ETH zürich

LQCD I: Lattice Gauge Theories

Prof. Marina Krstic Marinkovic, ETH Zürich

4-10 August 2024, Lyceum Alpinum, Zuoz

PSI Particle Physics Summer School – From Low to High: Particle Physics at the Frontier

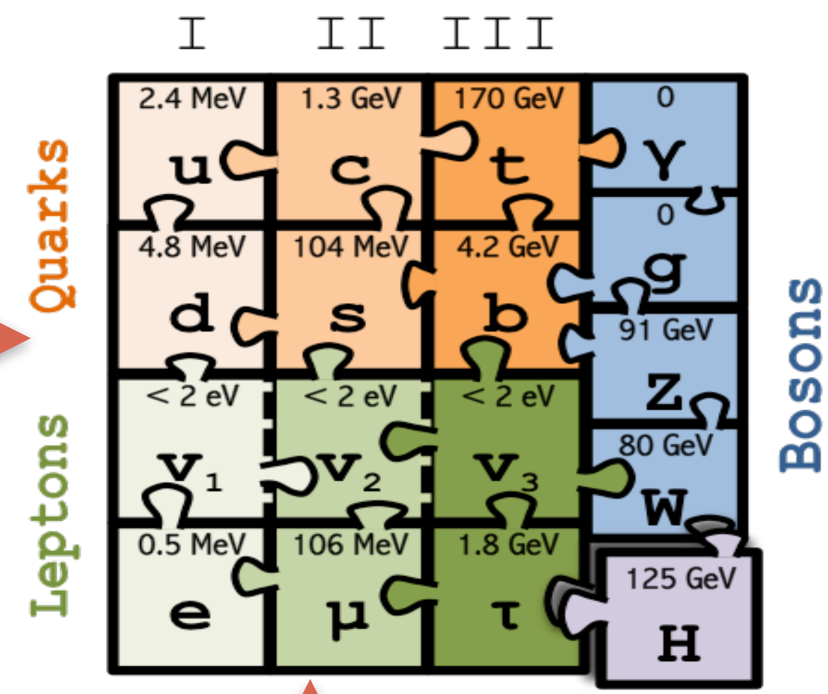
Scientific trajectory

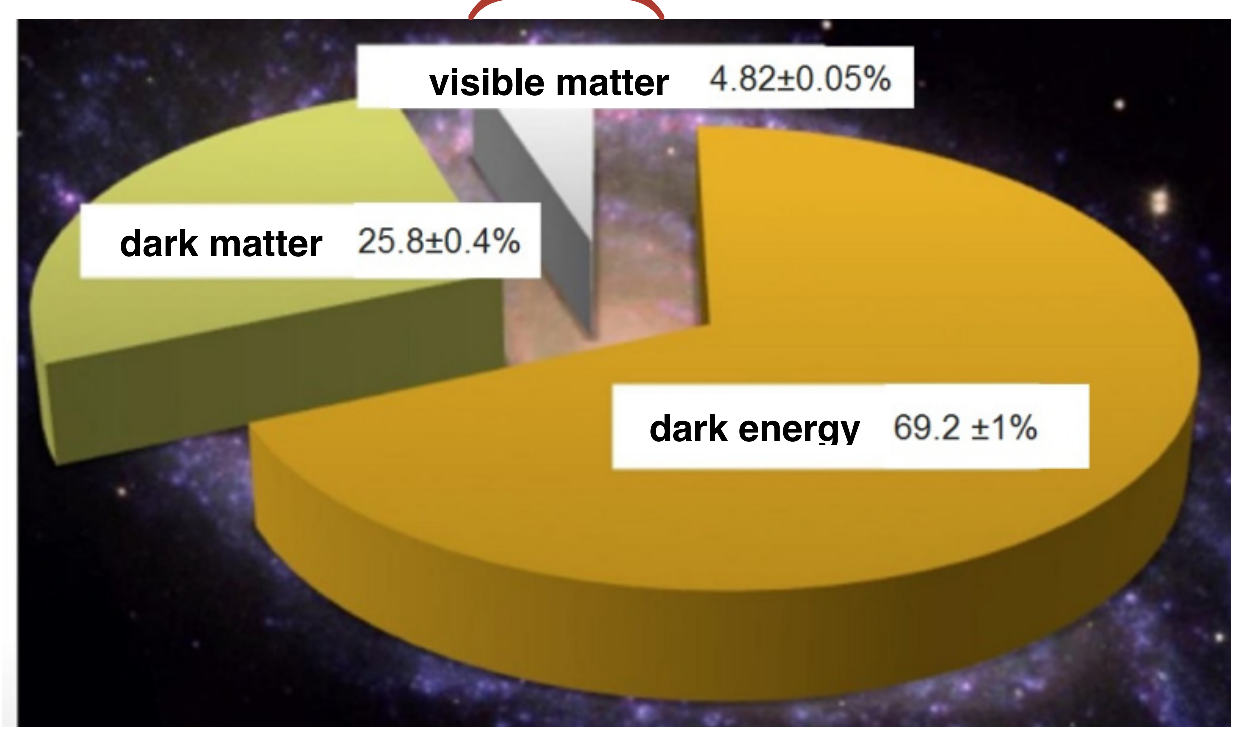
- ➡ **2006** DESY Zeuthen (Th. Particle Phys.) / University of Oslo (Quantum Chemistry)
- ➡ **2008 - 2009** MSc Thesis: DESY Zeuthen/Humboldt Universität zu Berlin, Germany
- ➡ **2009 - 2013** PhD Thesis: HU Berlin, Germany
- ➡ **2012 - 2014** Research Fellow: University of Southampton, UK
- ➡ **2014 - 2017** CERN Fellow (Visiting Scientist since October 2017)
- ➡ **2016 - 2019** Assistant Professor, Trinity College Dublin, Ireland
- ➡ **2020 - 2021** Junor-professorship at LMU Münich (3+3y)
- ➡ **2021 -** **Assistant Prof. in Computational Physics** (Tenure-track)

Building blocks of the universe

- **Four fundamental interactions:**
  - ➡ Electromagnetism
  - ➡ Weak
  - ➡ Strong
  - ➡ Gravity

Standard Model
of Particle Physics





[Planck CMB Data (2013 and 2015)]

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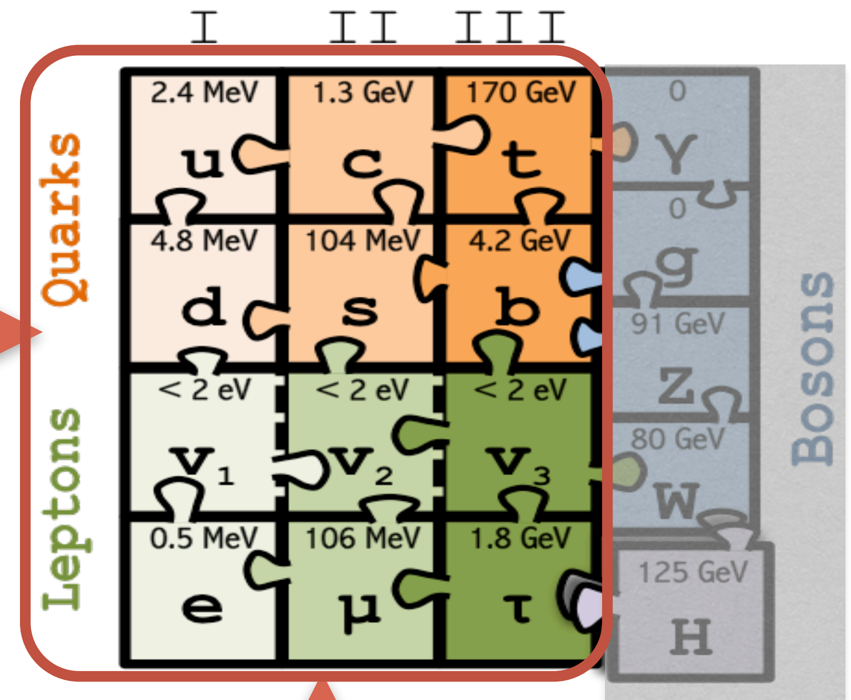

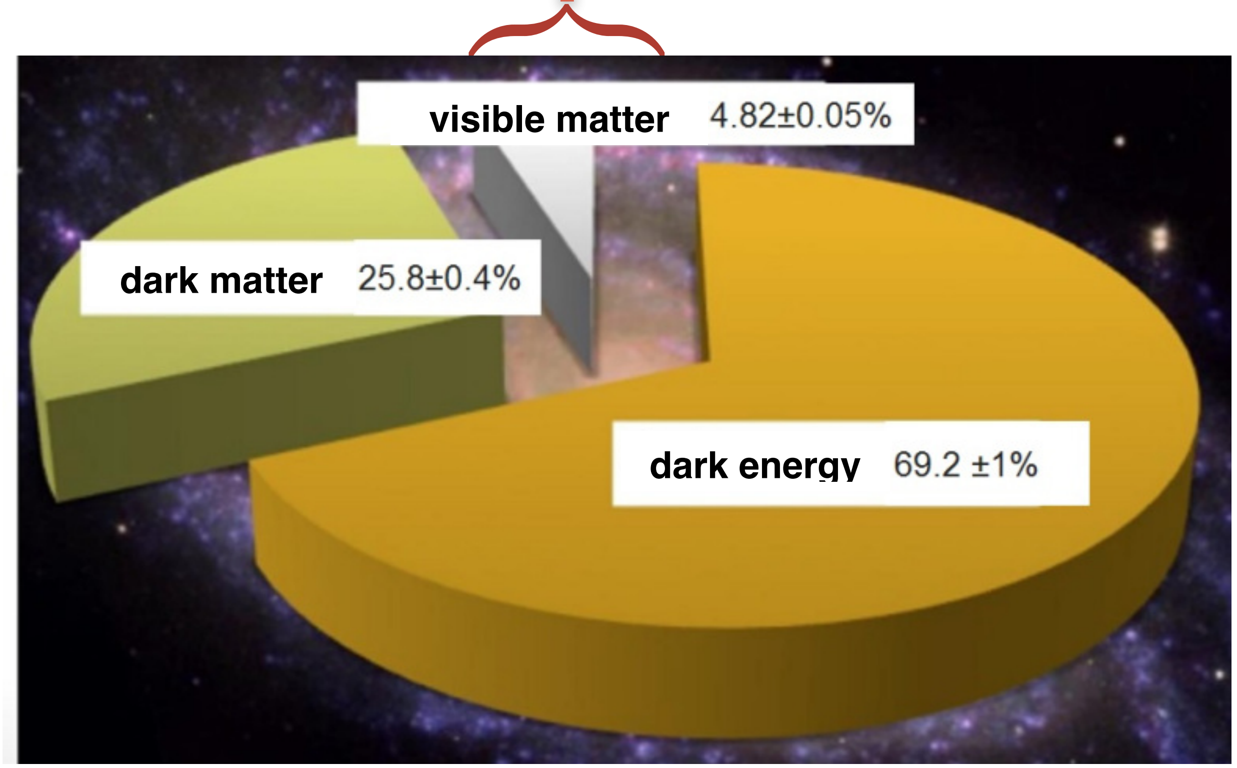

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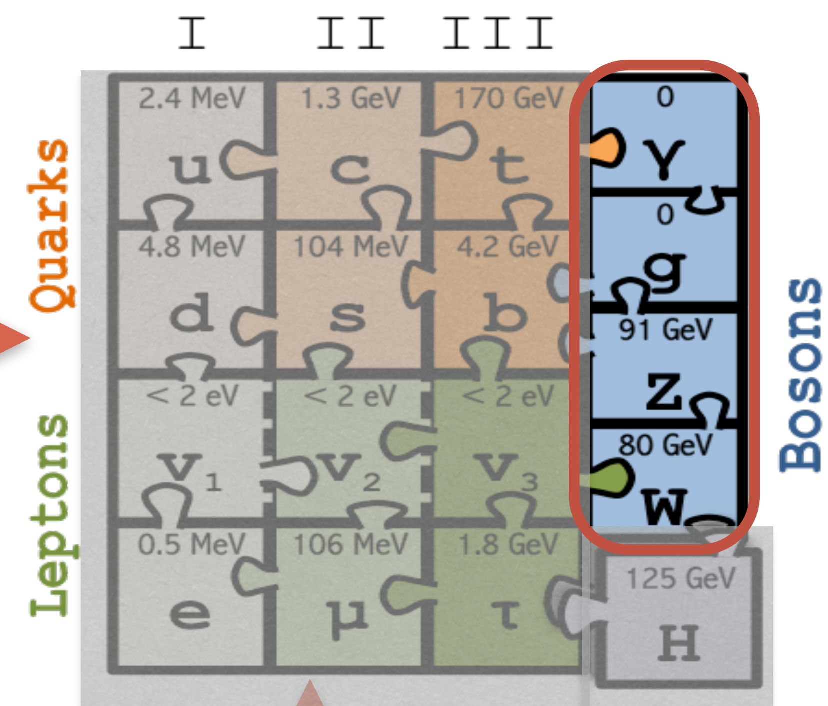

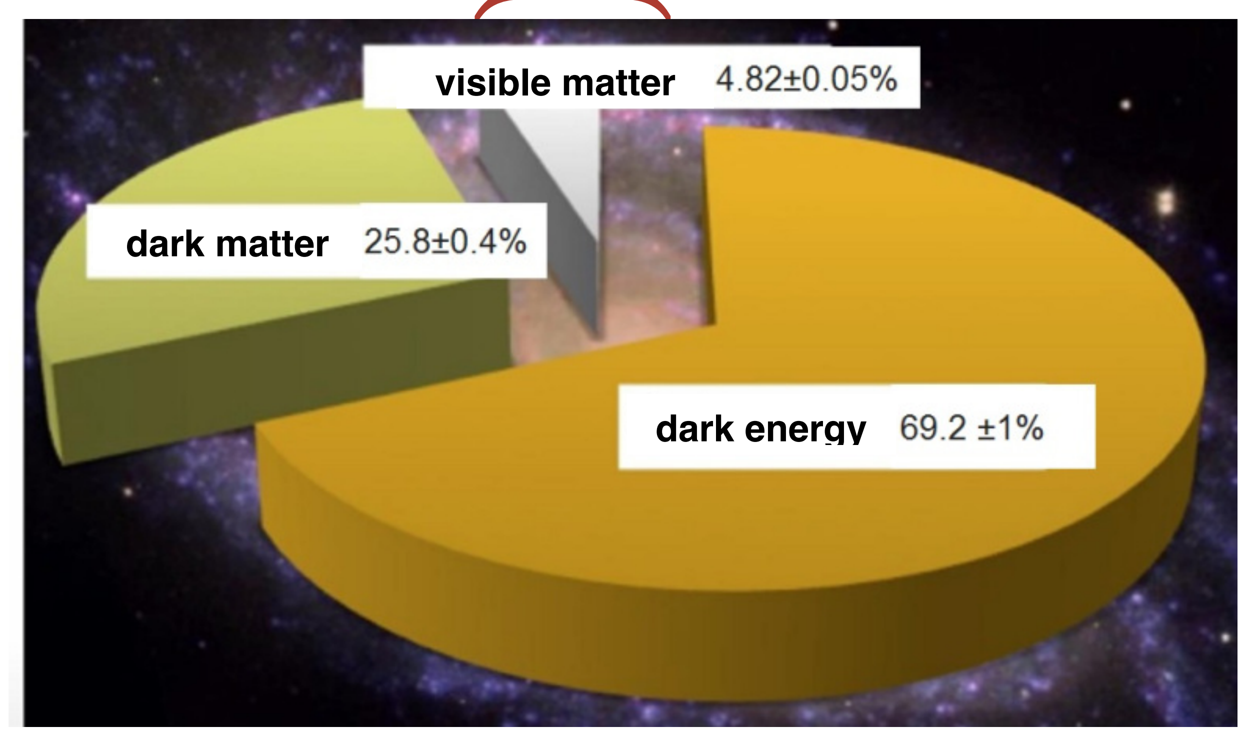

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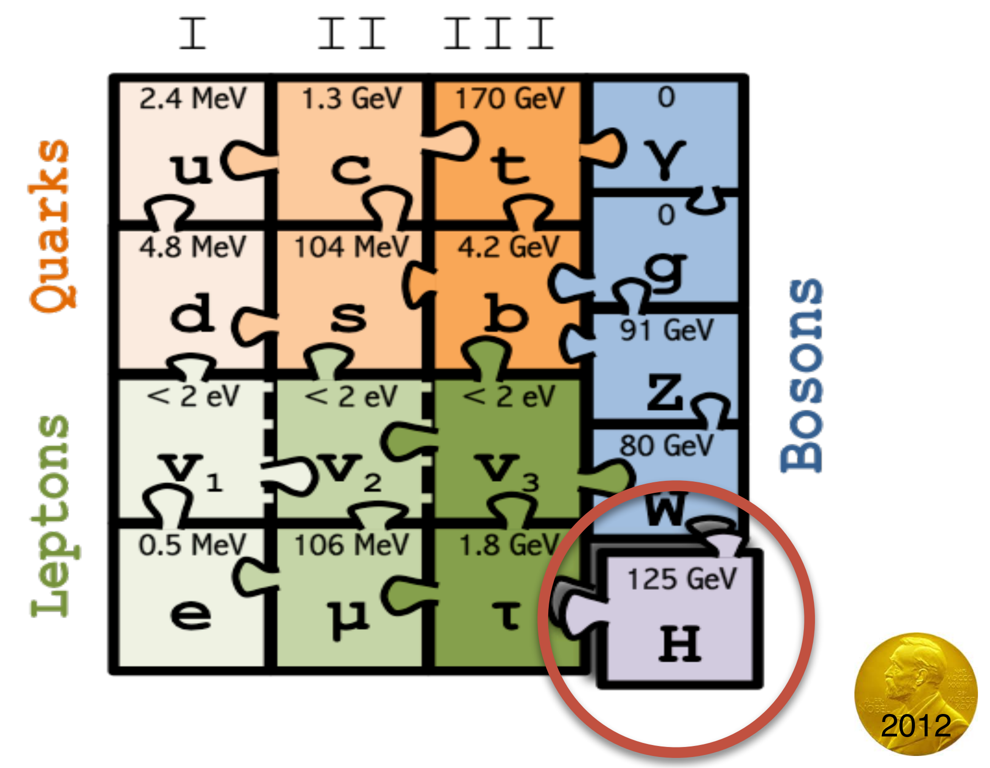

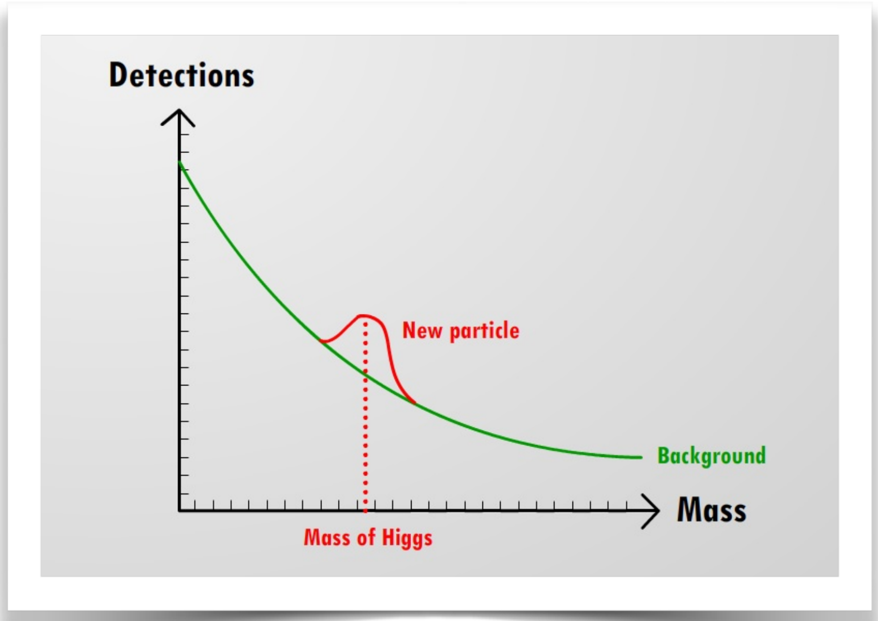

[https://blog.higgshunters.org/]

