Accelerator Physics & Modelling

Zuoz Summer School Lecture 2: Linear & Non-Linear Maps

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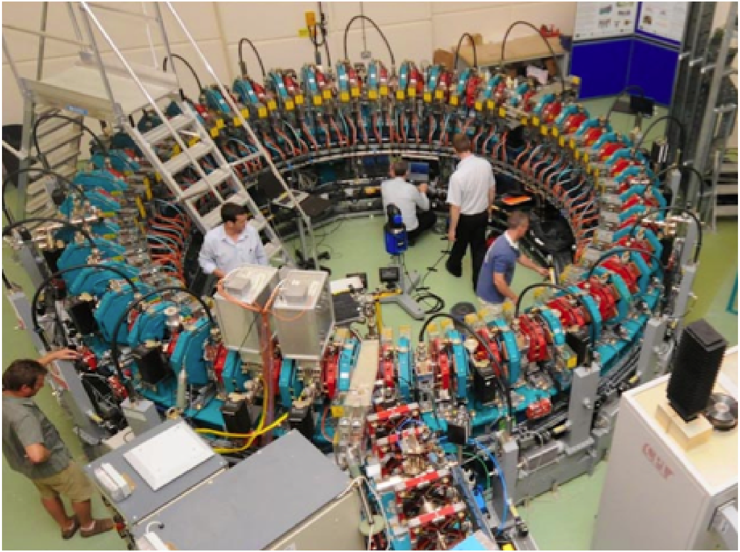
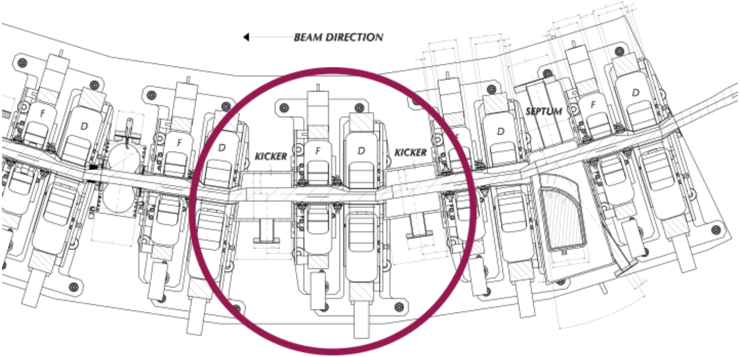
August 4-10, 2024

The Challenge



The Challenge



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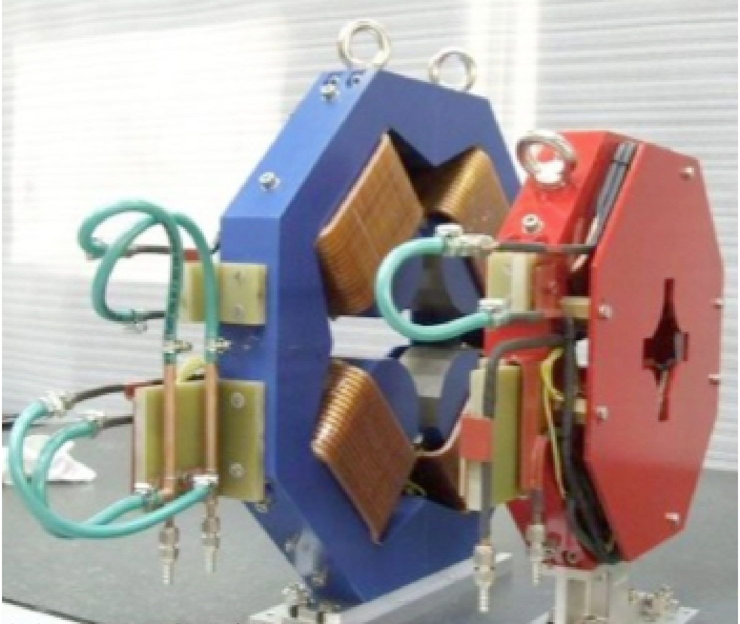


