

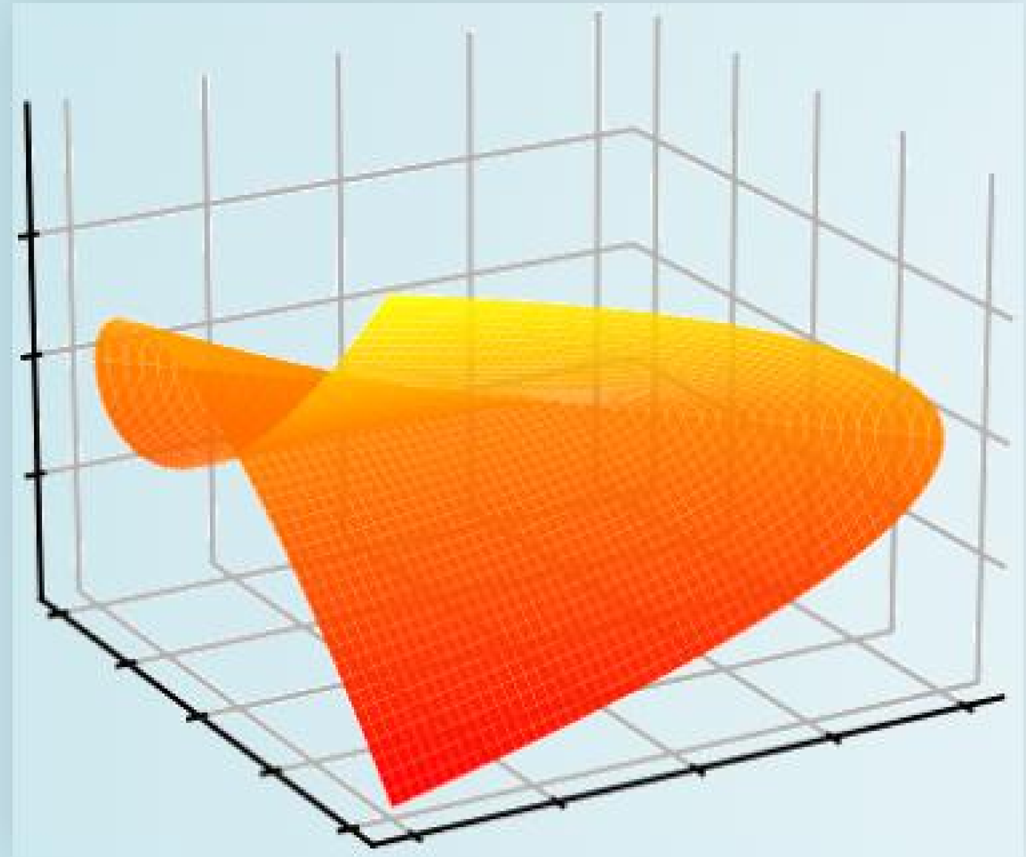
A function $c_i(x, y, z)$ may or may not have W_1 as a pole, depending on what happens on the orange surface