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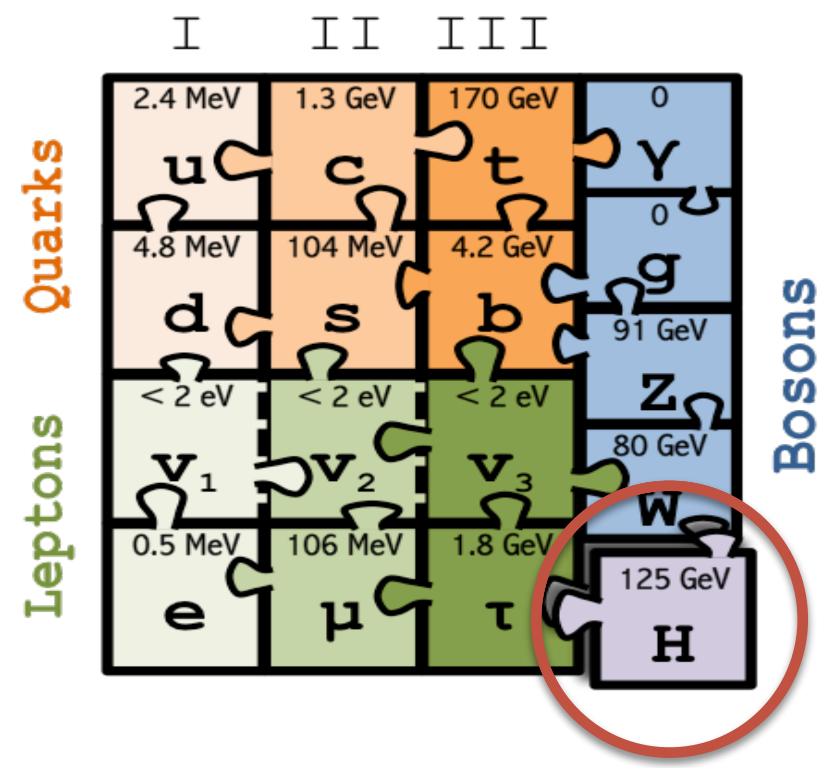

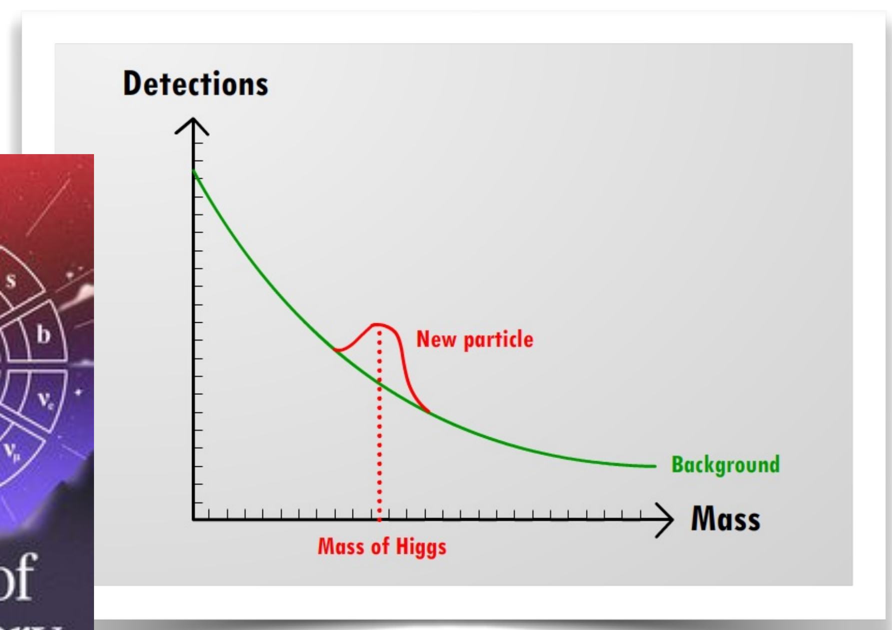

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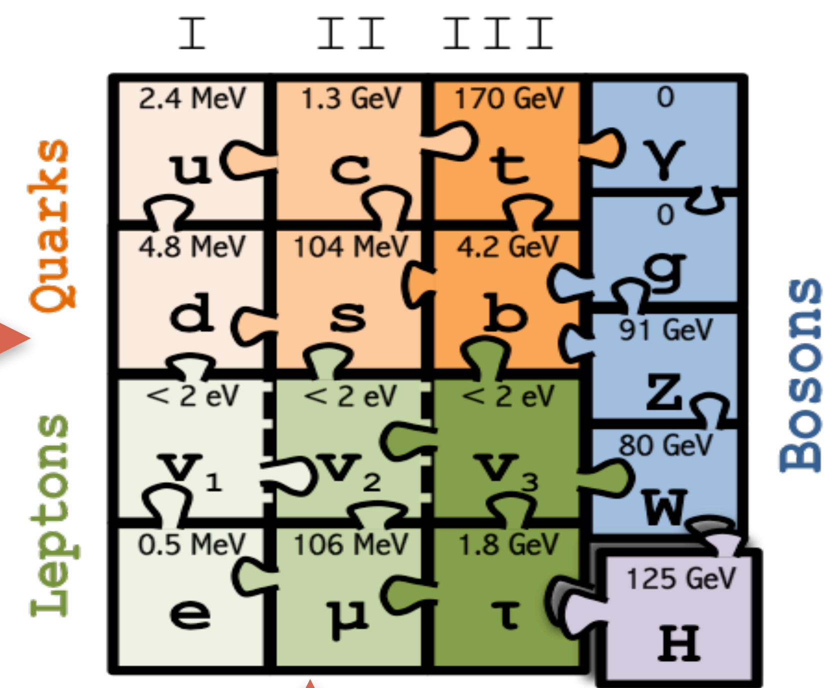
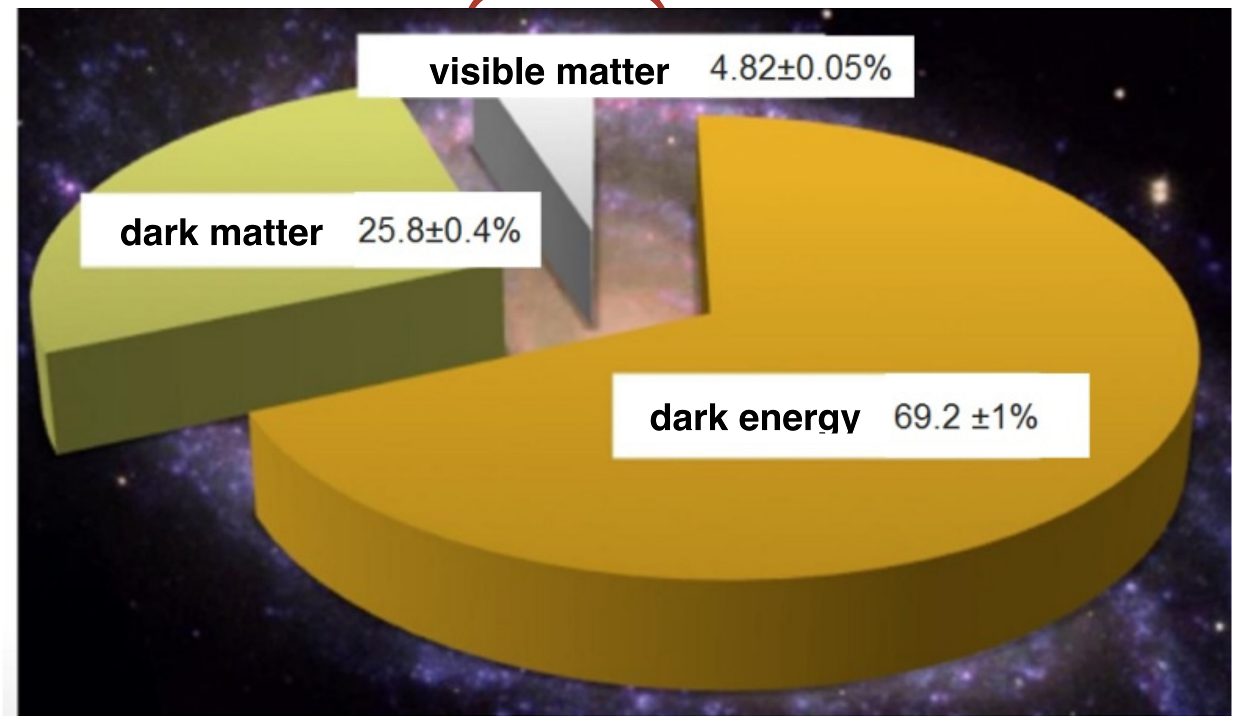
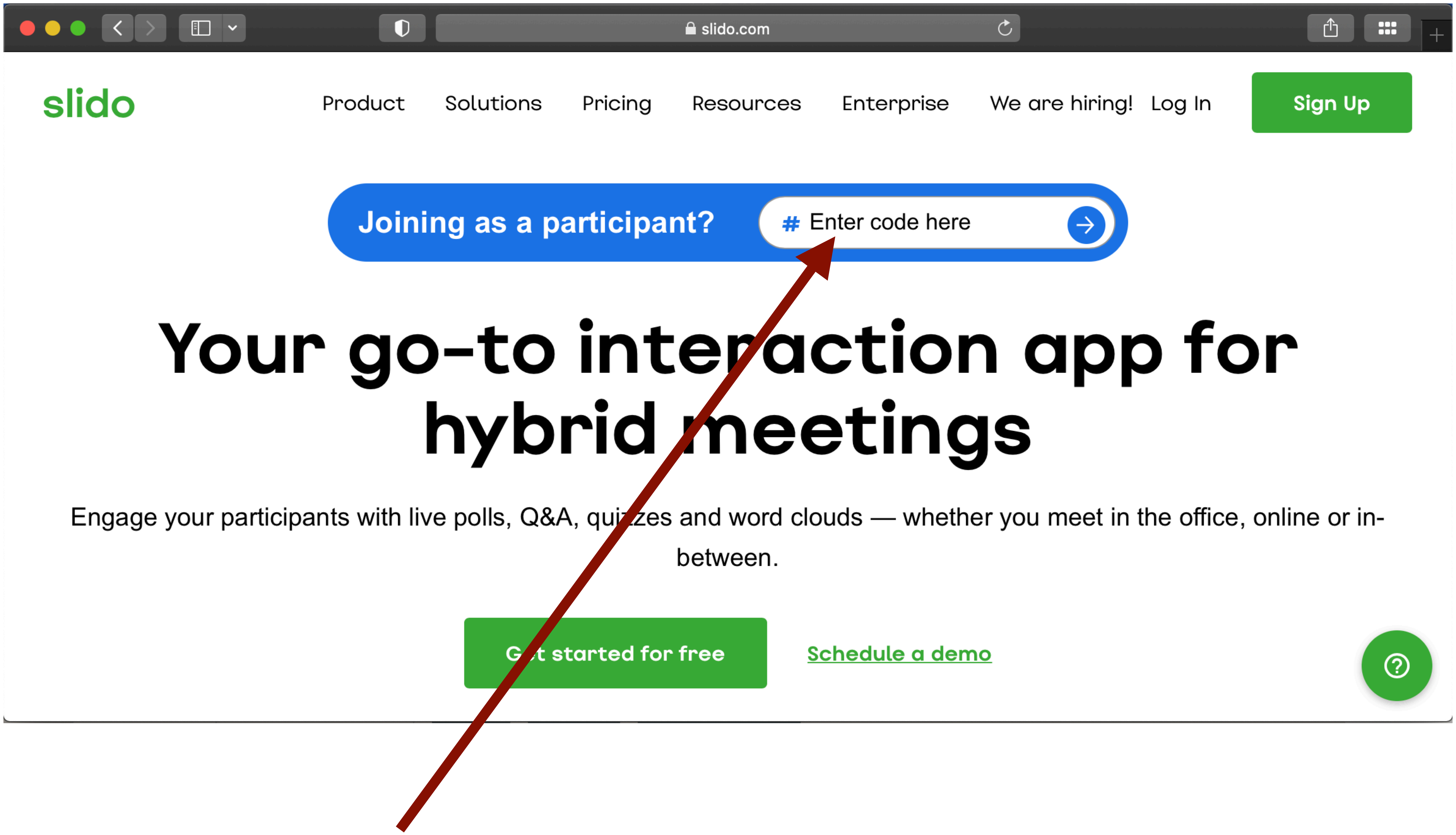




[Planck CMB Data (2013 and 2015)]

See lectures in this school:
[R. Gonzalez Suarez; Wed.,Fri.]
[P. F. Monni; Mon.,Wed.]
[V. Cirigliano; Tue.,Thurs.]
[A. Soter; Mon.,Tue.]

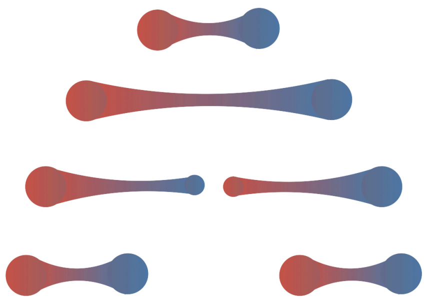
Go to slido.com



- **Poll code: #lattice1**

The strong interaction

- **Commonly accepted theory of strong interaction is Quantum Chromodynamics (QCD)**
- Fundamental parameters of **QCD:**
  - ➡ gauge coupling: g
  - ➡ quark masses: $m_u, m_d, m_s, \ldots$



- **Two exciting properties of QCD:**
  - ➡ color confinement
  - ➡ asymptotic freedom
- Strong coupling "constant":

$$\alpha_s(Q^2) = \frac{\bar{g}^2(Q^2)}{4\pi}$$

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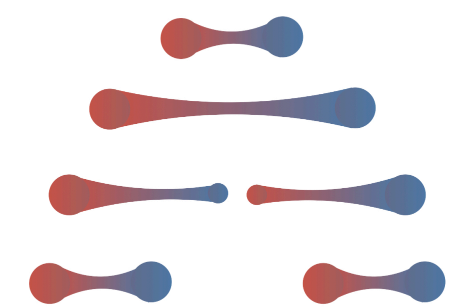

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[D. Gross, F. Wilczek; D. Politzer (1973)]

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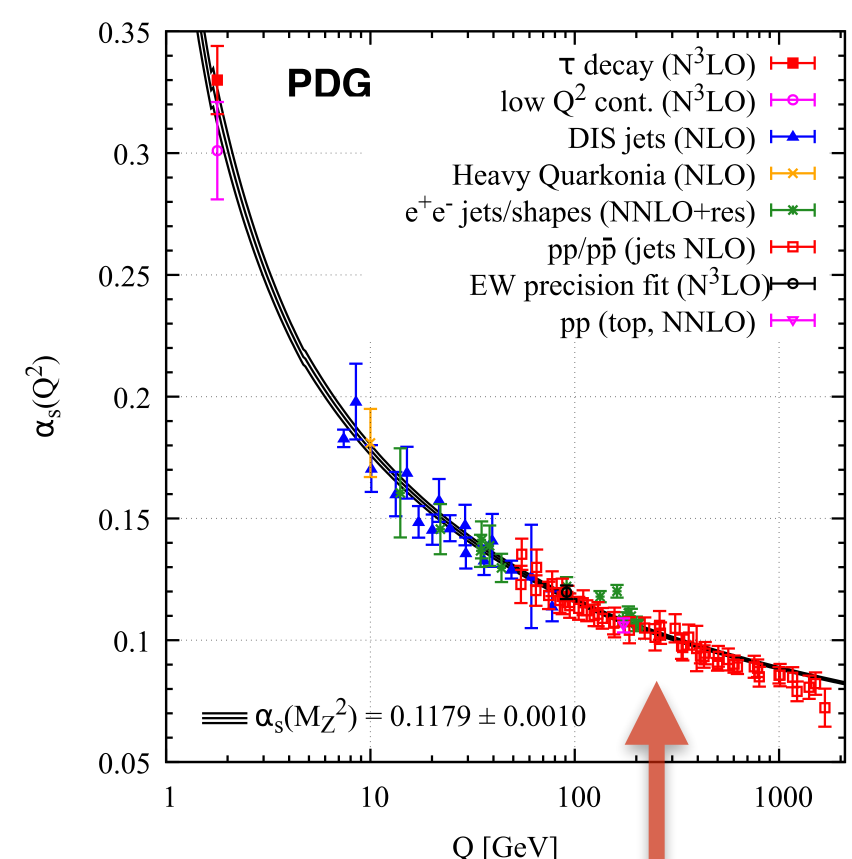
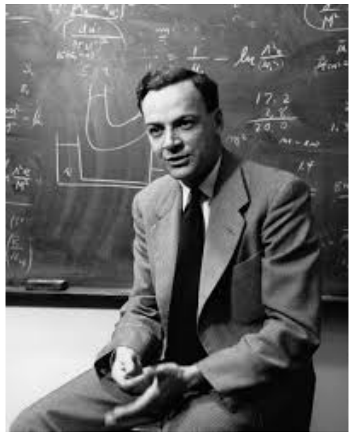
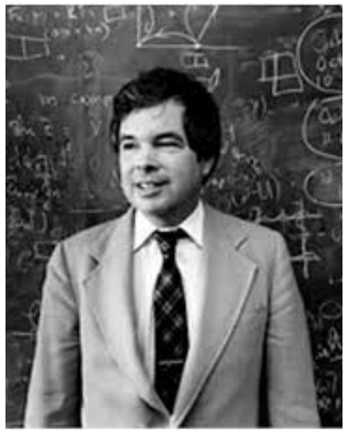


Non-perturbative treatment of QCD

[R. P. Feynman, "Space-Time Approach to Non-Relativistic Quantum Mechanics" Rev. Mod. Phys. 20, 367 (1948)]



[K. Wilson, "Confinement of quarks" *Phys. Rev.D*. **10** (8): 2445–245 (1974)]