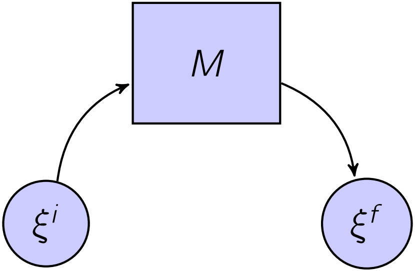
The Challenge

We have

- periodicity
- almost identical cells

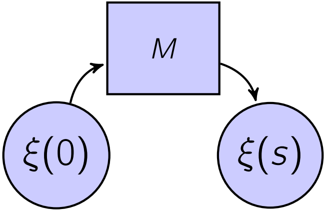
We want

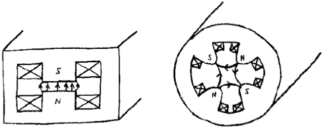
- study properties of the Lattice (stability)
- study dynamics of particles (long term tracking)



Action of Beam-Line Elements

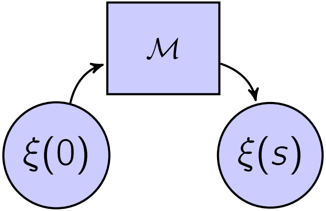
The action of each beam-line element can be described by a (symplectic) map M. Charged particle motion is Hamiltonian, and Hamiltonian flows generate symplectic maps



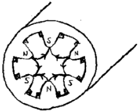
Linear Elements (M)  ...

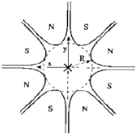
Action of Beam-Line Elements

The action of each beam-line element can be described by a (symplectic) map $\mathcal{M}$. Charged particle motion is Hamiltonian, and Hamiltonian flows generate symplectic maps



Non-Linear Elements ($\mathcal{M}$)





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Setup I

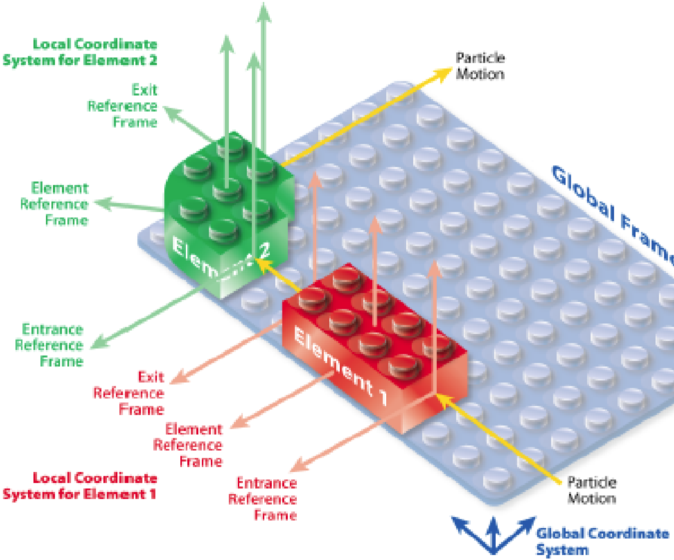
The interaction between charged particle and electromagnetic field depends only on the particle charge and velocity and on the field.

$$
\begin{aligned}
\text{Gauss' law} \qquad \nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} && (1)\\
\text{Faraday's law} \qquad \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && (2)\\
\text{Ampère's law} \qquad \nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} && (3)\\
\text{no mag. charges} \qquad \nabla \cdot \mathbf{B} &= 0 && (4)
\end{aligned}
$$

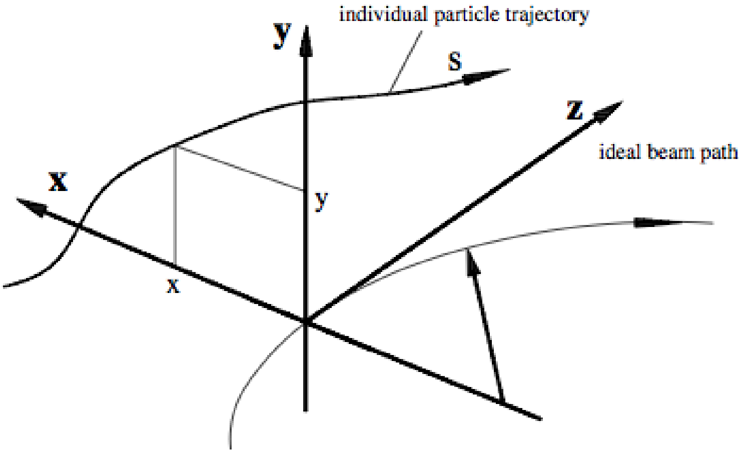
with $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{P} = \gamma m_0 \mathbf{v} + e\mathbf{A} = \mathbf{p} + e\mathbf{A}$ we obtain the "Hamiltonian" for particle accelerators