$$\lim_{W_j \to \epsilon} c_i(x, y, z) \sim \frac{1}{\epsilon^{q_{ij}}}$$

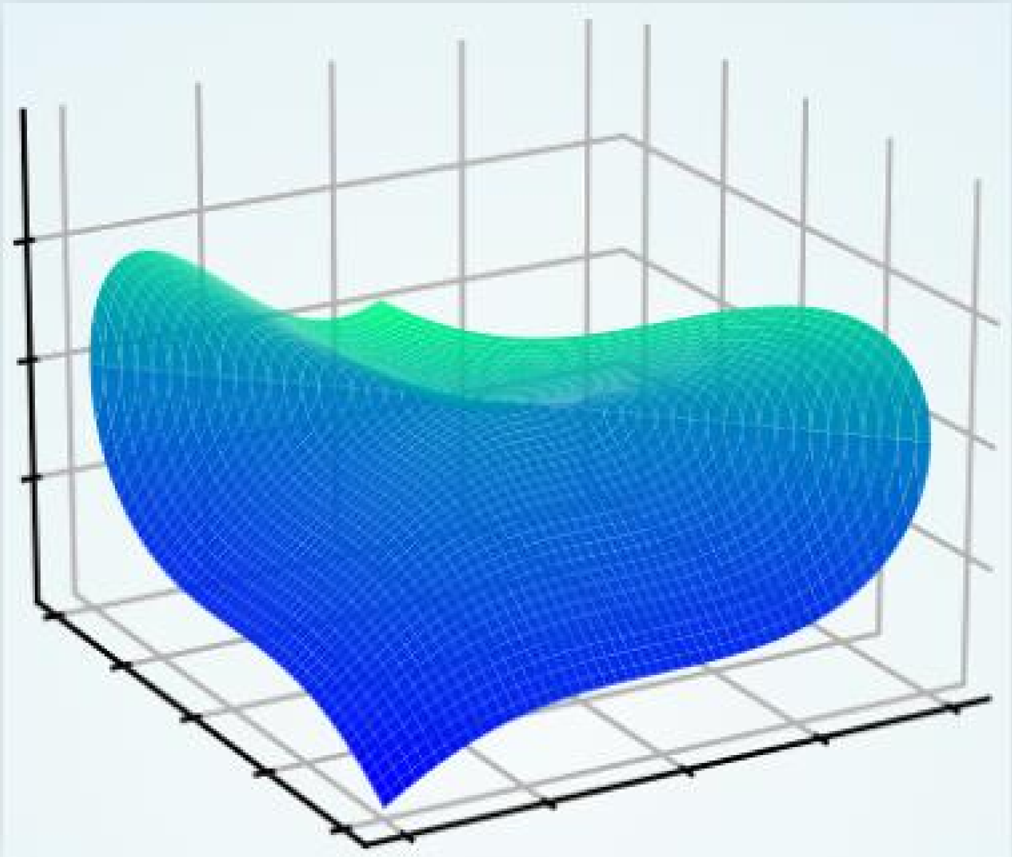
The LCD tells us about what happens on surfaces with one less dimension than the full space.

Multivariate Partial Fractions

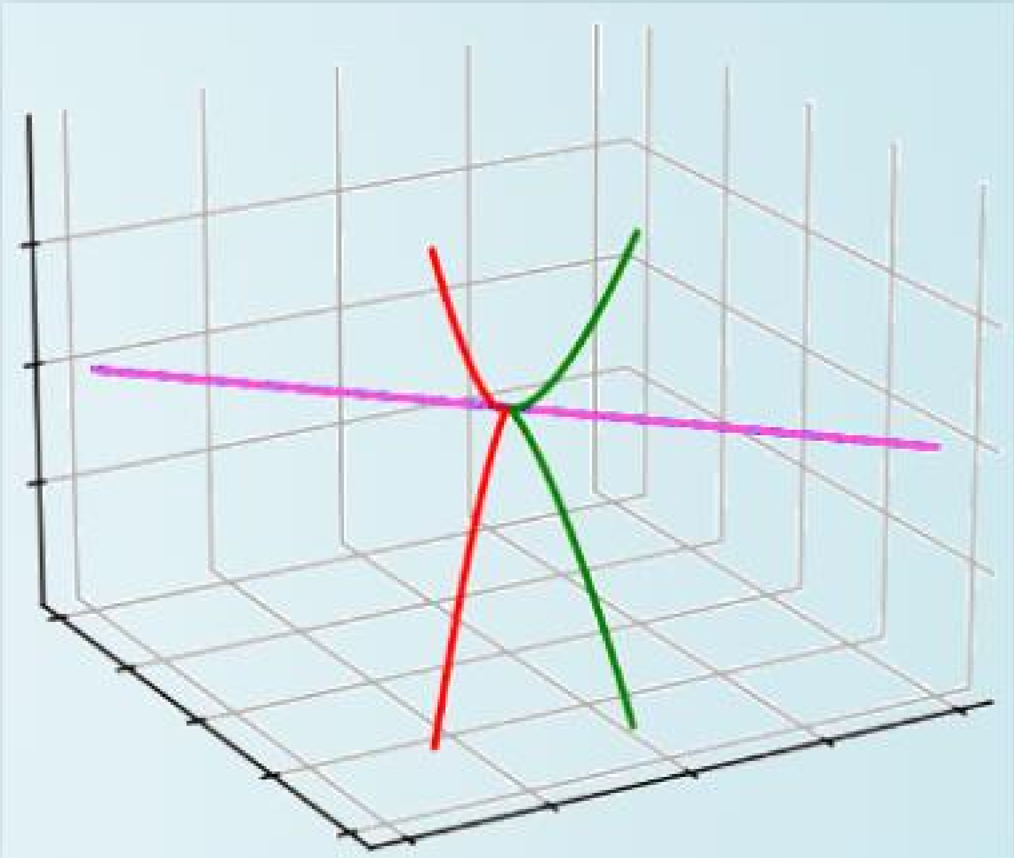
- To distinguish $\frac{1}{W_1 W_2}$ from $\frac{1}{W_1} + \frac{1}{W_2}$, look at $W_1 \sim W_2 \to \epsilon \ll 1$. Geometrically:



$W_1 = (xy^2 + y^3 - z^2)$



$W_2 = (x^3 + y^3 - z^2)$



$V(W_1) \cap V(W_2)$

- *Primary decompositions* of sets of polynomials (*ideals*), anogous to integers:

$$60 = 5 \times 3 \times 2^2$$

$$(xy^2 + y^3 - z^2, x^3 + y^3 - z^2) = (z^2, x + y) \cup (y^3 - z^2, x) \cup (2y^3 - z^2, x - y)$$

Partial-fraction decompositions tell us about the relations between poles.

Upcoming Results

- First two-loop computation in full color (N_c dependence) for $q\bar{q} \to \gamma\gamma\gamma$

Kinematics	**# Poles (W)**	**LCD Ansatz**	**Partial-Fraction Ansatz**	**Rational Functions**
5-point massless	30	29k	4k	$\sim$200 KB

- Updated two-loop leading-color amplitudes for $pp \to Wjj$, now in spinor helicity

Kinematics	**# Poles (W)**	**LCD Ansatz**	**Partial-Fraction Ansatz**	**Rational Functions**
5-point 1-mass	>200	>5M	$\sim$40k	$\sim$25 MB

First computed in Abreu, Febres Cordero, Ita, Klinkert, Page, Sotnikov (1.2 GB)

Try it yourself

```
pip install lips pyadic
```

Thanks for your attention!

Questions?

BACKUP SLIDES

ABSOLUTE VALUES ON THE RATIONALS

p-adic Numbers

- We have again a problem **in a finite field** 1 is not smaller than 2. In fact:

$$|x = 0|_{\mathbb{F}_p} = 0 \quad \text{and} \quad |x \neq 0|_{\mathbb{F}_p} = 1$$

  Can't easily take limits, without dividing by zero.

- A p-adic number $x \in \mathbb{Q}_p$ is Laurent expansion in powers of the prime p

$$x = a_{\nu_p} p^{\nu_p} + \cdots + a_{-1} p^{-1} + a_0 p^0 + a_1 p^1 + \ldots$$

- The p-adic absolute value is defined as (note the minus sign!)

$$|x|_{\mathbb{Q}_p} = p^{-\nu_p} \quad \Rightarrow \quad |p|_{\mathbb{Q}_p} < |1|_{\mathbb{Q}_p} < |\tfrac{1}{p}|_{\mathbb{Q}_p}$$

Retain integer arithmetics, while restoring the ability to take limits!