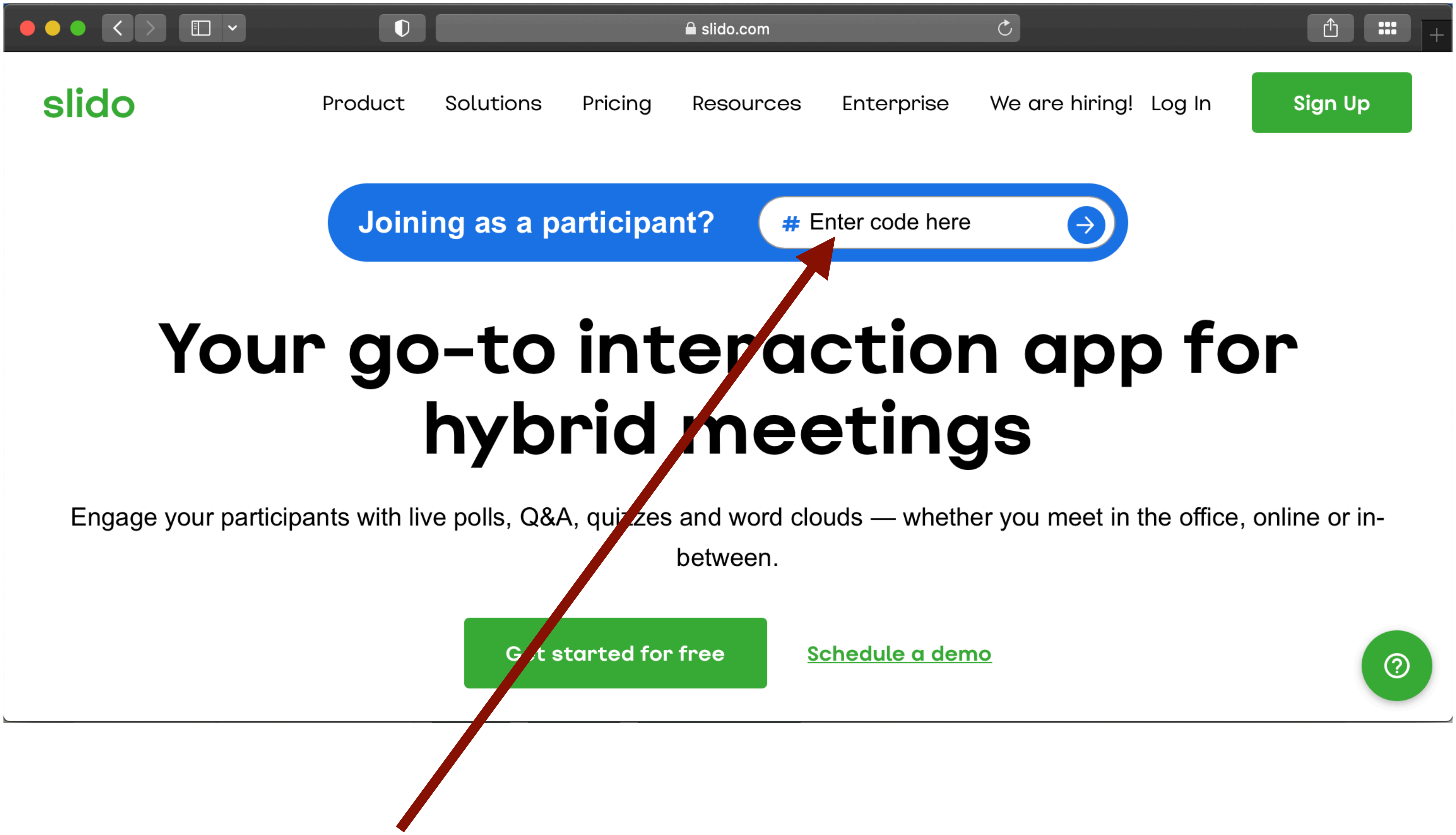


(1) Path Integral quantization

(2) Continuation to Euclidean time

(3) Lattice regularization

Go to slido.com



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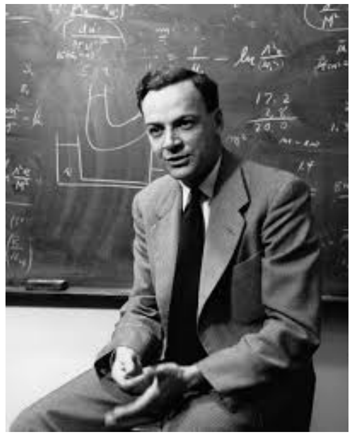

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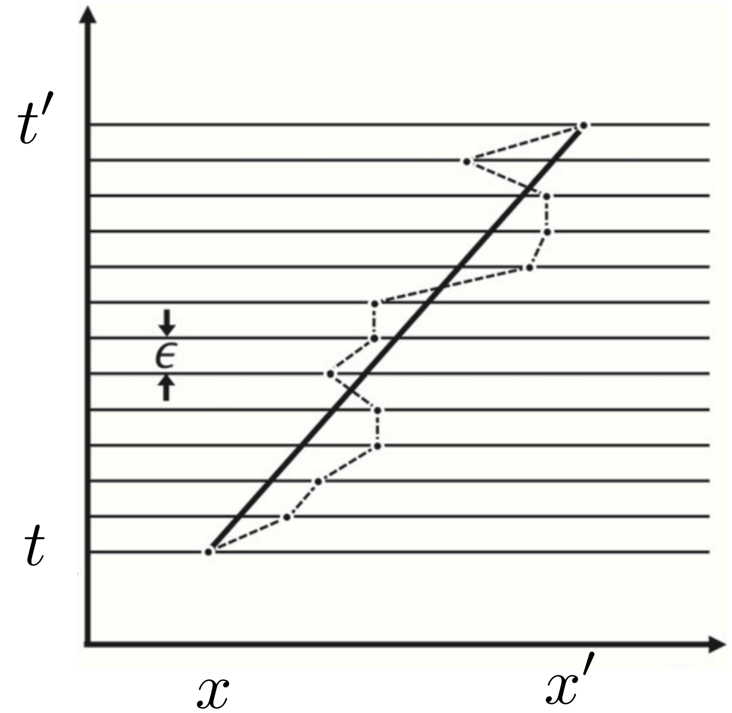
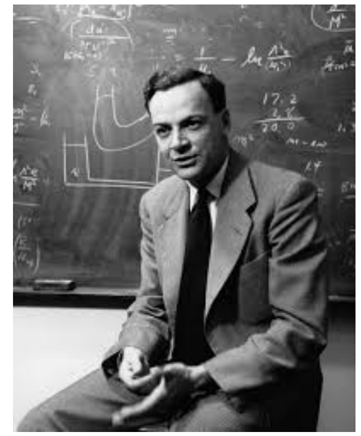
$$\langle x'|U(t',t)|x\rangle = \int dx_1 \, \langle x'|U(t',t_1)|x_1\rangle \, \langle x_1|U(t_1,t)|x\rangle$$



$$\langle x'|U(t',t)|x\rangle \; = \; \int \mathcal{D}x \; e^{\frac{i}{\hbar}S[x]}$$

Non-perturbative treatment of QCD



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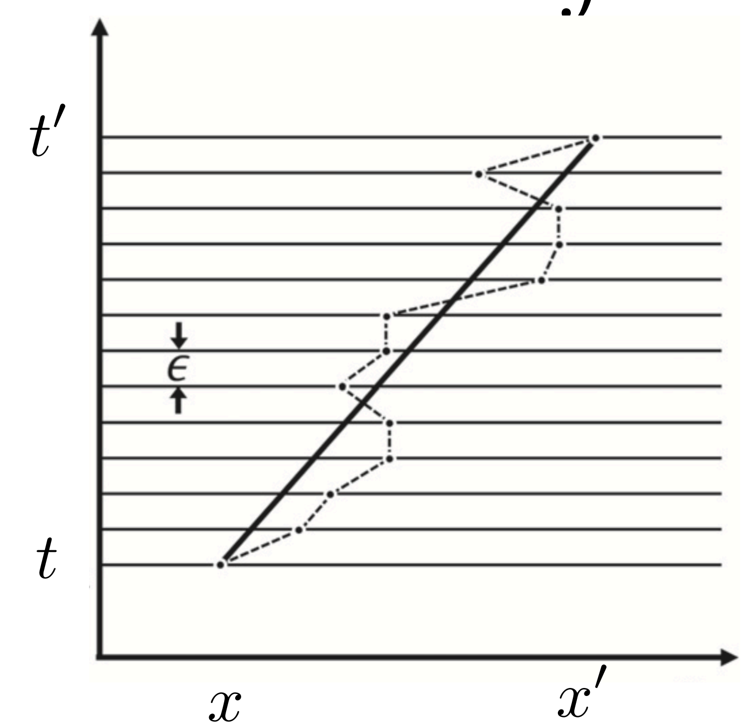
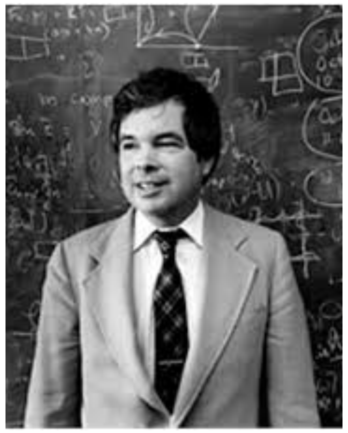


$$\epsilon \longrightarrow -i\,a$$

$$Z = \int \mathcal{D}x \, e^{-\frac{1}{\hbar} S_E[x]}$$

Non-perturbative treatment of QCD



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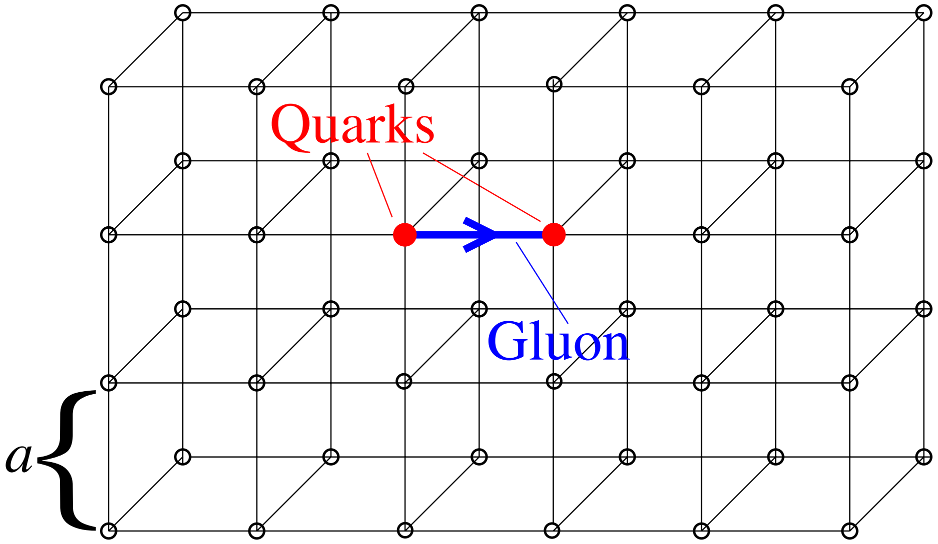

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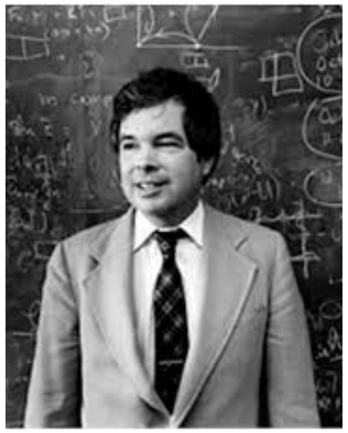

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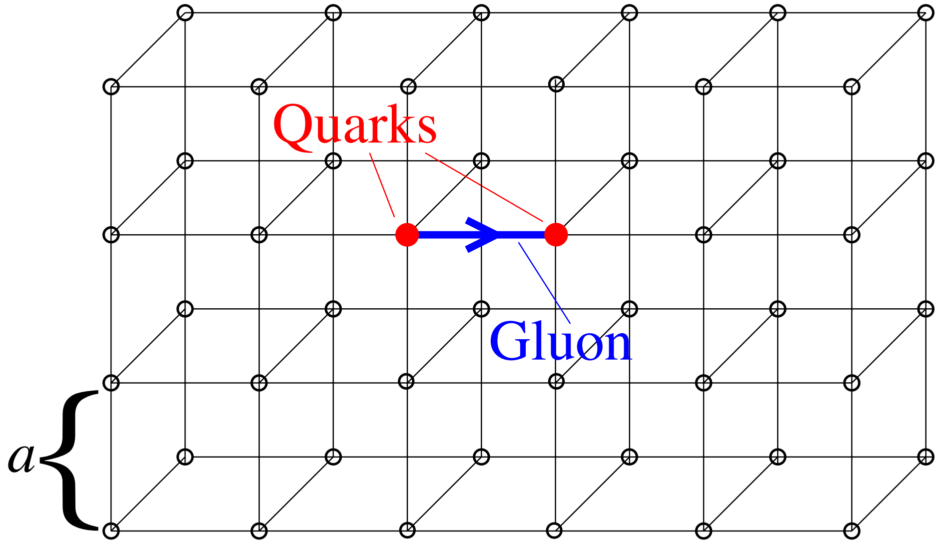
$$\mathcal{L}^E_{QCD} = \frac{1}{2g} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_{f=u,d,s,\ldots} \overline{\psi}_f \left\{ \gamma_\mu \left(\partial_\mu + i A^a_\mu T^a \right) + m_f \right\} \psi_f$$

$$S^E_{QCD} = \int d^4x \, \mathcal{L}^E_{QCD}$$



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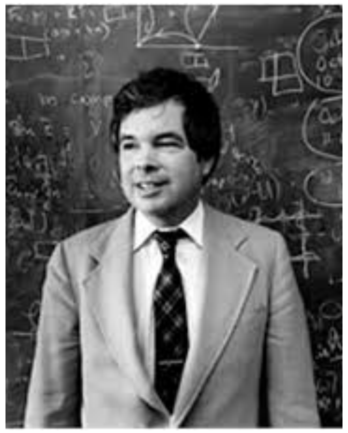

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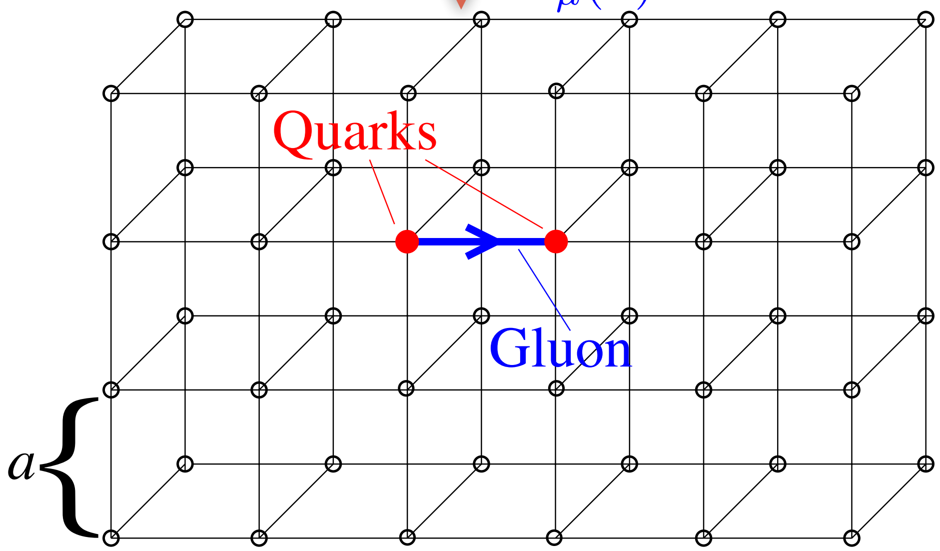
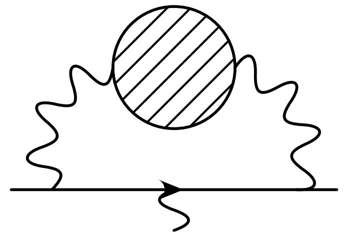
→ CLASSICAL STATISTICAL MECHANICS

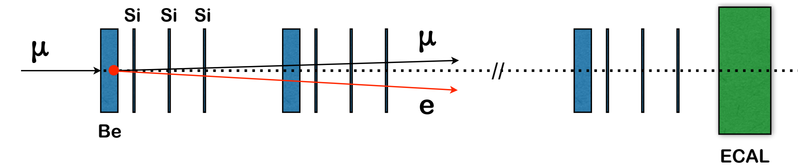
$$Z = \int \mathcal{D}U\, \mathcal{D}\psi\, \mathcal{D}\overline{\psi}\, e^{-S^E_{QCD}}$$

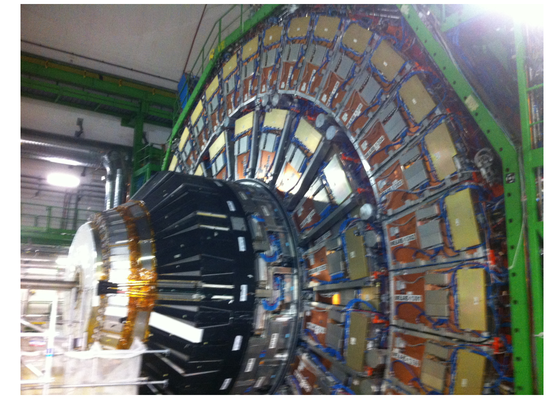
$$U_\mu(x) = e^{iagA_\mu(x)}$$



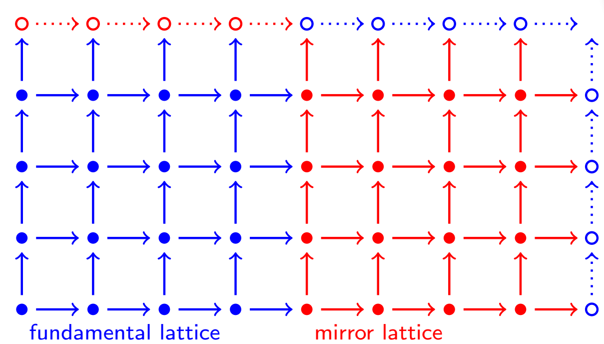
Physics (Theory&Experiment)