$$H = e\phi + \sqrt{c^2(\mathbf{P} - e\mathbf{A})^2 + m_0^2 c^4}.$$

Setup II


Local Coordinate System for Element 2
Exit Reference Frame
Element Reference Frame
Entrance Reference Frame
Particle Motion
Global Frame
Element 2
Element 1
Exit Reference Frame
Element Reference Frame
Entrance Reference Frame
Local Coordinate System for Element 1
Particle Motion
Global Coordinate System

Coordinate System I



- using Frenet–Serret coordinates, we are able to describe particle trajectories in $\mathbb{R}^3$ naturally

Coordinate System II

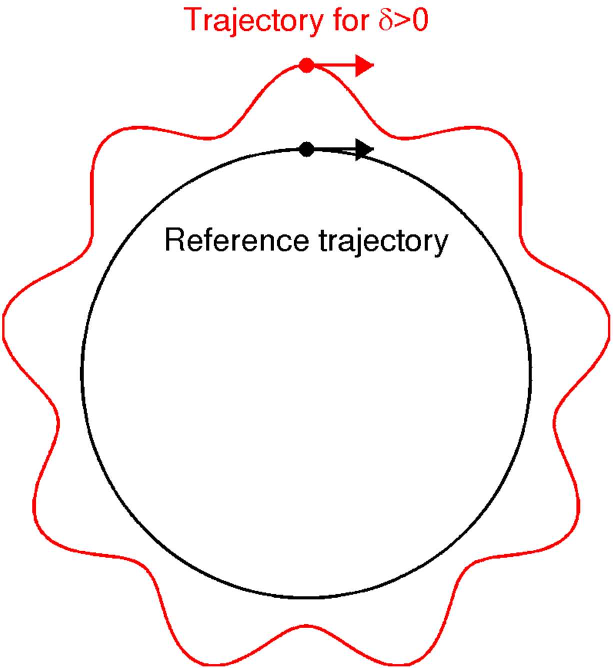
- assume we already know the ideal path $\mathbf{r}_0(z)$ hence it is only natural to transformed away the ideal path or the geometry of the design beam transport line which is already well known to us from the placement (LEGO) of the accelerator elements
- the new coordinates measure directly the deviation of any particles from the reference particle.

A trajectory may follow a path described by

$$\mathbf{r}(z) = \mathbf{r}_0(z) + d\mathbf{r}$$

where $d\mathbf{r}$ accounts for the deviation of the ideal path.
Defining unit vectors $\mathbf{u}$ curvature κ we are able to obtain

$$d\mathbf{r} = \mathbf{u}_x dx + \mathbf{u}_y dy + \mathbf{u}_z h dz, \text{ with } h = 1 + \kappa_x x + \kappa_y y.$$


Trajectory for δ>0
Reference trajectory

Constructing the Hamiltonian

Canonical Transformations

- $t \Rightarrow s$ change of independent variable
- make all quantities small w.r.t reference trajectory
- transform into the curvilinear Frenet–Serret

$$
\begin{aligned}
h &= \frac{1}{\rho} \\
H &= -\left(1 + hx\right) \times \\
&\quad \sqrt{\left(\frac{1}{\beta_0} + \delta - \frac{q\phi}{P_0 c}\right)^2 - (p_x - a_x)^2 - (p_y - a_y)^2 - \frac{1}{\beta_0^2 \gamma_0^2}} \\
&\quad - \left(1 + hx\right) a_s + \frac{\delta}{\beta_0}
\end{aligned}
\tag{5}
$$