PYTHON PACKAGES

pyAdic

- `Pyadic` provides flexible number types for finite fields and p-adic numbers in Python. Related algorithms, such as rational reconstruction are also implemented.

```
from pyadic import ModP
from fractions import Fraction as Q
ModP(Q(7, 13), 2147483647)
<<< 1817101548 % 2147483647
# Can also go back to rationals
from pyadic.finite_field import rationalise
rationalise(ModP(Q(7, 13), 2147483647))
<<< Fraction(7, 13)
```

Lips

- `Lips` is a phase-space generator and manipulator for 4-dimensional kinematics in any field, $\mathbb{C}, \mathbb{F}_p, \mathbb{Q}_p, \mathbb{Q}[i]$. It is particularly useful for spinor-helicity computations.

```python
from lips import Particles
from lips.fields.field import Field
# Random finite field phase space point, arbitrary multiplicity
multiplicity = 5
PSP = Particles(multiplicity, field=Field("finite field", 2 ** 31 - 1, 1), seed=0)
# Evaluate an arbitrary complicated expression
PSP("(8/3s23⟨24⟩[34])/(⟨15⟩⟨34⟩⟨45⟩⟨4|1+5|4])")
<<< 683666045 % 2147483647
```

- It can also be used to generate points in singular configuration.

SPINOR HELICITY

Representations of the Lorentz group

(Recall: $\mathfrak{so}(1,3)_\mathbb{C} \sim \mathfrak{su}(2) \times \mathfrak{su}(2)$)

(j_-, j_+)	dim.	name	quantum field	kinematic variable
(0,0)	1	scalar	h	m
(0,½)	2	right-handed Weyl spinor	$\chi_{R\,\alpha}$	λ_α
(½,0)	2	left-handed Weyl spinor	$\chi_L^{\dot{\alpha}}$	$\bar{\lambda}^{\dot{\alpha}}$
(½,½)	4	rank-two spinor/four vector	$A^\mu / A^{\dot{\alpha}\alpha}$	$P^\mu / P^{\dot{\alpha}\alpha}$
(½,0)⊕(0,½)	4	bispinor (Dirac spinor)	Ψ	u, v

Spinor Covariants

Weyl spinors are sufficient for massless particles:

$$\det(P^{\dot{\alpha}\alpha}) = m^2 \to 0 \quad \Longrightarrow \quad P^{\dot{\alpha}\alpha} = \bar{\lambda}^{\dot{\alpha}}\lambda^{\alpha}.$$

In terms of 4-momentum components we have:

$$\lambda_\alpha = \frac{1}{\sqrt{p^0+p^3}}\begin{pmatrix} p^0+p^3 \\ p^1+ip^2 \end{pmatrix}, \quad \lambda^\alpha = \epsilon^{\alpha\beta}\lambda_\beta = \frac{1}{\sqrt{p^0+p^3}}\begin{pmatrix} p^1+ip^2 \\ -p^0+p^3 \end{pmatrix}$$

$$\bar{\lambda}_{\dot{\alpha}} = \tfrac{1}{\sqrt{p^0+p^3}}\begin{pmatrix} p^0+p^3 \\ p^1-ip^2 \end{pmatrix}, \quad \bar{\lambda}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\lambda}_{\dot{\beta}} = \tfrac{1}{\sqrt{p^0+p^3}}\begin{pmatrix} p^1-ip^2 \\ -p^0+p^3 \end{pmatrix}$$

$$\bar{\lambda}_{\dot{\alpha}} = (\lambda_\alpha)^* \quad if \quad p^i \in \mathbb{R}; \qquad \bar{\lambda}_{\dot{\alpha}} \neq (\lambda_\alpha)^* \quad if \quad p^i \in \mathbb{C}$$