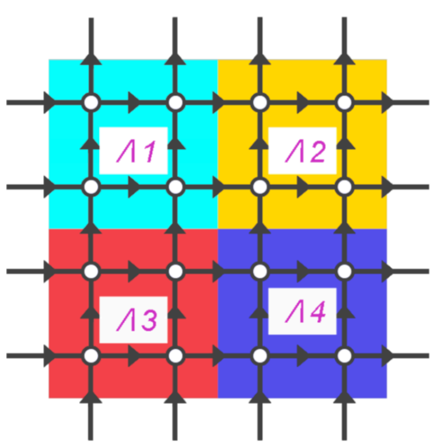




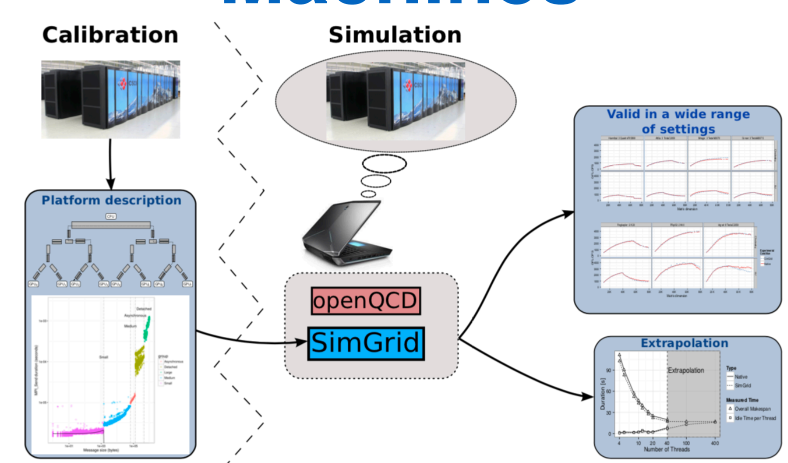
Lattice QCD



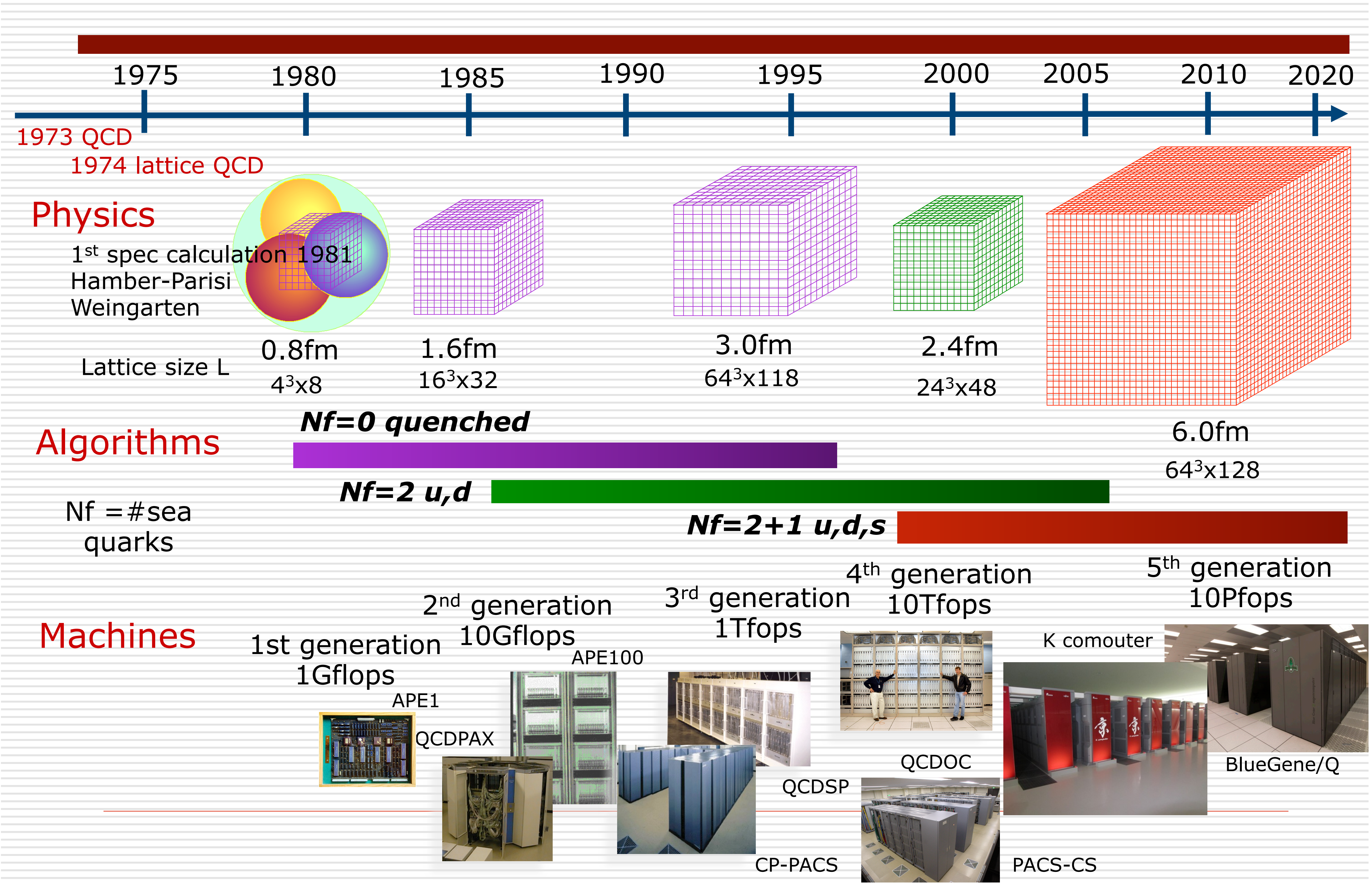
Computational Methods



Machines

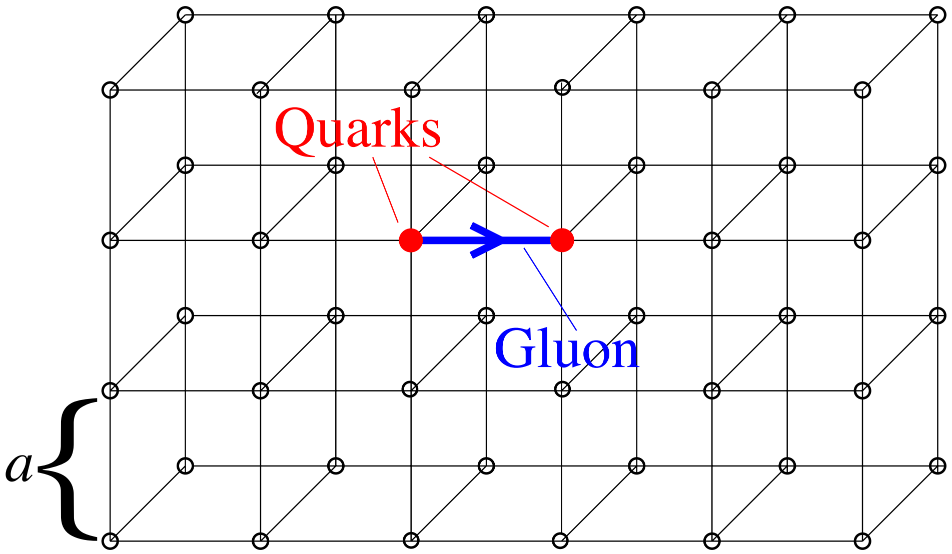


Development of Lattice QCD



[Credit: A. Ukawa, HPC Summer School (2013)]

Typical Lattice QCD Computation



https://www.cscs.ch/publications/photo-gallery/]

Altamira@IFCA, Santander

Wilson Cluster at Mainz U.

- physical lattice size: **~6fm**, spacing **0.05-0.1fm**
- **64^3 x 128** lattice —> **34 x 10^6** points
- Operator dimensions: **10^7 x 10^7** matrices
- **Advanced computational methods** are needed
- **Large computer resources**: multiple TFlop years!

Backstage of LGT Calculations

[bbc tyne in pictures backstage at theatre royal; image by www.bbc.co.uk]

Plan of the Lectures:

- **Lecture 1: Path Integral Quantization and scalar fields on the lattice;** *Why do we need to consider discrete space/time?*

- **Lecture 2: QCD and QED on the lattice; Computational methods for lattice field theories;** *Why is LQCD so comp. expensive? What are the limitations? Where can we move the needle with Lattice QCD+QED?*

Part 1:
Path Integral Quantization and scalar fields on the lattice

Outline

- Point Mechanics vs. Classical Field Theory
- Path Integral in Quantum Mechanics (real and Euclidean time)
- Scalar Field Theory on the lattice
- Analogy with Statistical Mechanics and continuum limit
- Spectrum of the Scalar Theory on the lattice

Classical Point Mechanics

- **Point Mechanics — 2nd Newton's law:**

$$m\frac{d^2x}{dt^2} = m\,\partial_t^2 x = F(x) = -\frac{dV(x)}{dx}$$

m **— mass of the classical non-relativistic point particle**

$V(x)$ **— external potential**

- **The principle of least action:**

$$S[x] = \int dt\ \mathcal{L}(x, \partial_t x)$$

- **Lagrange function:**

$$\mathcal{L}(x, \partial_t x) = \frac{m}{2}(\partial_t x)^2 - V(x)$$

- **Euler-Lagrange equation:**

$$\partial_t \frac{\delta \mathcal{L}}{\delta(\partial_t x)} - \frac{\delta \mathcal{L}}{\delta x} = 0$$

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Classical Field Theory

- **Generalization to systems with infinitely many d.o.f. (field values $\phi(\vec{x})$)**

- **Classical field e. o. m. for neutral scalars — Klein-Gordon eq.:** $\partial_\mu \partial^\mu \phi = -\frac{dV(\phi)}{d\phi}$

- **Again, the classical e. o. m obtained by minimizing the action:**

$$S[\phi] = \int d^4x \; \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu x)} - \frac{\delta \mathcal{L}}{\delta x} = 0$$

- **A simple example of interacting scalar field theory (** `$\phi^4$ — *theory*` **)**

$$V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

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Point Mechanics vs. Classical Field Theory

Point Mechanics	Field Theory
time t	space-time $x = (t, \vec{x})$
particle coordinate x	field value ϕ
particle path $x(t)$	field configuration $\phi(x)$
action $S[x] = \int dt\ L(x, \partial_t x)$	action $S[\phi] = \int d^4x\ \mathcal{L}(\phi, \partial_\mu \phi)$
Lagrange function $L(x, \partial_t x) = \frac{m}{2}(\partial_t x)^2 - V(x)$	Lagrangian $\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - V(\phi)$
equation of motion $\partial_t \frac{\delta L}{\delta(\partial_t x)} - \frac{\delta L}{\delta x} = 0$	field equation $\partial_\mu \frac{\delta L}{\delta(\partial_\mu \phi)} - \frac{\delta L}{\delta \phi} = 0$
Newton's equation $\partial_t^2 x = -\frac{dV(x)}{dx}$	Klein-Gordon equation $\partial_\mu \partial^\mu \phi = -\frac{dV(\phi)}{d\phi}$
kinetic energy $\frac{m}{2}(\partial_t x)^2$	kinetic energy $\frac{1}{2}\partial_\mu \phi \partial^\mu \phi$
harmonic oscillator potential $\frac{m}{2}\omega^2 x^2$	mass term $\frac{m^2}{2}\phi^2$
anharmonic perturbation $\frac{\lambda}{4}x^4$	self-interaction term $\frac{\lambda}{4}\phi^4$

[Credit: U.-J. Wiese https://inspirehep.net/literature/946884]

- **Point Mechanics vs. Classical Field Theory**

Path Integral in Quantum Mechanics: real time (I)

- **Time-dependent Schrödinger Eq.**

$$i\hbar\partial_t \left|\Psi(t)\right\rangle \; = \; H \left|\Psi(t)\right\rangle$$

- **Time evolution operator**

$$U(t',t) = e^{-\frac{i}{\hbar}H(t'-t)}; \qquad \left|\Psi(t')\right\rangle \; = \; U(t',t)\left|\Psi(t)\right\rangle$$

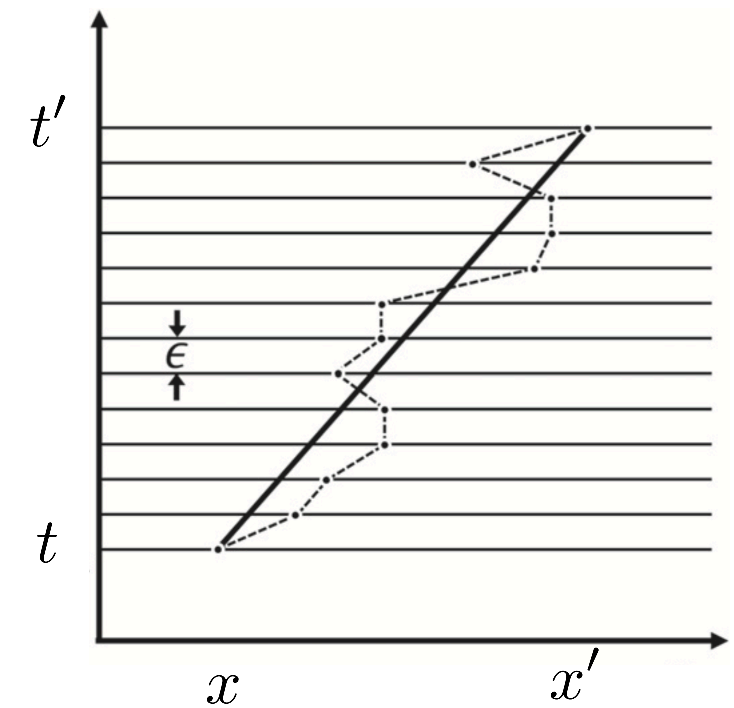
H — **time independent Hamilton operator**

- **Transition amplitude of a non-relativistic point particle — a propagator:**

$$\langle x'|U(t',t)|x\rangle$$

- **Contains information about the energy spectrum of the theory.**

Path Integral in Quantum Mechanics: real time (II)



- **Propagator from x to x'**

$$\langle x'|U(t',t)|x\rangle = \int dx_1 \, \langle x'|U(t',t_1)|x_1\rangle \, \langle x_1|U(t_1,t)|x\rangle$$

- **Divide time interval into N elementary steps of size**

$$[t',t]; \qquad t'-t = N\,\epsilon$$

- **Repeat previous procedure at all intermediate times**

$$\langle x'|U(t',t)|x\rangle = \int dx_1 \int dx_2 \, \ldots \int dx_{N-1} \, \langle x'|U(t',t_{N-1})|x_{N-1}\rangle \times \, \ldots \, \times \langle x_2|U(t_2,t_1)|x_1\rangle \times \langle x_1|U(t_1,t)|x\rangle$$

- **Take factor $\langle x_{i+1}|U(t_{i+1},t_i)|x_i\rangle$; assume single non-relativistic point particle**

$$H = \frac{p^2}{2m} + V(x)$$

- **Insert complete set of states + BCH formula:**

$$\langle x_{i+1}|U(t_{i+1},t_i)|x_i\rangle = \frac{1}{2\pi}\int dp \; e^{-\frac{i\epsilon p^2}{2m\hbar}} \; e^{-\frac{i}{\hbar}p(x_{i+1}-x_i)} \; e^{-\frac{i\epsilon}{\hbar}V(x_i)}$$

Path Integral in Quantum Mechanics: real time (III)

$$\langle x_{i+1}|U(t_{i+1},t_i)|x_i\rangle = \frac{1}{2\pi}\int dp\; e^{-\frac{i\epsilon p^2}{2m\hbar}}\; e^{-\frac{i}{\hbar}p(x_{i+1}-x_i)}\; e^{-\frac{i\epsilon}{\hbar}V(x_i)}$$

- $\int dp$ **ill-defined: integrand rapidly oscillating f-on**

- **For it to be well-defined:**

  1. **replace:** $\epsilon \longrightarrow \epsilon - i\,a; \qquad a \in \mathbb{R}$
  2. **evaluate** $\int dp\,\ldots$
  3. **take** $a \to 0$ **limit, one gets:**

$$\langle x_{i+1}|U(t_{i+1},t_i)|x_i\rangle = \sqrt{\frac{m}{2\pi i\hbar\epsilon}}\; e^{\frac{i}{\hbar}\epsilon\left[\frac{m}{2}\left(\frac{x_{i+1}-x_i}{\epsilon}\right)^2 - V(x_i)\right]}$$

- **Insert into original propagator:**

$$\langle x'|U(t',t)|x\rangle \;=\; \int \mathcal{D}x\; e^{\frac{i}{\hbar}S[x]}$$

$$\int \mathcal{D}x \;=\; \lim_{\epsilon\to 0}\sqrt{\frac{m}{2\pi i\hbar\epsilon}}^{\,N-1}\int dx_1\int dx_2\,\ldots\int dx_{N-1}$$

Path Integral in Quantum Mechanics: real time (IV)

$$\langle x'|U(t',t)|x\rangle = \int \mathcal{D}x \, e^{\frac{i}{\hbar}S[x]}$$

$$\int \mathcal{D}x = \lim_{\epsilon\to 0} \sqrt{\frac{m}{2\pi i\hbar\epsilon}}^{N-1} \int dx_1 \int dx_2 \ldots \int dx_{N-1}$$

- **The action is continuum limit of the discretised action:**

$$S[x] = \int dt \, \Big[\frac{m}{2}(\partial_t x)^2 - V(x)\Big] = \lim_{\epsilon\to 0} \sum_i \epsilon \Big[\frac{m}{2}\Big(\frac{x_{i+1}-x_i}{\epsilon}\Big)^2 - V(x_i)\Big]$$

- **Summary:**

  (i) **Integrate over all particle positions for each intermediate time** t_i

  (ii) **Amounts to integrating over all possible paths of a particle starting at** x **and ending at** x'

  (iii) **Each path weighted with oscillating phase**

  (iv) **Classical path: the smallest oscillations** $\Rightarrow$ **largest contribution to the P.I. (**$\hbar \to 0$ **class. limit)**

- **Note: definition of the path integral required an analytic continuation in time!**

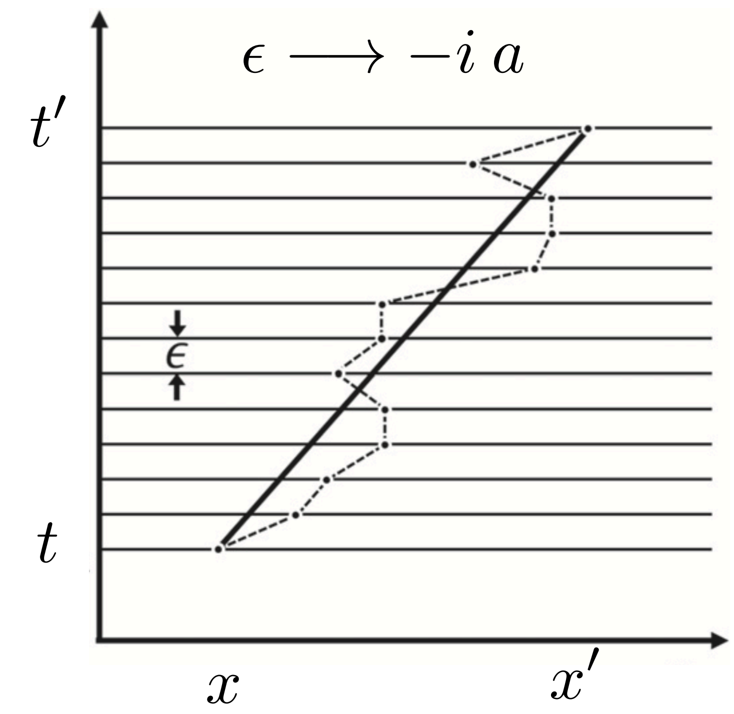
Euclidean Path Integral in Quantum Mechanics

- **Statistical partition function:** $Z = e^{-\beta H}; \quad \beta = \frac{1}{T}$
- $\beta = \frac{i}{\hbar}(t' - t) \quad \Longrightarrow \quad e^{-\beta H} \Leftrightarrow U(t, t')$

System at finite temperature T

$\Updownarrow$

System propagating in purely imaginary time



- **Repeat all the steps from the derivation in real time:**
  - **(i)** **Divide Euclidean time interval into N time steps:** $\beta = \frac{Na}{\hbar}$
  - **(ii)** **Insert complete set of position eigenstates**