Drift Space

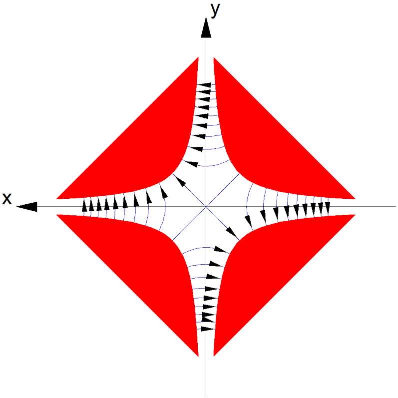
Set $\mathbf{A} = 0$ expanding the Hamiltonian (6) to second order in the dynamical variables

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}\frac{\delta^2}{\beta_0^2\gamma_0^2}$$

This is much simpler than Hamiltonians we have recently looked at and the equations of motion is very easy, and we find once again that the transfer matrix for a drift of length L is given by:

$$M = \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Hamiltonian Inside a Quadrupole I



$$B_x = b_2 \frac{y}{r_0} \quad B_y = b_2 \frac{x}{r_0} \quad B_s = 0$$

Hamiltonian Inside a Quadrupole II

- on the axis of the quadrupole, the field strength is zero.
- hence, choose the reference trajectory to lie along the axis
- we can work in a straight coordinate system.

The normalized vector potential reads with the normalized longitudinal component

$$a_s = q\frac{A_s}{P_0} = -\frac{1}{2}\frac{q}{P_0}\frac{b_2}{r_0}\left(x^2 - y^2\right)$$

and the normalised quadrupole gradient reads $k_1 = \frac{q}{P_0}\frac{b_2}{r_0}$.
Now the Hamiltonian can be written:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}} + \frac{1}{2}k_1\left(x^2 - y^2\right) \quad (6)$$

Hamiltonian Inside a Quadrupole III

Expanding the Hamiltonian (Eq. 6) to second order in the dynamical variables we construct the Hamiltonian:

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}k_1 x^2 - \frac{1}{2}k_1 y^2 + \frac{1}{2\beta_0^2\gamma_0^2}\delta^2$$

Remarks

- *this looks very much like the harmonic oscillator equation; for $k_1 > 0$ we have a "focusing" potential in x, and a "defocusing" potential in y*
- *In z there is no focusing of any kind*

$$M = \begin{pmatrix} \bullet & \bullet & 0 & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 0 & 0 \\ 0 & 0 & \bullet & \bullet & 0 & 0 \\ 0 & 0 & \bullet & \bullet & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{7}$$

$$M = \begin{pmatrix} \bullet & \bullet & 0 & 0 & 0 & \bullet \\ \bullet & \bullet & 0 & 0 & 0 & \bullet \\ 0 & 0 & \bullet & \bullet & 0 & 0 \\ 0 & 0 & \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 & 1 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Periodic, Uncoupled, Linear Beamlines I

- we have M_i
- we can calculate the trajectory of a charged particle along an element

Next is to study particle or beam "dynamics" by considering a beamline with N elements

$$\mathbf{M} = \prod_{i=1}^{N} \mathbf{M_i}$$

and

$$\mathbf{x}^f = \mathbf{M} \cdot \mathbf{x}^i.$$

Critical for this will be the fact that the transfer matrices are symplectic. This is why we went to so much trouble with Hamiltonian mechanics: exact solutions to Hamilton's equations are guaranteed to produce symplectic maps (Liouville's theorem).

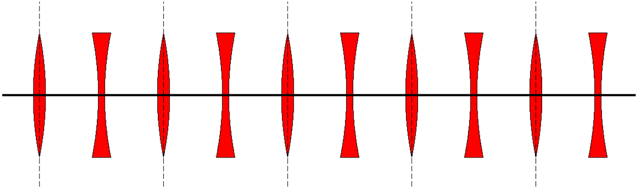
Periodic, Uncoupled, Linear Beamlines II

Tackling beamline optics for a general beamline is too big a step all at once. To provide a "gentler" introduction, we will begin by considering a linear beamline with two important properties.

1. The beamline is periodic: it consists of a repeated unit, or "cell", itself consisting of a given set of elements.
2. The beamline is uncoupled: the transfer matrices for each individual element are block-diagonal.

After developing this special (but very important) case, we shall return to the more general case.