Spinor Invariants

$$\langle ij \rangle = \lambda_i \lambda_j = (\lambda_i)^\alpha (\lambda_j)_\alpha \qquad [ij] = \bar{\lambda}_i \bar{\lambda}_j = (\bar{\lambda}_i)_{\dot{\alpha}} (\bar{\lambda}_j)^{\dot{\alpha}}$$

$$s_{ij} = \langle ij \rangle [ji]$$

$$\langle i \mid (j+k) \mid l] = (\lambda_i)^\alpha (\not{P}_j + \not{P}_k)_{\alpha\dot{\alpha}} \bar{\lambda}_l^{\dot{\alpha}}$$

$$\langle i \mid (j+k) \mid (l+m) \mid n \rangle = (\lambda_i)^\alpha (\not{P}_j + \not{P}_k)_{\alpha\dot{\alpha}} (\bar{\not{P}}_l + \bar{\not{P}}_m)^{\dot{\alpha}\alpha} (\lambda_n)_\alpha$$

$$tr_5(ijkl) = tr(\gamma^5 \not{P}_i \not{P}_j \not{P}_k \not{P}_l) = [i \mid j \mid k \mid l \mid i \rangle - \langle i \mid j \mid k \mid l \mid i]$$

FIVE-PARTON TWO-LOOP FINITE REMAINDERS

Example Simplifications

uubggg pmpmp Nf1 #3

```
-1/4+s34^3/(4*(-s15+s23+s34)^3)-(3*s34^2)/(4*(-s15+s23+s34)^2)+(3*s34)/(4*(-s15+s23+s34))-(3*s15*s34*s45)/(4*(-s15+s23+s34)*(s12+s23-s45)^2)+(3*s15*s34^2*s45)/(4*(-s15+s23+s34)^3*(s12+s23-s45))-(3*s15*s34*s45)/(4*(-s1
5+s23+s34)^2*(s12+s23-s45))+(3*s15^2*s34*s45^2)/(4*(-s15+s23+s34)^3*(s12+s23-s45)^2)+(s15^3*s45^3)/(4*(-s15+s23+s34)^3*(s12+s23-s45)^3)+(3*s15*s34*s45*(-s15+s34+s45))/(4*(-s15+s23+s34)^2*(s12+s23-s45)^2)+(3*s15^2*s45^
2*(-s15+s34+s45))/(4*(-s15+s23+s34)^2*(s12+s23-s45)^3)+(3*s15*s45*(s15^2+s34^2-3*s15*s45+s45^2+2*s34*(-s15+s45)))/(4*(-s15+s23+s34)*(s12+s23-s45)^3)+(-s23^3+3*s23^2*s45-3*s23*s45*(-s15+s45)+s45*(3*s15*(s15-s34)-6*s15*
s45+s45^2))/(4*(s12+s23-s45)^3)+(-(s12+s34)/(4*s12^2*(-s15+s23+s34))-s34^2/(4*s12^2*(-s15+s23+s34)*(s12+s23-s45))-(s15^2*s45^2*(-s15+s34+s45)^2)/(4*s12^2*(-s15+s23+s34)^3*(s12+s23-s45)^3)+(s15^2+s23^2-s34^2+s15*(s23-6
*s45)-3*s23*s45+3*s45^2)/(4*s12^2*(s12+s23-s45)^2)-(s34^4+s15^2*s45^2+2*s34^3*(-s15+s45)+4*s15*s34*s45*(-s15+s45)+s34^2*(s15+s45)^2)/(4*s12^2*(-s15+s23+s34)^3*(s12+s23-s45))+(s34*(-(s15*(s34-2*s45))+s34*(s34+s45)))/(2
*s12^2*(-s15+s23+s34)^2*(s12+s23-s45))-(s34*(s12*s34-s15*(s34-2*s45)+s34*(s34+s45)))/(4*s12^2*(-s15+s23+s34)^3)+(2*s12*s34+s15*(-s34+s45)+s34*(2*s34+s45))/(4*s12^2*(-s15+s23+s34)^2)+(s15*s45*(s15^2*(s34-s45)+s34*(s34+
s45)^2+s15*(-2*s34^2-s34*s45+s45^2)))/(2*s12^2*(s15-s23-s34)^3*(s12+s23-s45)^2)+(s15*s45*(s15^3-(s34+s45)^3-s15^2*(3*s34+4*s45)+s15*(3*s34^2+7*s34*s45+4*s45^2)))/(2*s12^2*(-s15+s23+s34)^2*(s12+s23-s45)^3)+(s15^3-9*s15
^2*s45+(2*s34-s45)*(s34+s45)^2+s15*(-3*s34^2+4*s34*s45+9*s45^2))/(4*s12^2*(-s15+s23+s34)*(s12+s23-s45)^2)+(s15^4+(s34+s45)^4-2*s15^3*(2*s34+5*s45)-2*s15*(s34+s45)^2*(2*s34+5*s45)+s15^2*(6*s34^2+24*s34*s45+19*s45^2))/
(4*s12^2*(s15-s23-s34)*(s12+s23-s45)^3)+(s15^3*(s34-3*s45)-s34*(s34+s45)^3-s15^2*(3*s34^2+s34*s45-8*s45^2)+s15*(3*s34^3+7*s34^2*s45+s34*s45^2-3*s45^3))/(4*s12^2*(-s15+s23+s34)^2*(s12+s23-s45)^2)+(-s15^3-s23^3+s34^3+4*
s34^2*s45+6*s34*s45^2+4*s45^3+s23^2*(s34+4*s45)+s15^2*(-s23+3*s34+8*s45)-s23*(s34^2+4*s34*s45+6*s45^2)-s15*(s23^2+3*(s34+2*s45)^2-2*s23*(s34+3*s45)))/(4*s12^2*(s12+s23-s45)^3))*tr5
```