- **The Euclidean Path Integral:**

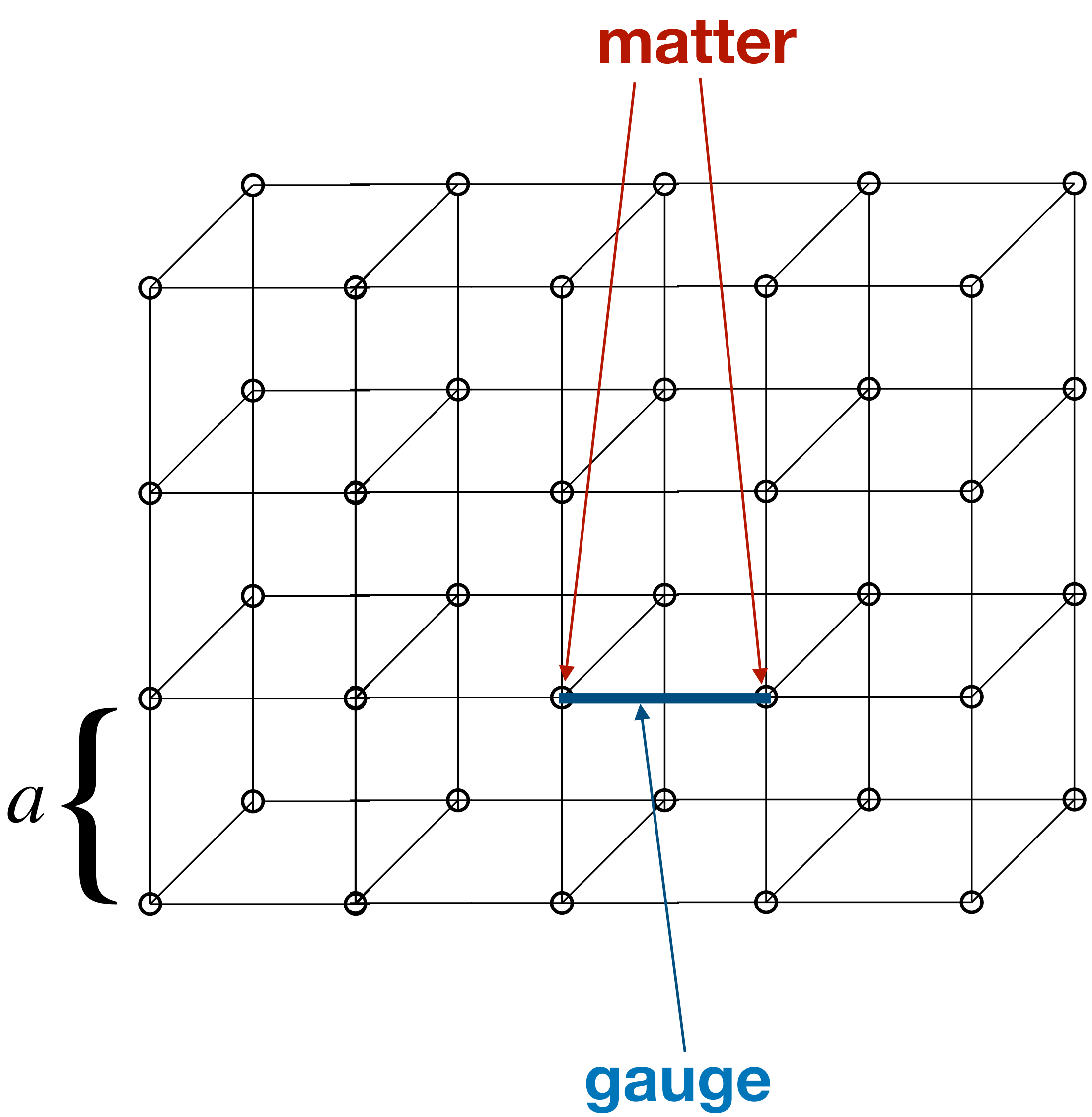
$$Z = \int \mathcal{D}x\, e^{-\frac{1}{\hbar} S_E[x]}$$

$$S_E[x] = \int dt\, \left[\frac{m}{2}(\partial_t x)^2 + V(x)\right] = \lim_{a\to 0} \sum_i a\left[\frac{m}{2}\left(\frac{x_{i+1} - x_i}{a}\right)^2 + V(x_i)\right]$$

$$\int \mathcal{D}x = \lim_{a\to 0} \sqrt{\frac{m}{2\pi i \hbar a}}^{\,N} \int dx_1 \int dx_2 \ldots \int dx_N$$

Scalar fields on the lattice

- **Quantum field theories beyond tree level are plagued by UV divergencies**
- **Defining the theory on a lattice introduces a minimum length**



- **Lattice spacing acts as an UV cut-off:** $\Lambda \sim \frac{1}{a}$

$$Z = \int \mathcal{D}\phi \; e^{-S_E[\phi]}$$

$$\int \mathcal{D}\phi = \prod_x \int d\phi(x)$$

$$S_E[\phi] = \int d^D x \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!}\phi^4\right]$$

Scalar field theory

- **Discretize space-time and define matter fields on the sites of the lattice:**

$$\begin{aligned} x &\longrightarrow an = (an_1, an_2, an_3, an_4); \quad n_\mu \in \mathbb{Z} \\ \phi(x) &\longrightarrow \phi(an) \end{aligned}$$

- **Momentum integrals are restricted to the first Brillouin zone:**

$$\begin{aligned} \int d^4x &\longrightarrow a^4 \sum_{n_\mu} \equiv \sum_x \\ \int \frac{d^4k}{2\pi} &\longrightarrow \int_{|k|<\frac{2\pi}{a}} \frac{d^4k}{2\pi} \equiv \int_k \end{aligned}$$

- **Discretize the derivatives:**

$$\begin{aligned} \partial_\mu \phi(x) &\longrightarrow \nabla_\mu \phi(x) = \frac{1}{a}\left[\phi(x + a\hat{\mu}) - \phi(x)\right] \\ &\longrightarrow \nabla^*_\mu \phi(x) = \frac{1}{a}\left[\phi(x) - \phi(x - a\hat{\mu})\right] \end{aligned}$$

Lattice action

- **Discretized action (bare parameters)**

$$S[\phi] = \sum_x \big[\frac{1}{2}\nabla_\mu\phi(x)\nabla_\mu\phi(x) + \frac{1}{2}m_0^2\phi(x)^2 + \frac{\lambda_0}{4!}\phi(x)^4\big]$$

- **Scalar propagator**

$$\Delta(k)^{-1} = \sum_\mu \big[\frac{2}{a}sin(\frac{k_\mu a}{2})\big]^2 + m_0^2$$
$$= k^2 + m_0^2 + \mathcal{O}(a^2)$$

- **Space time symmetry:** $O(4) \longrightarrow H(4)$

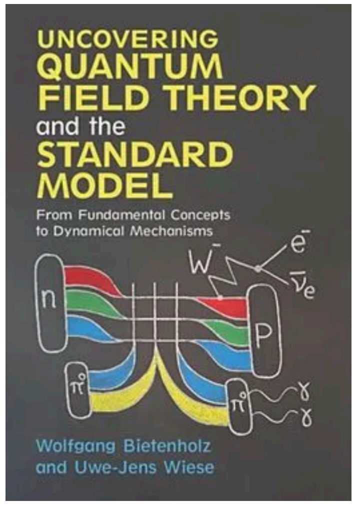
- **Rotation symmetry recovered in the continuum limit**

From P.I. in Quantum Mech. to Statistical Stat. Mech.

Quantum mechanics	Classical statistical mechanics
Euclidean time lattice	d-dimensional spatial lattice
elementary time step a	crystal lattice spacing
particle position x	classical spin variable s
particle path $x(t)$	spin configuration s_x
path integral $\int \mathcal{D}x$	sum over configurations $\prod_x \sum_{s_x}$
Euclidean action $S_E[x]$	classical Hamilton function $\mathcal{H}[s]$
Planck's constant $\hbar$	temperature T
quantum fluctuations	thermal fluctuations
kinetic energy $\frac{1}{2}(\frac{x_{i+1}-x_i}{a})^2$	neighbor coupling $s_x s_{x+1}$
potential energy $V(x_i)$	external field energy $\mu B s_x$
weight of a path $\exp(-\frac{1}{\hbar}S_E[x])$	Boltzmann factor $\exp(-\mathcal{H}[s]/T)$
vacuum expectation value $\langle \mathcal{O}(x) \rangle$	magnetization $\langle s_x \rangle$
2-point function $\langle \mathcal{O}(x(0))\mathcal{O}(x(t)) \rangle$	correlation function $\langle s_x s_y \rangle$
energy gap $E_1 - E_0$	inverse correlation length $1/\xi$
continuum limit $a \to 0$	critical behavior $\xi \to \infty$

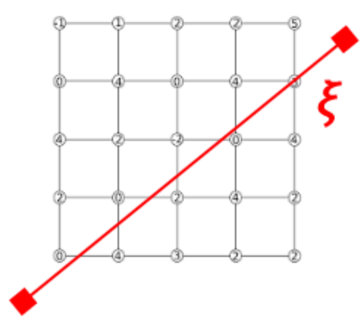
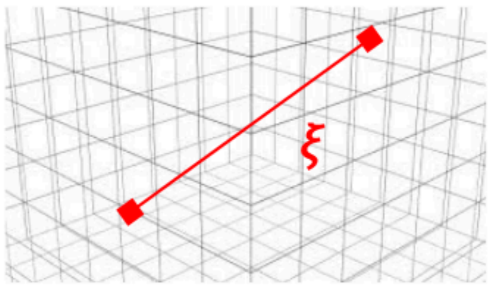
[Credit: U.-J. Wiese https://inspirehep.net/literature/946884]

[See also:]



[To appear on 31 August 2024]

From Statistical Mech. to Quantum Field Theories

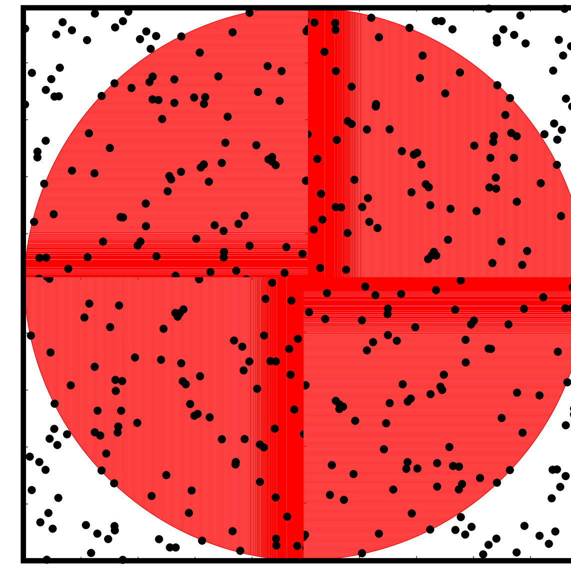
Classical Statistical Mechanics		Quantum Field Theories	
Partition function	$Z_\beta = \sum_\sigma e^{-\beta H(\sigma)}$	Feynman Path Integral	$\mathcal{Z} = \int D[U]\, e\, [(-1 / g^2)\, S(U)]$
Inverse temperature	$\boldsymbol{\beta \sim 1 / T}$	Inverse gauge coupling	$\mathbf{\sim 1 / g^2}$
Correlation functions	$< \sigma(x)\, \sigma(y)> \sim e^{-\|x-y\|/\xi}$	2-point functions	$< Tr(U(p)_x)\; TrU(p)_y > \sim e^{-\|x-y\| r}$
Inverse correlation length	$\mathbf{1 / \xi}$	Particle mass: **m**	
2nd order phase transition $\boldsymbol{\xi / a \rightarrow \infty}$ with **a** fixed		Continuum limit: $\mathbf{m\, a \rightarrow 0}$ with **m** fixed	

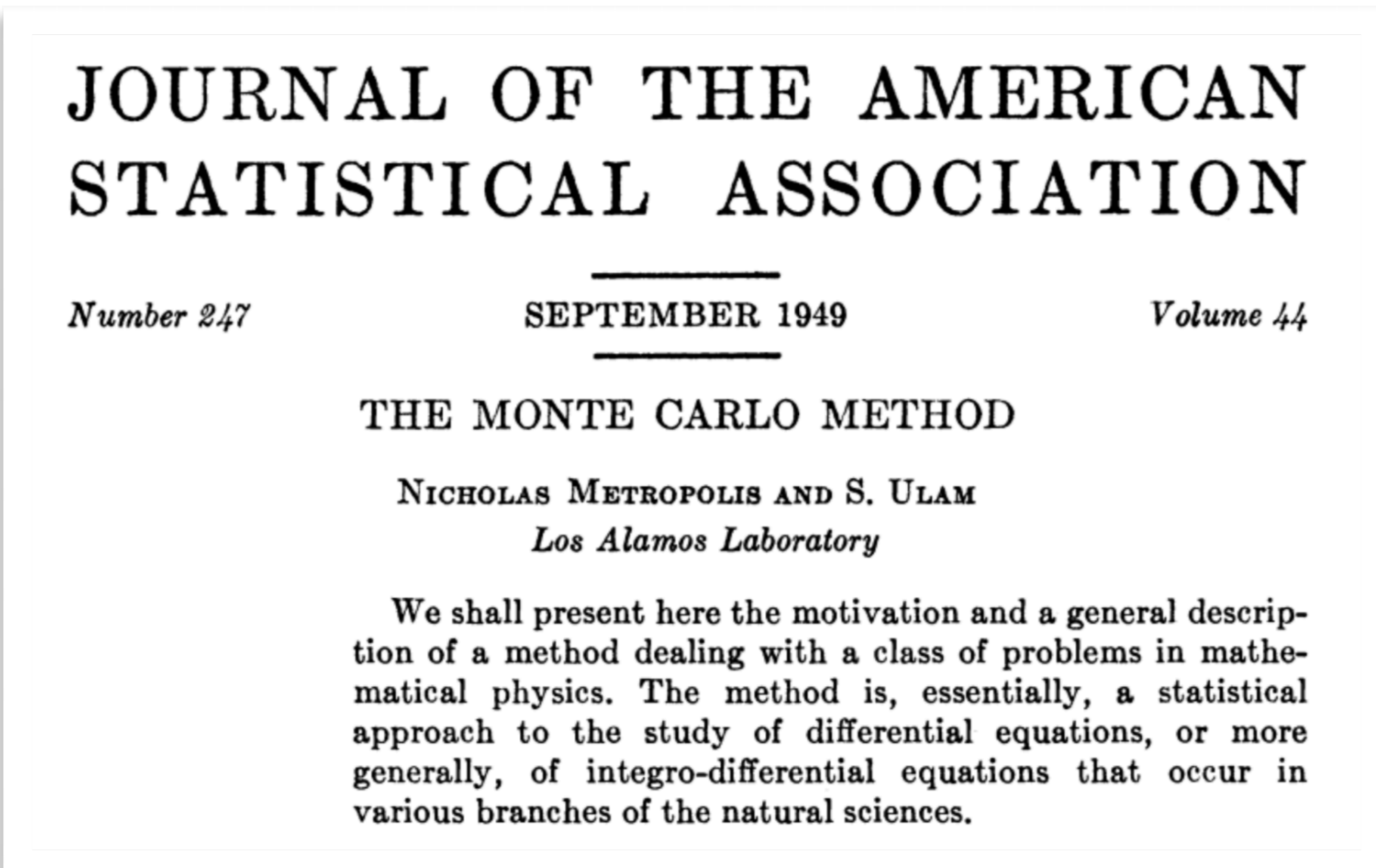
2-d spatial lattice with physical lattice spacing ***a***

4-d space-time lattice with unphysical cut-off ***a***

A tool for Lattice Simulations: *Monte Carlo* methods

- **Monte Carlo: a numerical method for estimating high-dimensional integrals by random sampling**





JOURNAL OF THE AMERICAN STATISTICAL ASSOCIATION

Number 247 SEPTEMBER 1949 *Volume 44*

THE MONTE CARLO METHOD

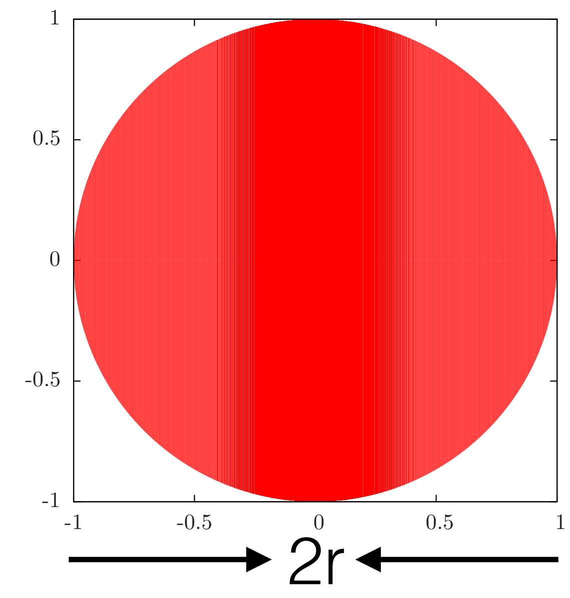
NICHOLAS METROPOLIS AND S. ULAM
Los Alamos Laboratory

We shall present here the motivation and a general description of a method dealing with a class of problems in mathematical physics. The method is, essentially, a statistical approach to the study of differential equations, or more generally, of integro-differential equations that occur in various branches of the natural sciences.