Periodic, Uncoupled, Linear Beamlines III



Remarks

- *if we understand the dynamics in one (periodic) cell, we understand the optics in the entire beamline*
- *proof by the use of the Floquet's Theorem of ODE's.*

Transfer Matrix for a Thin Quadrupole I

We shall use the "thin lens" approximation for the quadrupoles. That is, we shall take the limit $L \to 0, k_1 L \to \frac{1}{f}$ where L is the length of the quadrupole, k_1 is the normalized quadrupole gradient, and f is a constant (the "focal length" of the quadrupole).

$$M_Q(f) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1/f & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1/f & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

Transfer Matrix for a FODO Cell I

For the horizontally focusing quadrupole, we write:

$$f = 2f_0$$

since the cell starts half-way through a horizontally focusing quadrupole; for half a quadrupole, the focal length is twice that for a full quadrupole. For the vertically focusing quadrupole:

$$f = -f_0.$$

The total transfer matrix is:

$$M = M_Q(2f_0) \cdot M_D(L) \cdot M_Q(-f_0) \cdot M_D(L) \cdot M_Q(2f_0) \qquad (9)$$

Performing the matrix multiplications in (9), we find the transfer matrix for a full FODO cell:

Transfer Matrix for a FODO Cell II

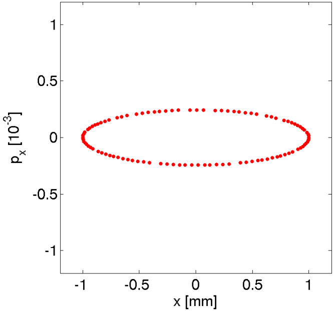
$$M = \begin{pmatrix} 1-\frac{L^2}{2f_0^2} & \frac{L}{f_0}(L+2f_0) & 0 & 0 & 0 & 0 \\ \frac{L}{4f_0^3}(L-2f_0) & 1-\frac{L^2}{2f_0^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\frac{L^2}{2f_0^2} & -\frac{L}{f_0}(L-2f_0) & 0 & 0 \\ 0 & 0 & -\frac{L}{4f_0^3}(L+2f_0) & 1-\frac{L^2}{2f_0^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{2L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

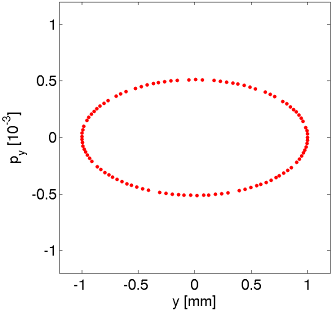
Let us consider the case $L = 1\,\mathrm{m}$, $f_0 = \sqrt{2}\,\mathrm{m}$. Take a particle with initial coordinates at the start of a FODO cell:

$$x = 1\,\mathrm{mm}, \quad p_x = 0, \quad y = 1\,\mathrm{mm}, \quad p_y = 0 \tag{11}$$

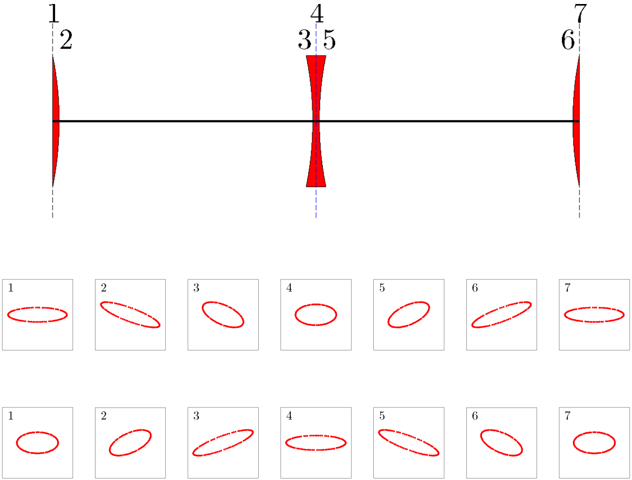
Now track the particle through 100 FODO cells by applying the transfer matrix (10) to the vector $\mathbf{x}^i$, and plot p_x, x and p_y, y:

Transfer Matrix for a FODO Cell III





Transfer Matrix for a FODO Cell IV



Horizontal (top) and vertical (bottom) phase space through a FODO cell.