is equal to

$$-\frac{[32]^3[41]^3}{2[31]^3[42]^3}$$

ggggg mpmpp Nf1 # 9

is equal to

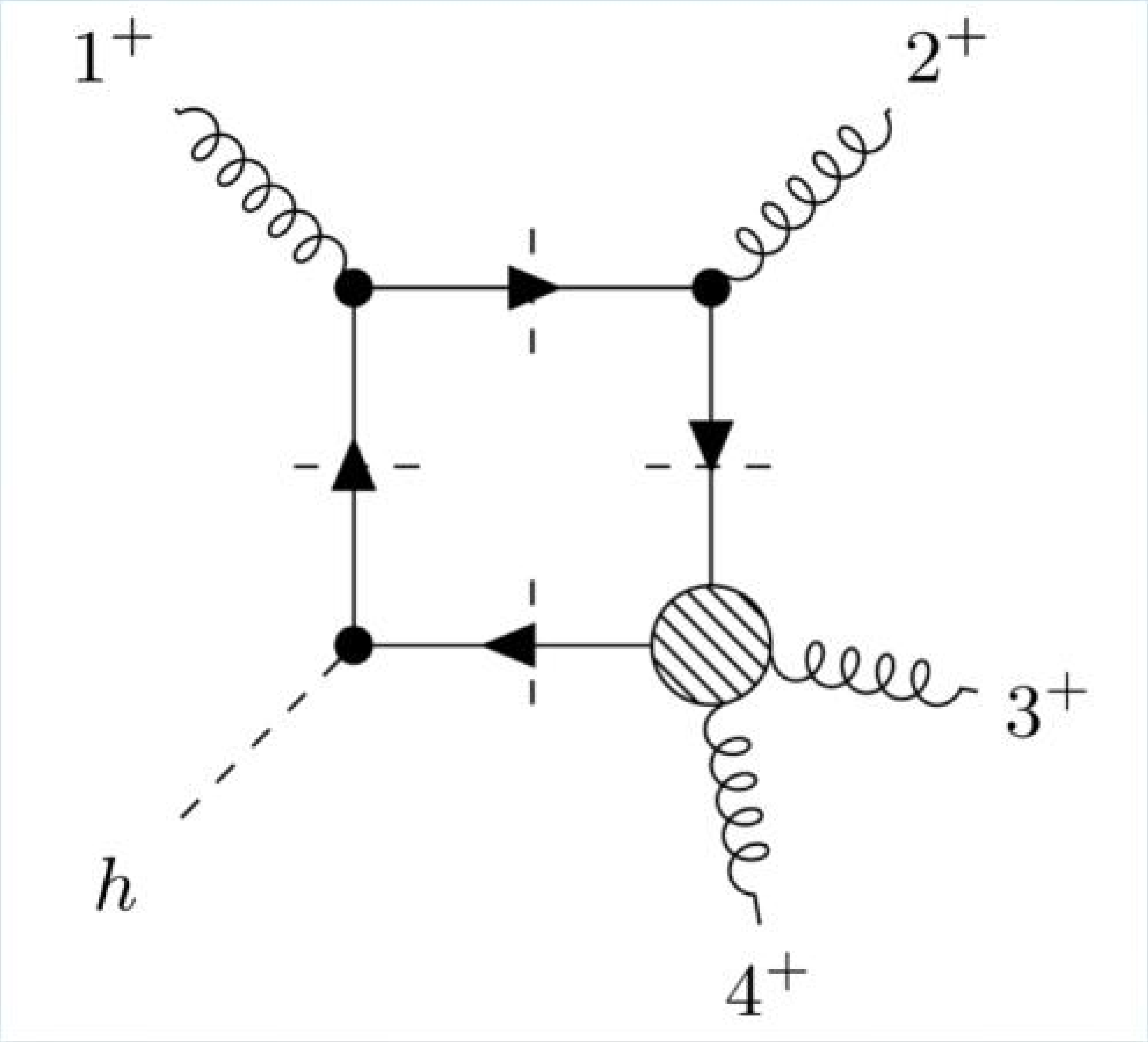
$$-1\frac{[12]^3[15][23]\langle 25\rangle^3[35]^3}{[13]^4[25]\langle 5|1+2|5]^3}+\frac{97}{12}\frac{[12]^4\langle 25\rangle[35]^4}{[13]^4[25]^3\langle 5|1+2|5]}+\frac{13}{3}\frac{[12]^4\langle 15\rangle[15][35]^4}{[13]^4[25]^4\langle 5|1+2|5]}+\frac{1}{4}\frac{[12]^4\langle 15\rangle[15]\langle 25\rangle[35]^4}{[13]^4[25]^3\langle 5|1+2|5]^2}$$