A tool for Lattice Simulations: Monte Carlo methods

- **Ratio of the surface area of the circle and the square:**

  ➡ $$\frac{S_{circle}}{S_{square}} = \frac{\pi r^2}{4r^2}$$



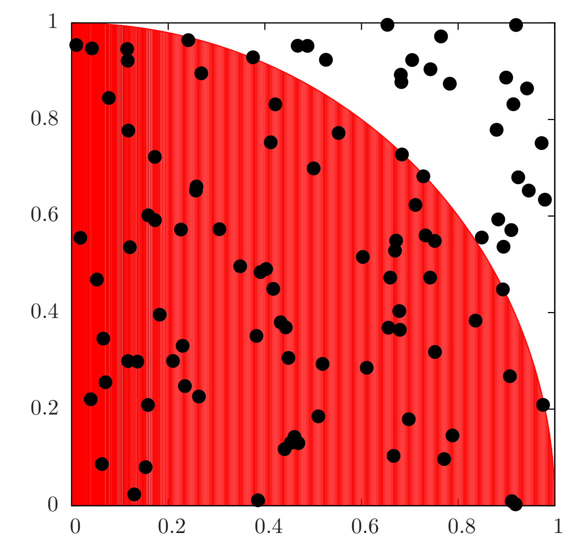
- **We can then express:**

  ➡ $$\pi = \frac{4S_{circle}}{S_{square}}$$

- **If we know S_{circle}/S_{square}, we know the value of π**

- Throw N random points on the surface of the square

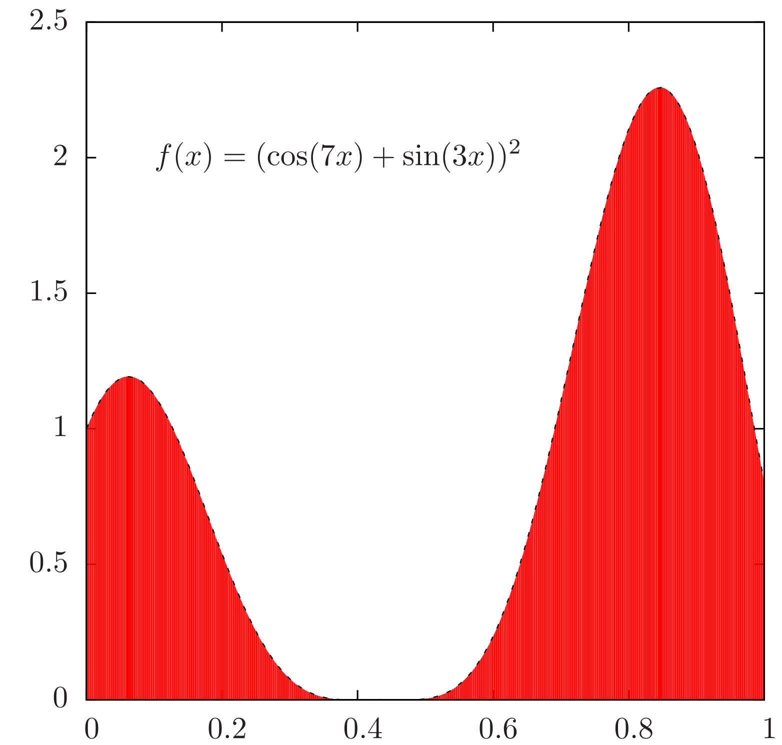
  ➡ $$\pi = \frac{4S_{circle}}{S_{square}} \approx \frac{N_{inside}}{N}$$

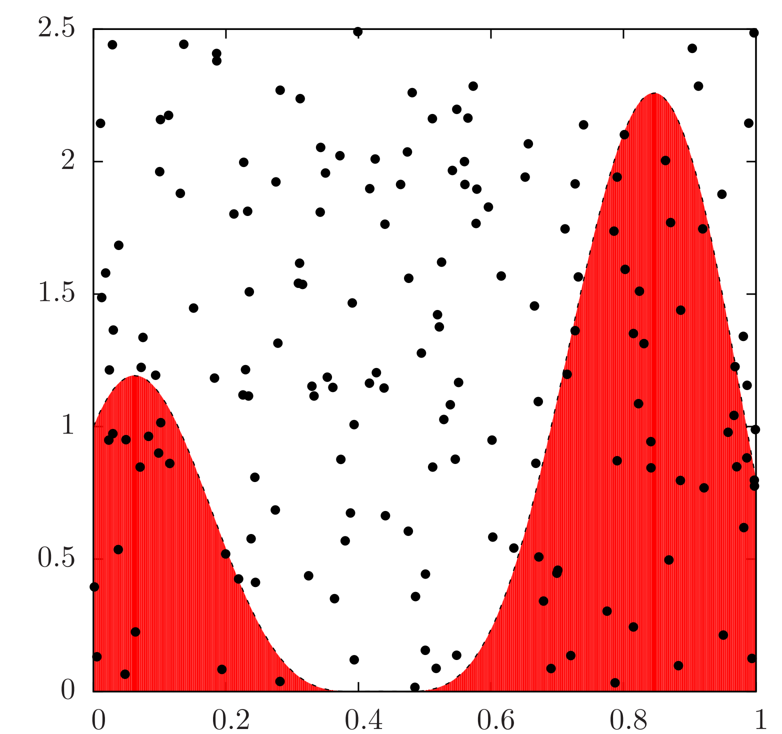


- For N - large, the value of **π** will be very well approximated

A tool for Lattice Simulations: Monte Carlo methods

- **Calculating 1-dim integral:**
  - $$I = \int_0^1 f(x)dx$$
- **Again, throw N random points on the rectangular surface:**
- **Count those under the value of the function** $f(x)$
- **Then the value of the integral is obtained by:**
  - $$I \approx \langle I \rangle_N = 2.5 \times 1 \times \frac{N_{inside}}{N}$$





- **For larger N, better and better approximation of the integral**
- **Monte Carlo error in d-dim:** $\approx \frac{1}{\sqrt{N}}$
- **Numerical integration error in d-dim:** $\approx \frac{1}{N^{2/d}}$
- **For d>4, Monte Carlo is better than Numerical Integration!**

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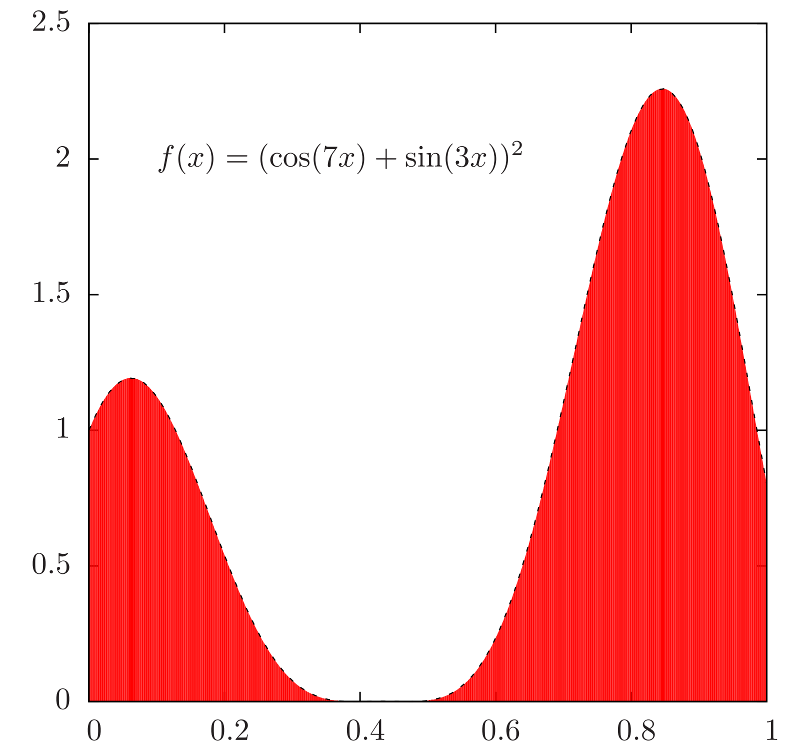

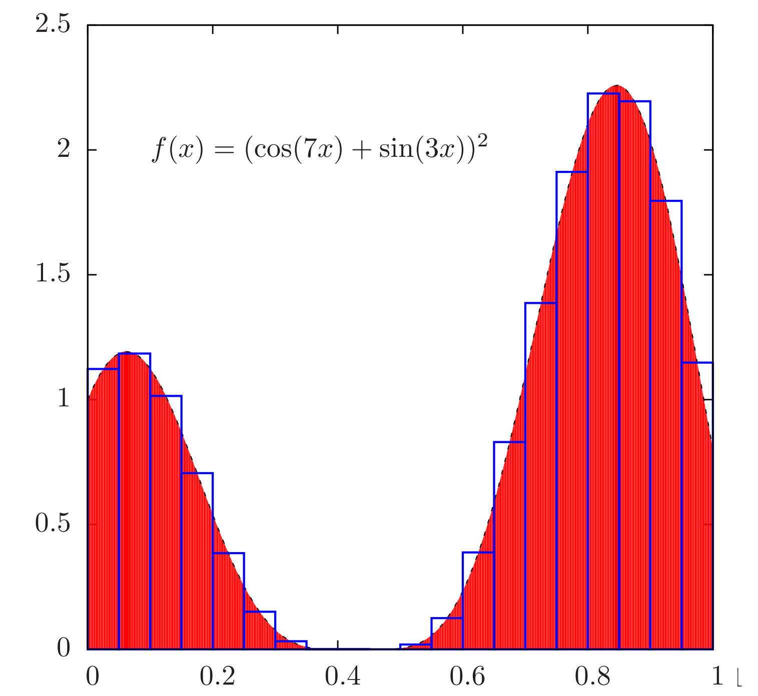

Expectation values in Monte Carlo

- **Ising model Hamiltonian (without external magnetic field):**

$$H = -J \sum_{\langle x,y \rangle} s_x s_y; \qquad \langle x, y \rangle - \text{nearest neighbours}$$

- **Generate configurations of spins with probability** $\sim e^{-\frac{H}{T}}$ **(importance sampling)**

- **Expectation value of the observables as averages over ensemble of spin configurations**

$$\langle O \rangle \approx \bar{O} = \frac{1}{N_{cnfg}} \sum_{k=1}^{N_{cnfg}} O[\bar{s}_k] + \mathcal{O}\left(\frac{1}{\sqrt{N_{cnfg}}}\right)$$

- **How do we generate the ensemble** $\{\bar{s}_k\}$ **?**

Numerical Simulations

- **Numerical approach exploits the analogy:**

Euclidean QFT ⟷ **Statistical Mechanics**

$$\langle O[\phi]\rangle = \frac{1}{Z}\int \mathcal{D}\phi \; e^{-S_E[\phi]}\, O[\phi]$$

- **Path integrals are computed by importance sampling**

$$\mathcal{P}[\phi_i] \propto e^{-S_E[\phi]}$$

- **Generate ensemble of field configurations** $\{\phi_i\}$

- **Expectation values** $\langle O[\phi]\rangle$ **are averages over the ensemble**

$$\langle O\rangle \approx \bar{O} = \frac{1}{N_{cnfg}}\sum_{i=1}^{N_{cnfg}} O[\phi] + \mathcal{O}\left(\frac{1}{\sqrt{N_{cnfg}}}\right)$$

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Markov processes

- **Recursive procedure that generates $\{\phi_i\}$ with specific algorithm s.t. aimed distribution is asymptotically obtained**

$$\{\phi_0\} \longrightarrow \{\phi_1\} \longrightarrow \{\phi_2\} \longrightarrow \cdots \longrightarrow \{\phi_i\} \longrightarrow \{\phi_{i+1}\} \longrightarrow \ldots$$

- **Markov chains that converge exponentially to the equilibrium distr.**

- **The configurations $\{\phi_i\}$ are correlated by construction**

$$Var\,[\bar{O}] \;=\; Var[O]\,\Big(\frac{2\tau_O}{N_{cnfg}}\Big)$$

— integrated autocorrelation time

- **The error of the estimator scales as $\dfrac{1}{\sqrt{N_{cnfg}}}$ (Monte Carlo)**

- **The variance of the actual observable**

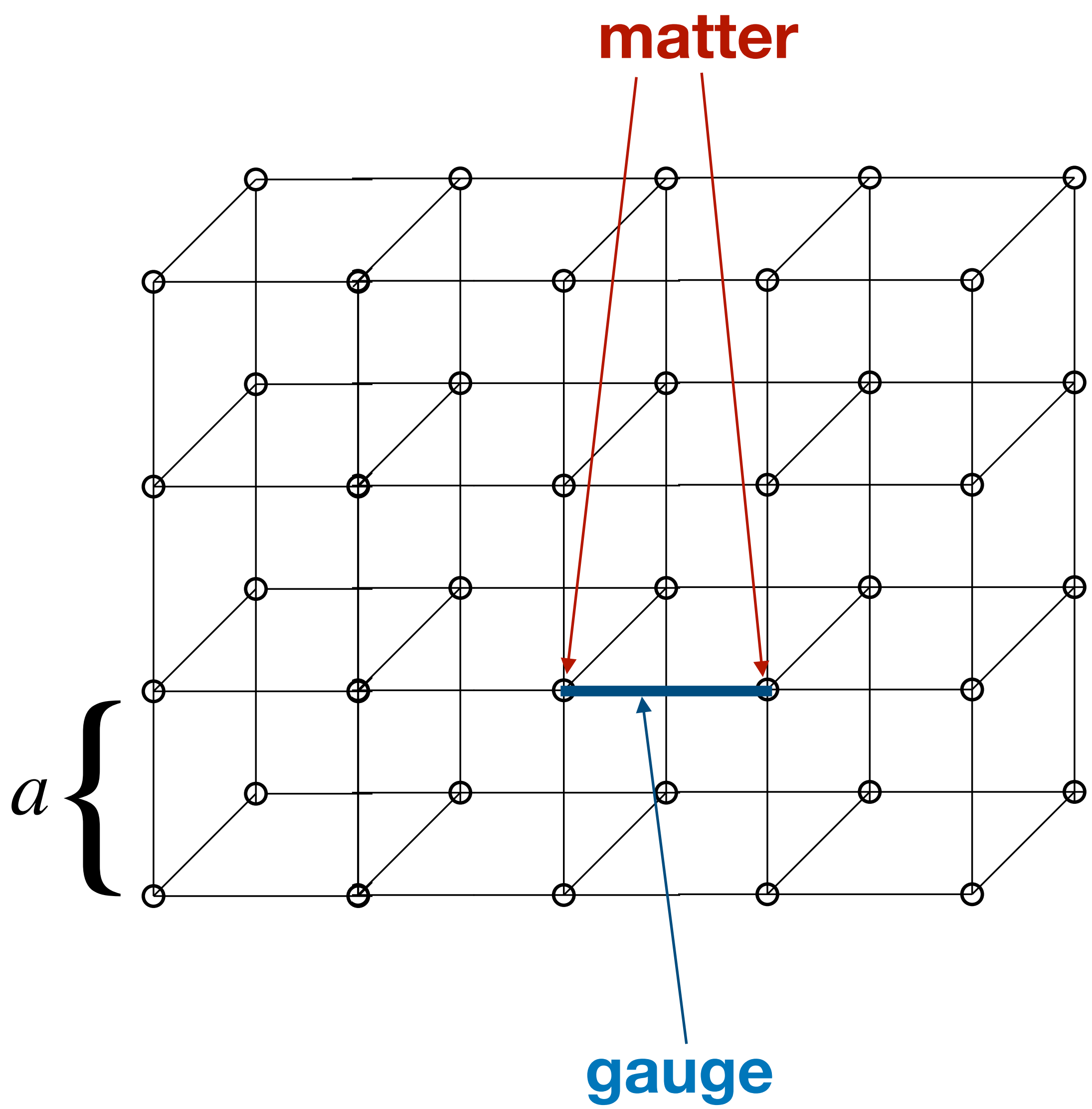
$$Var\,[O] \;=\; \langle\big(O - \langle O\rangle\big)^2\rangle$$

→ **property of QFT itself, should not depend of the Markov chain.**

QFT on the lattice: symmetries

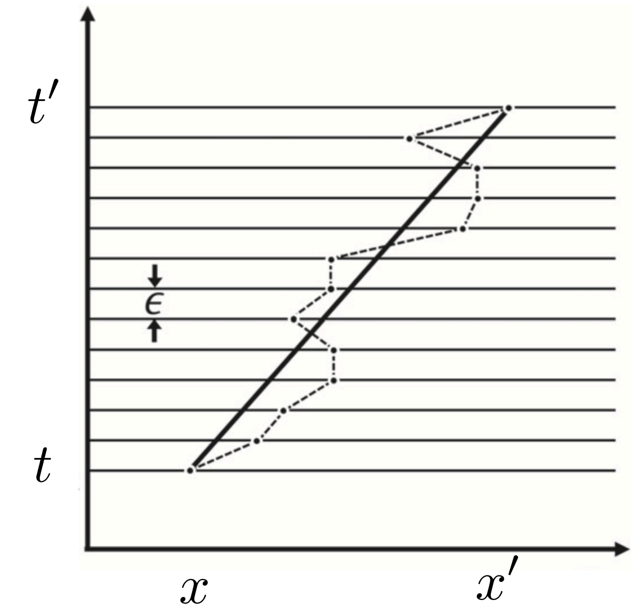
- **Translational Symmetry:**
  Broken to discrete symmetry, but nicely restored in the continuum limit

- **Rotational Symmetry:**
  Similar to translational symmetry. Finite number of irreducible representations (quantum numbers) instead of spin, but correct states are obtained in the continuum limit

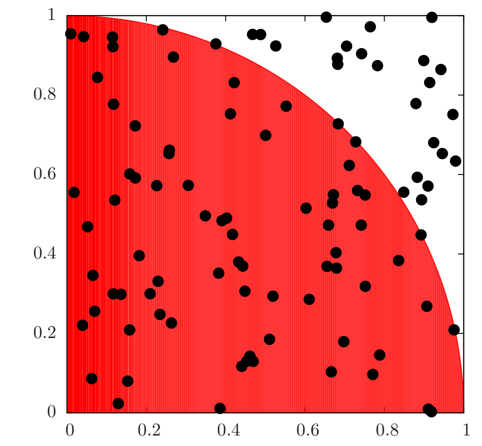


In the first lecture:

- We have learned/reminded ourselves how to quantize:
  - ➡ Quantum Mechanics
  - ➡ Scalar Field Theory



- Mapping to Classical Statistical Mechanics

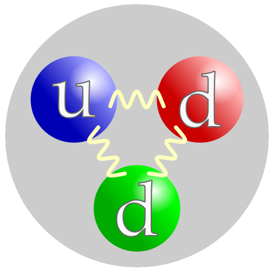


- Monte Carlo Sampling of Spin Systems $\longrightarrow$ Quantum Fields

- Monte Carlo Errors, other sources of errors

In the lecture on Friday:

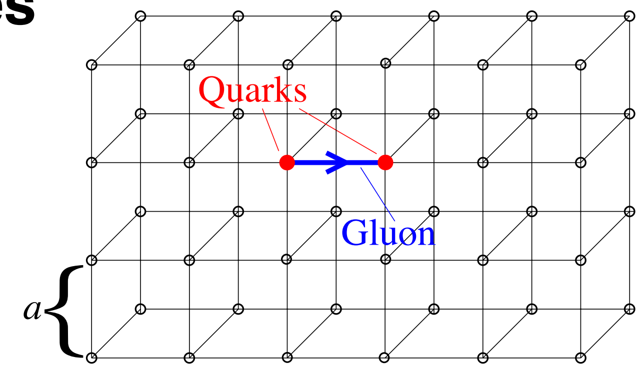
- We shall generalize this approach to Quantum Chromodynamics (QCD)

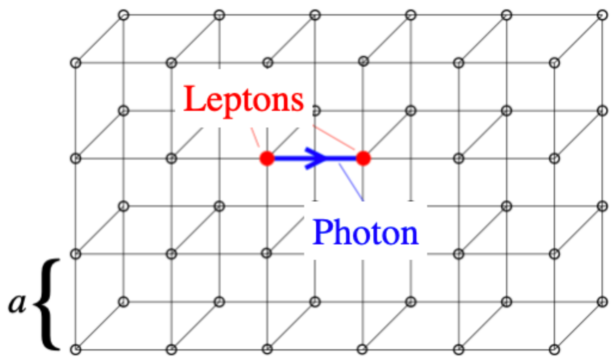


- We shall see how the numerical sampling is done in practice
  - ➡ for scalar field theory
  - ➡ for QCD

- Why is QCD numerically so expensive?