Stability of the MAP I

Ansatz: the map is written in general form:

$$M_2 = \begin{pmatrix} \cos\mu_x + \alpha_x \sin\mu_x & \beta_x \sin\mu_x \\ -\gamma_x \sin\mu_x & \cos\mu_x - \alpha_x \sin\mu_x \end{pmatrix} = \mathbb{1}\cos\mu + J\sin\mu \tag{12}$$

For a FODO cell, the horizontal part of the transfer matrix is, from (10):

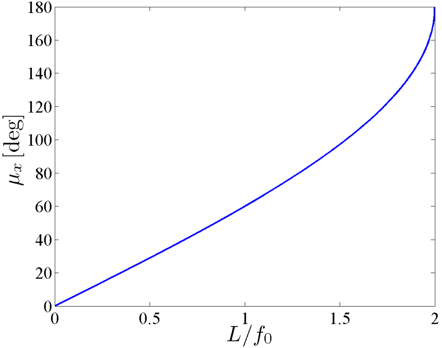
$$M_2 = \begin{pmatrix} 1 - \frac{L^2}{2f_0^2} & \frac{L}{f_0}(L + 2f_0) \\ \frac{L}{4f_0^3}(L - 2f_0) & 1 - \frac{L^2}{2f_0^2} \end{pmatrix} \tag{13}$$

With (12) and (13), we find that the phase advance across the cell is given by:

$$\cos\mu_x = 1 - \frac{L^2}{2f_0^2} \tag{14}$$

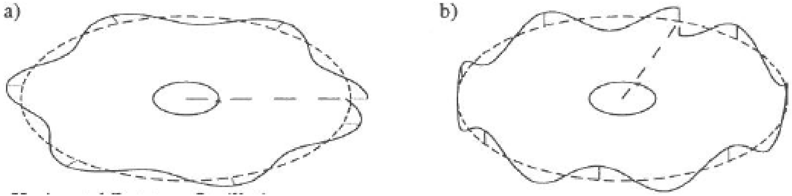
Note that if $L/f_0 > 2$, the cell is unstable.

Stability of the MAP II

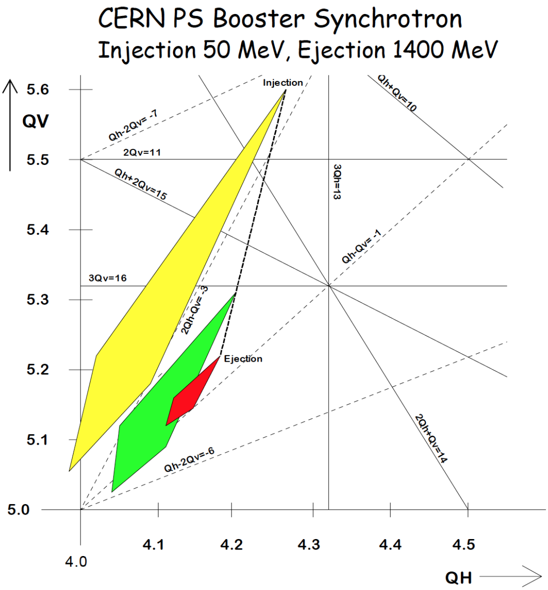


Tune

$$Q_x = \frac{1}{2\pi} \int_C \mu_x(s) ds$$



Tune-Diagram



Hamiltonian for a Sextupole I

A sextupole field can be derived from the vector potential:

$$A_x = 0, \quad A_y = 0, \quad A_s = -\frac{1}{6}\frac{P_0}{q}\,k_2\left(x^3 - 3xy^2\right). \tag{15}$$

and the sextupole strength is denoted by k_2.

$$H = -\sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}} + \frac{1}{6}\,k_2\left(x^3 - 3xy^2\right) + \frac{\delta}{\beta_0}. \tag{16}$$

Remark

- *The equations of motion are non-linear*
- *the Hamiltonian is non integrable*

To track a particle through a sextupole, we have to take one of two approaches:

Hamiltonian for a Sextupole II

1. integrate the eqm numerically (e.g. using a Runge-Kutta, Leap-Frog, or other suitable schemes) or
2. make some approximations that will enable us to write down an approximate map in closed form

Drawbacks of numerical integration:

- they tend to be rather slow
- often, we are interested in tracking tens of thousands of particles, thousands of times around storage rings consisting of thousands of elements.
- we shall make some approximations that will enable us to write down a map in closed form
- There are various ways to do this, we use the idea of Lie transformations

Hamiltonian for a Sextupole III

- Lie transformations provide a means to construct a dynamical map in closed form, even from a Hamiltonian that is non integrable
- again it is necessary to make some approximations, and these need to be understood in some detail

Using Lie operator notation, we can write the map for a particle moving through the sextupole as:

$$\boxed{\xi(s) = e^{-:H:\,s}\xi(0)} \tag{17}$$

with $\xi = (\mathbf{q}, \mathbf{p})^T$. Since the Lie transformation evolves the dynamical variables according to Hamilton's equations (for the Hamiltonian H) the map expressed in the form (17) is necessarily symplectic. Since application of a Lie transformation just involves differentiation and summation (of an infinite series) we can, in principle, apply the map in this form, for any Hamiltonian.

Sextupole map: What about symplecticity

Map to second order in s.

$$x_f = x_i + \frac{px_i s}{\sqrt{1 - px_i^2}} - \frac{k_2 x_i^2 s^2}{4\left(1 - px_i^2\right)^{\frac{3}{2}}} + \mathcal{O}(s^3), \tag{18}$$

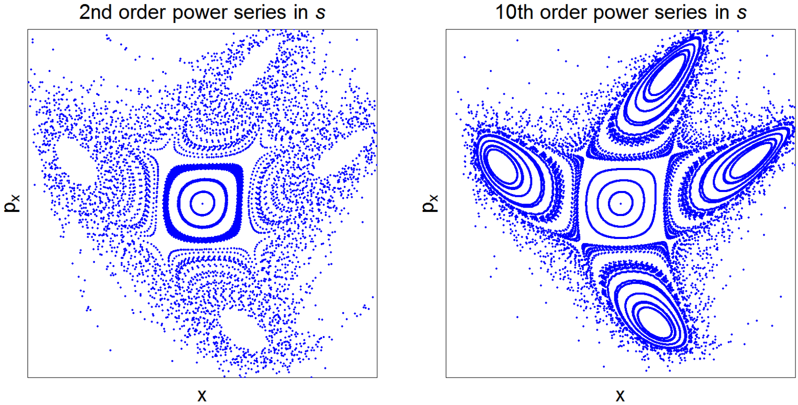
$$px_f = px_i - \frac{1}{2} k_2 x_i^2 s - \frac{k_2\ x_i px_i\ s^2}{2\sqrt{1 - px_i^2}} + \mathcal{O}(s^3). \tag{19}$$

The higher order terms get increasingly complicated and difficult to interpret.

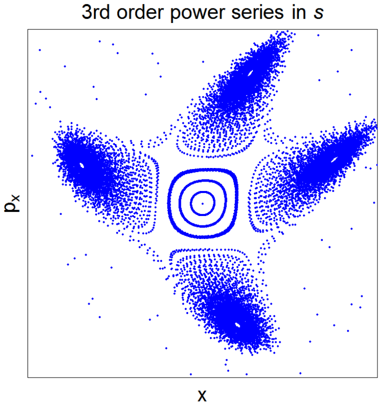
How important is the symplectic error?

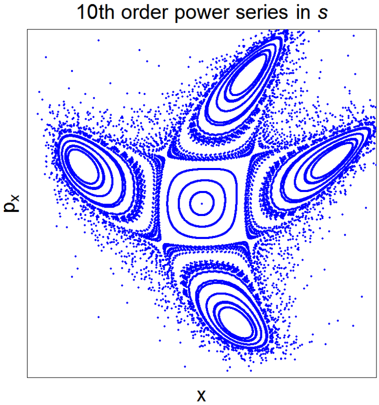
The following plots compare the results using sextupole maps of different orders in s. Note that we use a linear phase advance of $0.246 \times 2\pi$ between sextupoles, a sextupole length of 0.1 m, and a sextupole strength $k_2 = -6000\,\mathrm{m}^{-3}$.