$$-\frac{3}{2}\frac{[12]^2\langle 25\rangle^2[25][35]^2}{[13]^2[25]\langle 5|1+2|5]^2}+\frac{7}{4}\frac{[12]^3\langle 25\rangle^2[35]^3}{[13]^3[25]\langle 5|1+2|5]^2}-\frac{43}{3}\frac{[12]^3\langle 25\rangle[35]^3}{[13]^3[25]^2\langle 5|1+2|5]}-\frac{25}{3}\frac{[12]^3\langle 15\rangle[15][35]^3}{[13]^3[25]^3\langle 5|1+2|5]}$$

$$-\frac{3}{2}\frac{[12]\langle 25\rangle[25][35]}{[13][25]\langle 5|1+2|5]}+4\frac{[12]^2\langle 25\rangle[35]^2}{[13]^2[25]\langle 5|1+2|5]}-\frac{15}{2}\frac{[12]^2[35]^2}{[13]^2[25]^2}+\frac{7}{2}\frac{[12][35]}{[13][25]}-\frac{2}{3}$$

HIGGS + 4-PARTON AMPLITUDE (@ FINITE TOP-MASS)

Example of cut diagram



Only singularity involving m_{top} (from pentagon contributions)

$$16|S_{1\times2\times3\times4}| = -s_{12}\, s_{23}\, s_{34}\, \langle 1|2+3|4]\, \langle 4|2+3|1] + m_{top}^2\, tr_5(1234)^2$$

We can generate point near this singularity in a similar fashion.

Structure of the coefficients

The massive external leg (the Higgs) is easily accomodated by considering it as a pair of massless particles (think decay products).
In the end all dependance on P_{Higgs} is removed by using momentum conservation.

The coefficients are Taylor expasions in m_{top}:

$$C^{(0)} + m_{top}^2 C^{(2)}.$$

with $C^{(0)}$ and $C^{(2)}$ resabling the six-gluon coefficients.