- QED corrections to hadronic observables





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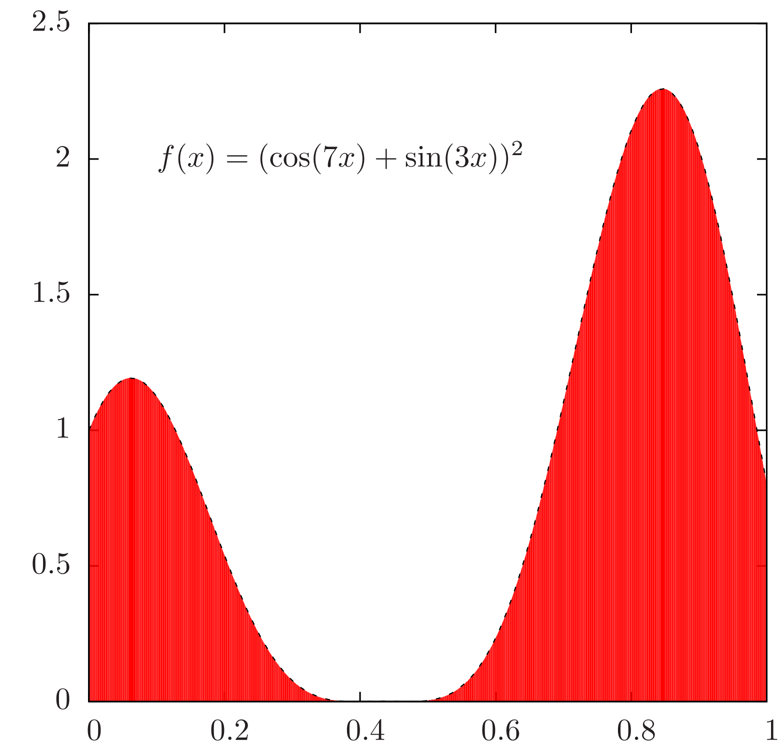

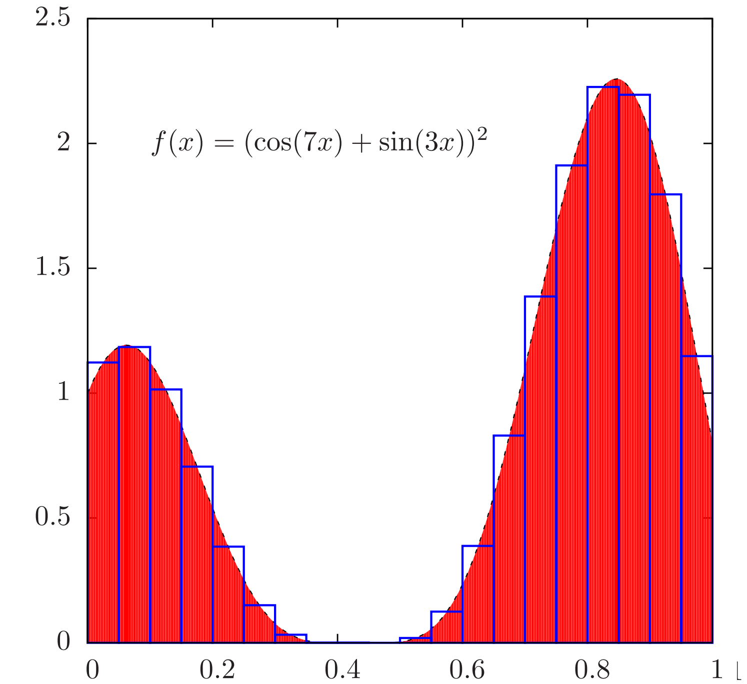

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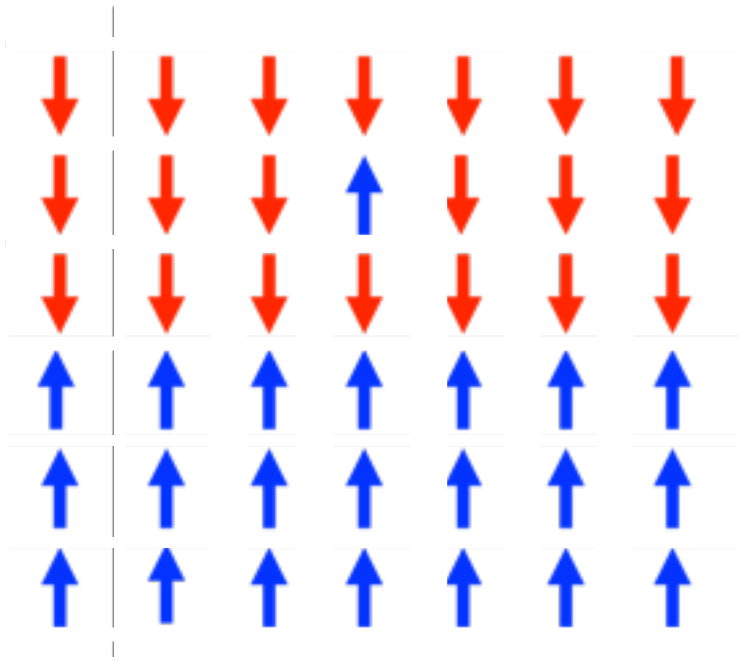
- **For d>4, Monte Carlo is better than Numerical Integration!**

Classical statistical Mechanics: Ising Model

→ CLASSICAL HAMILTON FUNCTION:

$$\mathcal{H}[S] = -J \sum_{\langle x y \rangle} S_x S_y - \mu B \sum_x S_x$$

* $J > 0$; FERROMAGNETIC COUPLING CONST.
* μ COUPLING TO AN EXTERNAL MAG. FIELD B



Classical statistical Mechanics: Ising Model

- CLASSICAL PARTITION F-ON:

$$Z = \int \mathcal{D}s \, e^{-\mathcal{H}[s]/T}$$

$$= \prod_x \sum_{s_x = \pm 1} e^{-\frac{1}{T}\mathcal{H}[s]}$$

- THERMAL AVERAGES:

  * MAGNETIZATION: $\langle s_x \rangle = \frac{1}{Z} \prod_x \sum_{s_x=\pm 1} s_x \, e^{-\frac{\mathcal{H}[s]}{T}}$

  * SPIN CORRELATION F-ON:

$$\langle s_x s_y \rangle = \frac{1}{Z} \prod_x \sum_{s_x = \pm 1} s_x s_y \, e^{-\frac{\mathcal{H}[s]}{T}}$$

Expectation values in the MC approach: spin systems

- **Ising model Hamiltonian (without external magnetic field):**

$$H = -J \sum_{\langle x,y \rangle} s_x s_y; \qquad \langle x, y \rangle - \text{nearest neighbours}$$

- **Generate configurations of spins with probability** $\sim e^{-\frac{H}{T}}$ **(importance sampling)**

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- **Path integrals are computed by importance sampling**

$$\mathcal{P}[\phi_i] \propto e^{-S_E[\phi]}$$

- **Generate ensemble of field configurations** $\{\phi_i\}$

- **Expectation values** $\langle O[\phi] \rangle$ **are averages over the ensemble**

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- **Recursive procedure that generates $\{\phi_i\}$ with specific algorithm s.t. aimed distribution is asymptotically obtained**

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- **The configurations $\{\phi_i\}$ are correlated by construction**

$$Var\,[\bar{O}] \;=\; Var[O]\,\Big(\frac{2\tau_O}{N_{cnfg}}\Big)$$

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- **The variance of the actual observable**

$$Var\,[O] \;=\; \langle \big(O - \langle O \rangle\big)^2 \rangle$$

→ **property of QFT itself, should not depend of the Markov chain.**

Detailed Balance

- **Probability for a random transition $\phi \to \phi'$:**

$$R(\phi' \leftarrow \phi)$$

- **Sequence of probabilities in Markov chain, $P^{k+1}(\phi')$ depends only on $P^k(\phi)$**

$$P^{k+1}(\phi') = \int [d\phi] \, P^k(\phi) \, R(\phi' \leftarrow \phi)$$

- **Probabilities of ϕ' being initial or final configuration in a random update are equal**

$$\sum_{\phi} R(\phi' \leftarrow \phi) P(\phi) = \sum_{\phi} R(\phi \leftarrow \phi') P(\phi')$$

- **The above is known as a "balance equation"**

Detailed Balance

- **Calculate the sum using normalization of R**

$$0 \leq R(\phi \leftarrow \phi') \leq 1, \qquad \sum_{\phi} R(\phi \leftarrow \phi') = 1$$

- **Obtain that equilibrium distribution $P(\phi')$ is a fixed point of the Markov Process**

$$\sum_{\phi} R(\phi' \leftarrow \phi) P(\phi) = P(\phi')$$

- **The solution of the balance equation can be obtained if *detailed balance* is fulfilled**

$$R(\phi' \leftarrow \phi) P(\phi) = R(\phi \leftarrow \phi') P(\phi')$$

The MR^{2}T^2 algorithm

THE JOURNAL OF CHEMICAL PHYSICS VOLUME 21, NUMBER 6 JUNE, 1953

Equation of State Calculations by Fast Computing Machines

NICHOLAS METROPOLIS, ARIANNA W. ROSENBLUTH, MARSHALL N. ROSENBLUTH, AND AUGUSTA H. TELLER,
Los Alamos Scientific Laboratory, Los Alamos, New Mexico

AND

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(Received March 6, 1953)

A general method, suitable for fast computing machines, for investigating such properties as equations of state for substances consisting of interacting individual molecules is described. The method consists of a modified Monte Carlo integration over configuration space. Results for the two-dimensional rigid-sphere system have been obtained on the Los Alamos MANIAC and are presented here. These results are compared to the free volume equation of state and to a four-term virial coefficient expansion.

- **The Metropolis Algorithm:**
- **Given ϕ, propose ϕ' in a reversible, area-preserving way**
- **Accept the proposal as new entry with probability p, otherwise add ϕ to the chain again.**
- $$\mathrm{p} = \min\left(1, \frac{\pi(\phi')}{\pi(\phi)}\right)$$

The MR²T² algorithm

THE JOURNAL OF CHEMICAL PHYSICS VOLUME 21, NUMBER 6 JUNE, 1953

Equation of State Calculations by Fast Computing Machines

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- **The Metropolis Algorithm:**
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- **Accept the proposal as new entry with probability p, otherwise add ϕ to the chain again.**
- $\mathrm{p} = \min(1, \frac{\pi(\phi')}{\pi(\phi)})$

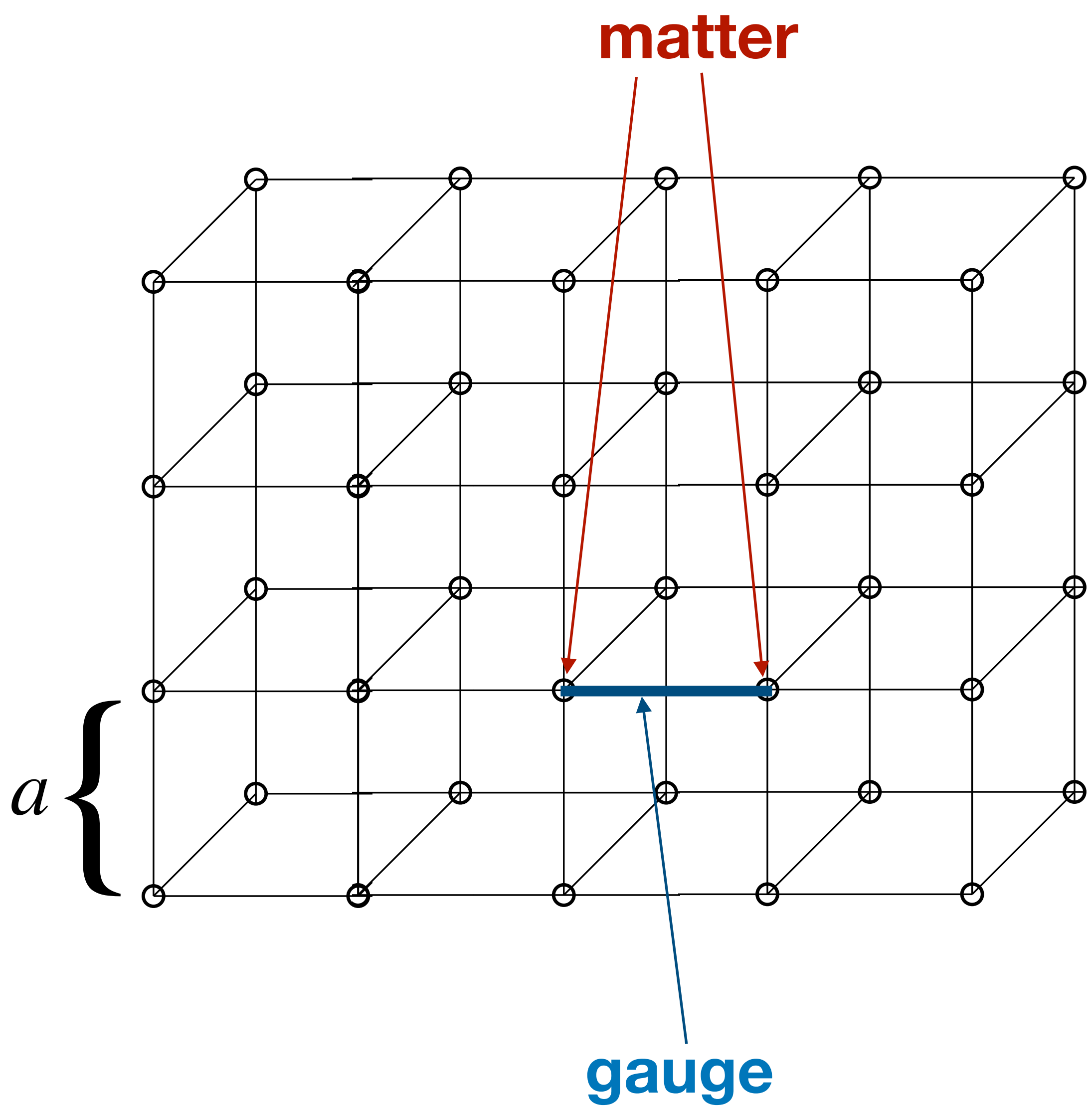
- **The Metropolis Algorithm for Scalar F.T.:**
- **Given ϕ propose ϕ'**
- **Choose a random number $r \in [0,1]$**
- **If $S_E(\phi') < S_E(\phi)$, always accept, otherwise add ϕ' to the chain again if**

$$e^{-\Delta S} > r;$$
$$\Delta S = S_E(\phi') - S_E(\phi)$$

QFT on the lattice: symmetries

- **Translational Symmetry:**
  Broken to discrete symmetry, but nicely restored in the continuum limit

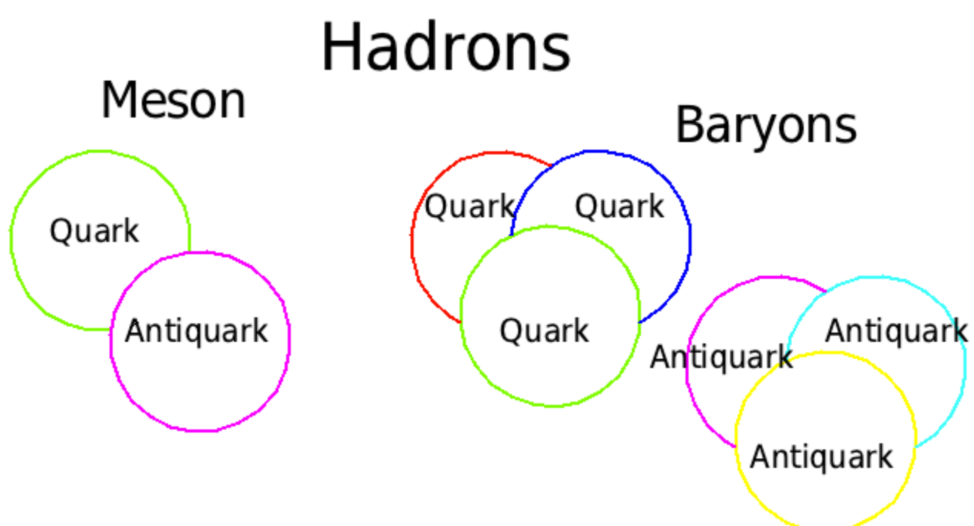
- **Rotational Symmetry:**
  Similar to translational symmetry. Finite number of irreducible representations (quantum numbers) instead of spin, but correct states are obtained in the continuum limit



Strong interaction

$\longrightarrow$ quarks ($q_i = u, d, s, c, b, [t]$) elementary constituens of:

- mesons ($q\bar{q}$)
- barions (qqq),($\bar{q}\bar{q}\bar{q}$)



Quark / Antiquark	Symbol		Charge/*e*		Baryon number, B		Strangeness, S	
up	u	$\bar{u}$	+2/3	-2/3	1/3	-1/3	0	0
down	d	$\bar{d}$	-1/3	+1/3	1/3	-1/3	0	0
charm	c	$\bar{c}$	+2/3	-2/3	1/3	-1/3	0	0
strange	s	$\bar{s}$	-1/3	+1/3	1/3	-1/3	-1	1
top	t	$\bar{t}$	+2/3	-2/3	1/3	-1/3	0	0
bottom	b	$\bar{b}$	-1/3	+1/3	1/3	-1/3	0	0

revisionworld

COLOUR: SU(3) symmetry

Quantum Chromodynamics

- QCD Lagrangian

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_{f=u,d,s,\ldots} \overline{\psi}_f \left\{ i\gamma_\mu \left(\partial_\mu - i g_s A^a_\mu T^a\right) - m_f \right\} \psi_f$$

$$S = \int d^4x \, \mathcal{L}_{QCD}$$

- Covariant derivative: $D_\mu = \partial_\mu - i g_s A_\mu$

Quantum Chromodynamics

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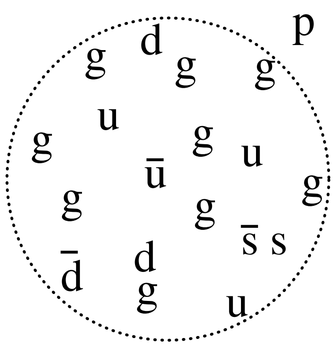
- Covariant derivative: $D_\mu = \partial_\mu - i g_s A_\mu$

- Bare QCD parameters ($N_f + 1$)
  - gauge coupling g_s
  - quark masses m_u, m_d, ...