Sextupole map: Lie transformation approach

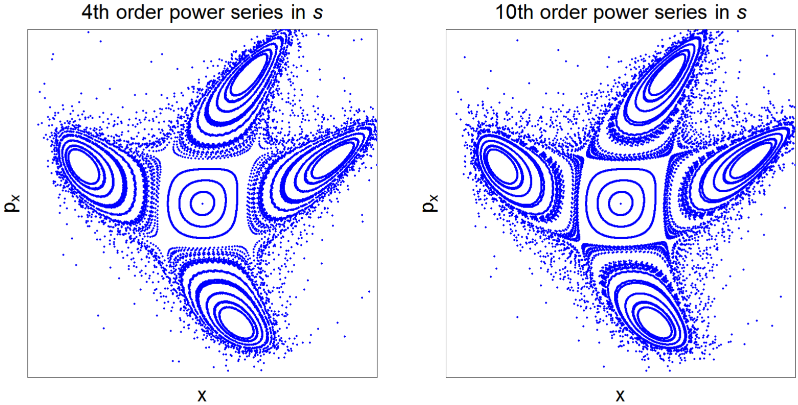


Sextupole map: Lie transformation approach

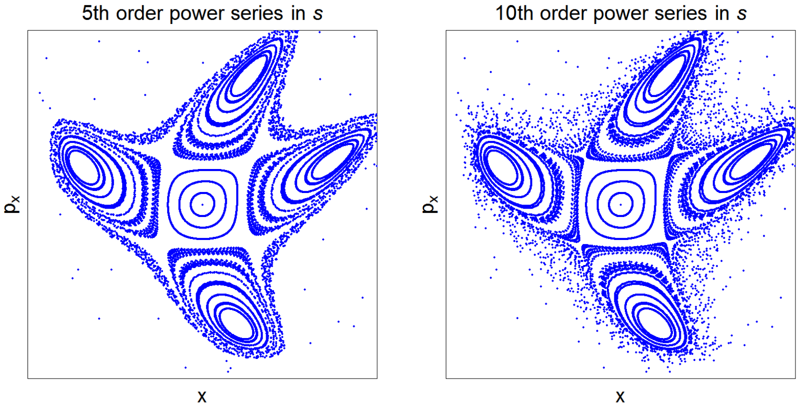




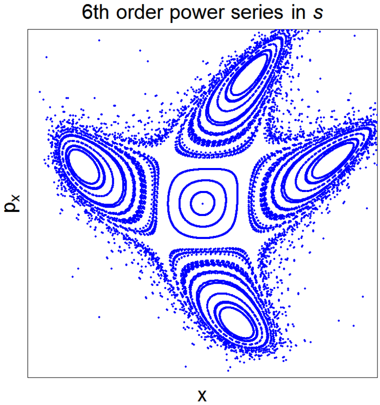
Sextupole map: Lie transformation approach

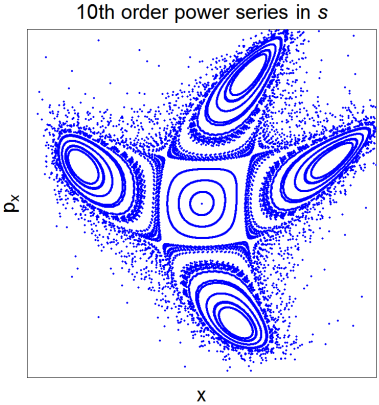


Sextupole map: Lie transformation approach



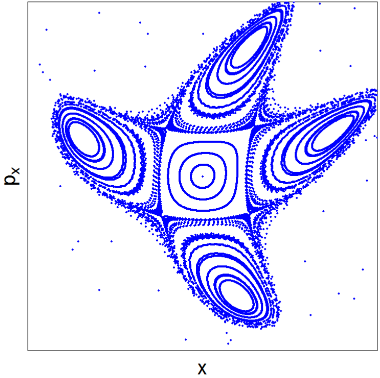
Sextupole map: Lie transformation approach





Sextupole map: Lie transformation approach

8th order power series in *s*



10th order power series in *s*