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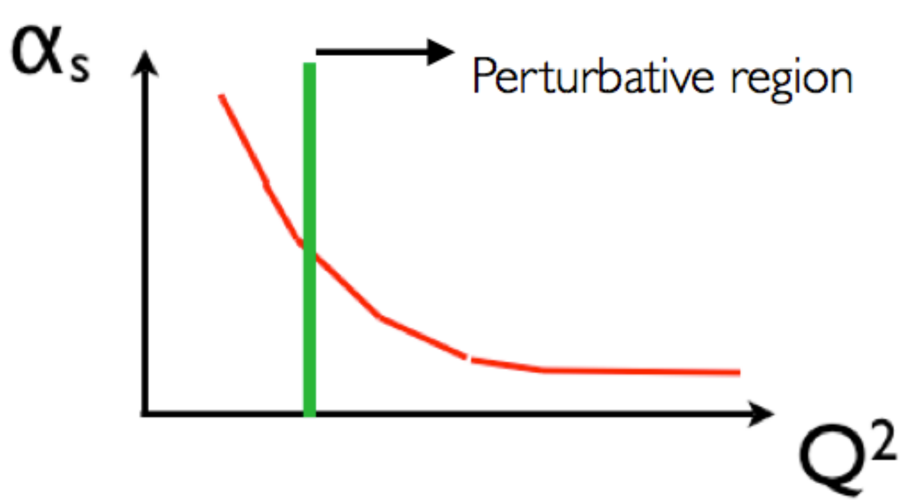


- Euclidean QCD Lagrangian ($t \equiv x^E \leftrightarrow -ix_0$)

$$\mathcal{L}_{QCD} = \frac{1}{2g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_{f=u,d,s,\ldots} \overline{\psi}_f \left\{ \gamma_\mu \left(\partial_\mu + i A^a_\mu T^a\right) + m_f \right\} \psi_f$$

$$S_{QCD} = \int d^4x \, \mathcal{L}_{QCD}$$

Path Integrals in Quantum Chromodynamics

- Each specific field configuration:

$$P(\psi, \overline{\psi}, A) \sim e^{-S(\psi,\overline{\psi},A)}$$

- Expectation value of an operator $O(\psi, \overline{\psi}, A)$:

$$\begin{aligned}\langle O(\psi, \overline{\psi}, A)\rangle &= \langle\langle O(\psi, \overline{\psi}, A)\rangle_F\rangle_G \\ &= \frac{1}{Z}\int \mathcal{D}[A]\mathcal{D}[\overline{\psi}, \psi]\, e^{-S(\psi,\overline{\psi},A)} O(\psi, \overline{\psi}, A) \\ Z &= \int \mathcal{D}[A]\mathcal{D}[\overline{\psi}, \psi]\, e^{-S(\psi,\overline{\psi},A)}\end{aligned}$$

Lattice regularization of QCD

- Divergencies in continuum QCD $\rightarrow$ **regularization** is necessary!
- One possible regularization:
  Introduce **momentum ultraviolet-cutoff** $\Leftrightarrow$ minimum distance (FT)
- If required also: **local gauge symmetry** $\rightarrow$ **Lattice QCD**
- Finite number of integrals over fields ($\int d^4x \ \rightarrow \ a^4 \sum_n$)
- Computable with the help of Monte Carlo techniques

Lattice regularization of QCD

$$S_{QCD}[\psi,\bar\psi,A] = S_G + S_F = \frac{1}{2g}\,F_{\mu\nu}F_{\mu\nu} + \int d^4x\;\bar\psi(x)\,\left[\gamma_\mu\,\left(\partial_\mu + iA_\mu(x)\right) + m\right]\,\psi(x)$$

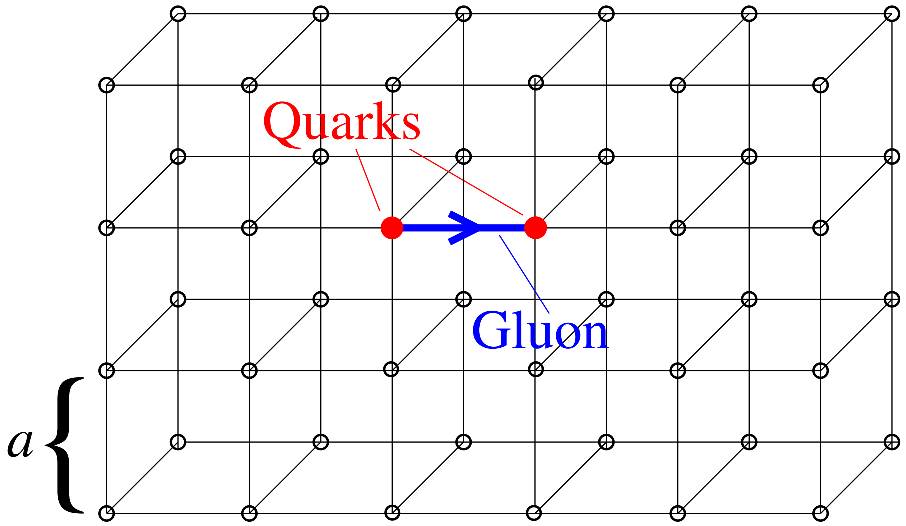
- Discretization prescription:

$$x \longrightarrow n = (n_1, n_2, n_3, n_4)\;\; n_1 = 0, \ldots, N-1$$

$$\psi(x), \bar\psi(x) \longrightarrow \psi(n), \bar\psi(n)$$

$$\int d^4x \ldots \longrightarrow a^4 \sum_n \ldots$$

$$\partial_\mu\;\psi(x) \longrightarrow \frac{\psi(n+\hat\mu) - \psi(n-\hat\mu)}{2a} + \mathcal{O}(a^2)$$



Naive Lattice Fermion Action

- Simple example - free fermion field ($A_\mu = 0$):

  $S_F^0[\psi,\bar{\psi}] = \int d^4x\ \bar{\psi}(x)\ (\gamma_\mu\ \partial_\mu + m)\ \psi(x)$

- Symmetrically discretized partial derivative:

$$\partial_\mu \psi(na) = \frac{\psi((n+\hat{\mu})) - \psi((n-\hat{\mu}))}{2a} + \mathcal{O}(a^2)$$

- Naive lattice ansatz for free fermion action:

$$S_F\left[\psi,\overline{\psi}\right] = a^4 \sum_{n\in\Lambda} \overline{\psi}(n) \left(\sum_{\mu=1}^{4} \gamma_\mu \frac{\psi(n+\hat{\mu}) - \psi(n-\hat{\mu})}{2a} + m\psi(n) \right)$$

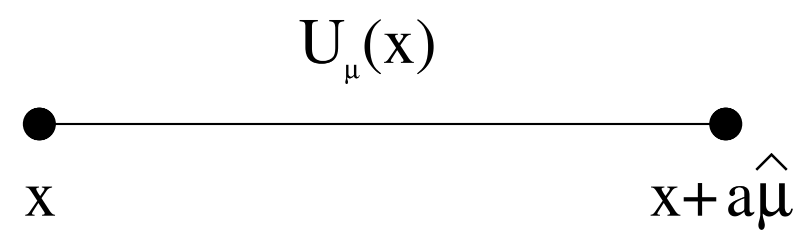
- **Let us examine gauge invariance** $\rightarrow$

Gauge Invariance

- $\Omega(n) \in SU(3)$:

$$\begin{aligned} \psi'(n) &= \Omega(n)\psi(n) \\ \overline{\psi}'(n) &= \overline{\psi}(n)\Omega(n)^\dagger \\ \overline{\psi}'(n)\psi'(n+\hat{\mu}) &= \overline{\psi}(n)\Omega(n)^\dagger\Omega(n+\hat{\mu})\psi(n+\hat{\mu}) \quad (!) \end{aligned}$$

- (!) not gauge invariant

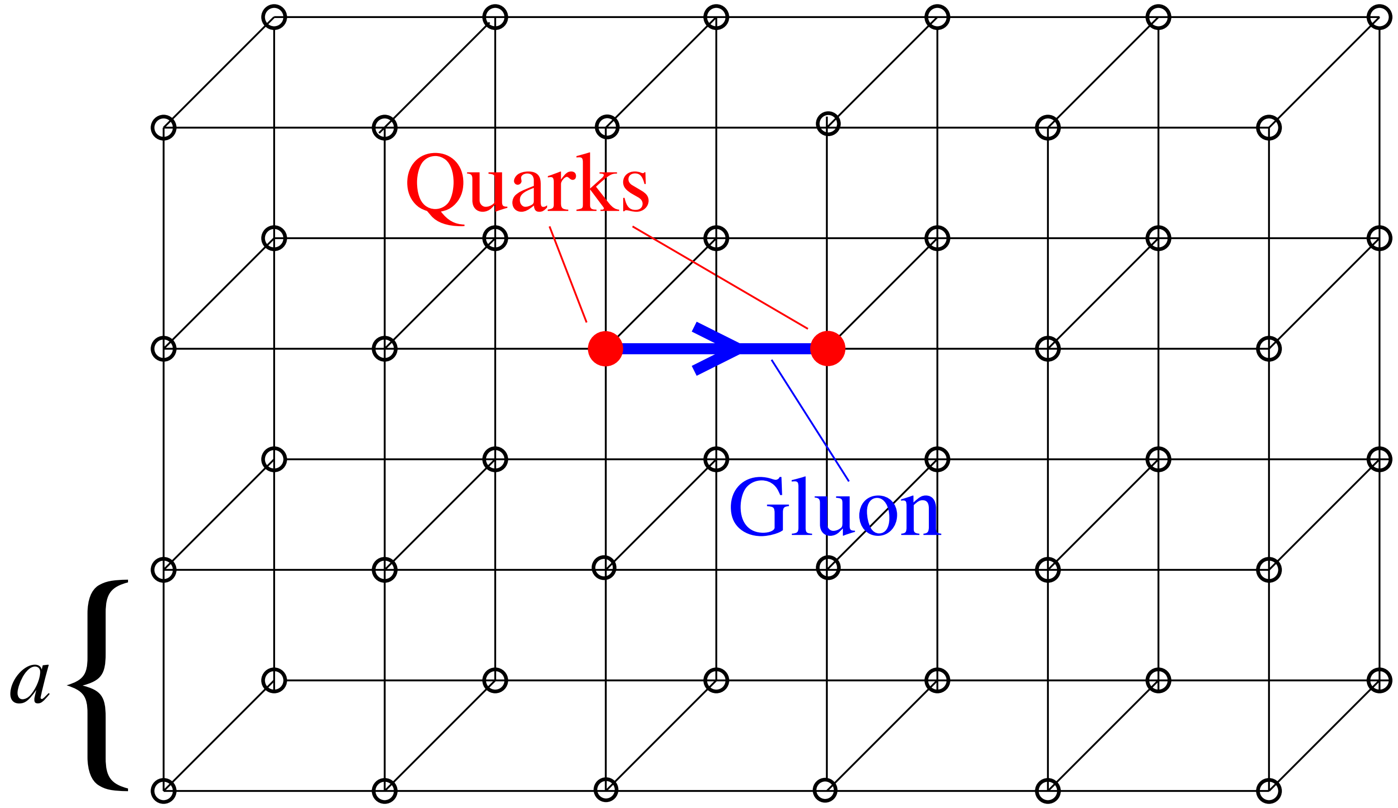


- Introduce **link variables** $\mathbf{U}_\mu(\mathbf{n})$:

$$\begin{aligned} U'_\mu(n) &= \Omega(n)U_\mu(n)\Omega(n+\hat{\mu})^\dagger \\ \overline{\psi}'(n)U'_\mu(n)\psi'(n+\hat{\mu}) &= \overline{\psi}(n)U_\mu(n)\psi(n+\hat{\mu}) \end{aligned}$$

- $U_\mu(n) \;\rightarrow\;$ fundamental gluonic variables on the lattice

Quark and gluon fields on the lattice



Quarks $\sim \overline{\psi}(n), \psi(n)$

Gluons $\sim$ "Link variables" $\sim$ Parallel transporter $\sim U_\mu(n) = e^{iagA_\mu}$

Lattice fermion action (I)

- Fermionic action: $S_F = a^4 \sum_f \overline{\psi}(n)\, D(n,m)\, \psi(m)$
- Naive fermion action

$$D(n,m) = m\delta_{n,m} + \frac{1}{2a} \sum_{\mu=\pm 1}^{\pm 4} \gamma_\mu U_\mu(n)\, \delta_{n+\hat{\mu},m}$$

- Propagator in momentum space:

$$\tilde{D}(p)^{-1} = \frac{m\mathbf{1} - ia^{-1} \sum_{\mu=\pm 1}^{\pm 4} \gamma_\mu sin(p_\mu a)}{m^2 + a^{-2} \sum_{\mu=\pm 1}^{\pm 4} sin(p_\mu a)^2}$$

- Important: case of massless fermions, $m = 0$:

$$\tilde{D}(p)^{-1}|_{m=0} = \frac{-ia^{-1} \sum_\mu \gamma_\mu sin(p_\mu a)}{a^{-2} \sum_\mu sin(p_\mu a)^2}$$

- Unphysical poles at $p_\mu = \frac{\pi}{a}$
- Unwanted **doublers**: obtainded 16 instead of 1 fermionic particles!

Lattice fermion action (II)

- **Wilson Dirac matrix** D_W

$$D_W(n,m) = \left(m + \tfrac{4}{a}\right)\delta_{n,m} - \tfrac{1}{2a}\sum_{\mu=\pm 1}^{\pm 4}\left(1-\gamma_\mu\right)U_\mu(n)\,\delta_{n+\hat{\mu},m}$$

- Wilson term: shifting the mass of the doublers to infinity, as $a \to 0$

- Only the physical pole, no doublers!

- Problem: Additional term **breaks chiral symmetry** explicitly

- **No-Go Theorem** on the lattice [Nielsen & Ninomiya, 1981]:
  simple action without doublers $\leftrightarrow$ broken chiral symmetry

- Different choices of lattice derivatives
  $\to$ $O(a), O(a^2), \ldots$ discretization errors

  $\to$ different rates to approach continuum limit

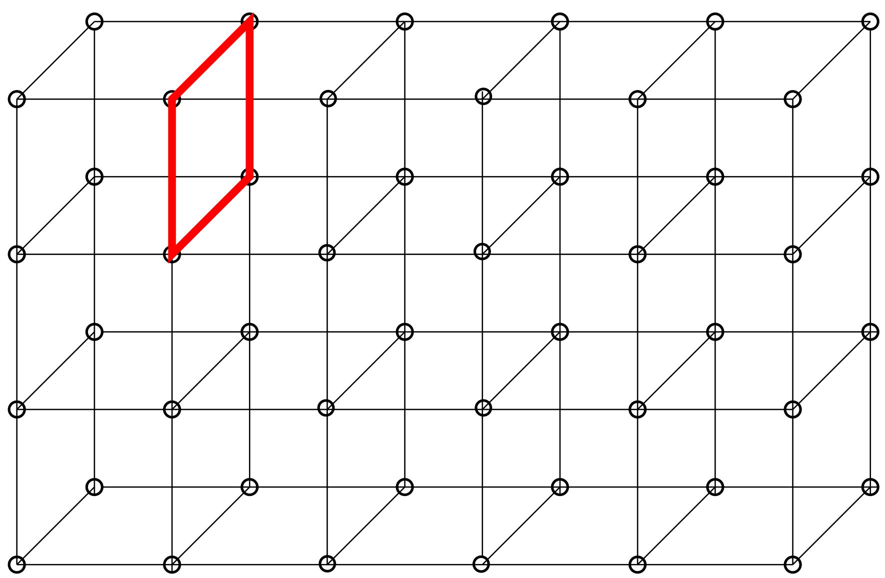
Lattice gauge action (I)

- $S_G = \frac{1}{2g} Tr\ F_{\mu\nu}(x)\ F_{\mu\nu}(x)$
- Need gauge invariant object: trace over closed loop of gauge links
- Smallest possible closed loop: **Plaquette**

$$U_{\mu\nu}(n) = U_\mu(n) U_\nu(n+\hat{\mu}) U_{-\mu}(n+\hat{\mu}+\hat{\nu}) U_{-\nu}(n+\hat{\nu})$$
$$= U_\mu(n) U_\nu(n+\hat{\mu}) U_\mu(n+\hat{\nu})^\dagger U_\nu(n)^\dagger$$

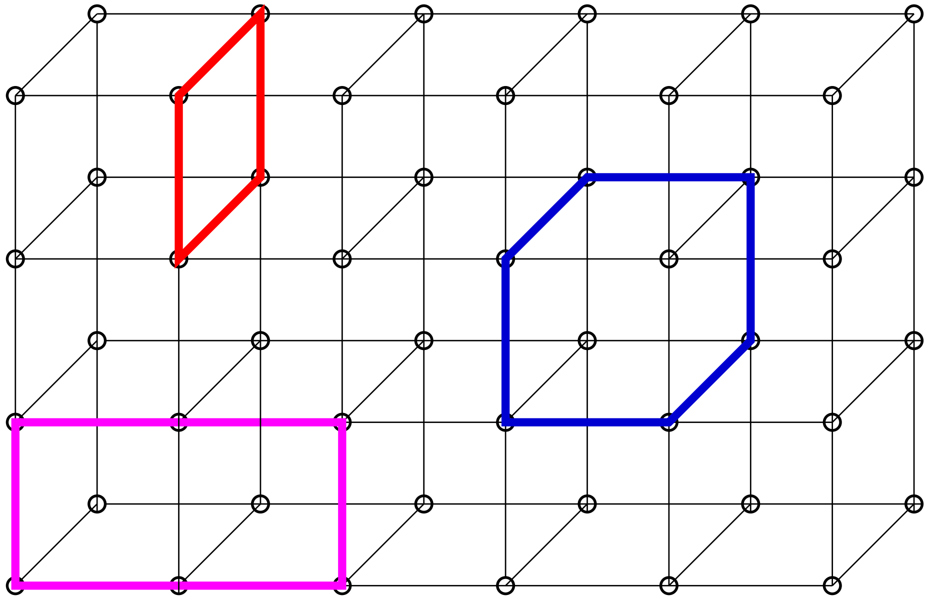
- **Wilson gauge action:**

$$S_g \sim \sum_n \sum_{\mu<\nu} \ \mathrm{Re\ tr} \left[1 - U_{\mu\nu}(n)\right]$$



Lattice gauge action (II)

- Improvement: taking into account larger Wilson loops



- All in the same universality class:

  $\longrightarrow$ converge to $\mathrm{tr}\,\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right]$ in the continuum limit

  $\longrightarrow$ improvement reduces the discretization errors!

- Lattice artefacts in scaling behaviour:

  $\longrightarrow$ Wilson gauge action: $O(a^2)$

  $\longrightarrow$ Luscher-Weisz: $O(a^4)$ [K. Symanzik, 1981; Luscher and Weisz, 1985]

Recipe for Lattice QCD Computation

(1) Generate ensembles of field configurations using Monte Carlo

(2) Average over a set of configurations: $\langle O \rangle \approx \bar{O} = \frac{1}{N_{cnfg}} \sum_{i=1}^{N_{cnfg}} O[U] + \mathcal{O}(\frac{1}{\sqrt{N_{cnfg}}})$

- Compute correlation function of fields, extract Euclidean matrix elements or amplitudes
- Computational cost dominated by quarks: inverses of large, sparse matrix

(3) Extrapolate to continuum, infinite volume, physical quark masses (now directly accessible)

- Lattice QCD path integrals are computed by **importance sampling**: $\mathcal{P}[U_i] \propto e^{-S_E[U]}$
- Markov chain: $\{U_0\} \longrightarrow \{U_1\} \longrightarrow \{U_2\} \longrightarrow \cdots \longrightarrow \{U_i\} \longrightarrow \{U_{i+1}\} \longrightarrow \ldots$
- The configurations are correlated by construction: $Var\,[\bar{O}] = Var[O]\,(\frac{2\tau_O}{N_{cnfg}})$

The cost of dynamical fermions

$$S^E_{QCD} = S_G[U] + S_f[U,\psi,\overline{\psi}]$$

$$\langle O[\psi,\overline{\psi},U]\rangle = \frac{1}{Z}\int \mathcal{D}U\mathcal{D}\psi\mathcal{D}\overline{\psi}e^{-S_G[U]-S_f[U,\psi,\overline{\psi}]}O[\psi,\overline{\psi},U]$$

$$\int \mathcal{D}\,\psi\;\mathcal{D}\,\overline{\psi}\;e^{-\overline{\psi}\,(\gamma_\mu D_\mu + m_q)\,\psi} \approx det\,(\gamma_\mu D_\mu + m_q)$$

- Fermions represented by Grassmann variables in P.I: expensive to manipulate on a computer (classical)
- Easy to evaluate integral over fermionic (anti-commuting) fields: $\eta_i\eta_j = -\,\eta_j\eta_i$
- Integration rules: $\int d\eta = 0;\quad \int d\eta\,\eta = 1$

The cost of dynamical fermions

$$\langle O[\psi,\overline{\psi},U]\rangle = \frac{1}{Z}\int \mathcal{D}\,U\;e^{-S_G[U]}\,[det\,(\gamma_\mu D_\mu + m_q)]^{N_f}\;O[\psi,\overline{\psi},U]$$

$$\int \mathcal{D}\,\phi\,\mathcal{D}\,\phi^\dagger\;e^{-\phi^\dagger\,(D^\dagger(m_q)D(m_q))^{-1}\,\phi} \quad\propto\quad [det\,(\gamma_\mu D_\mu + m_q)]^2$$

- Determinant: non-local object on the lattice —> virtually impossible to compute exactly!

- Computational cost of solving:

$$\chi = (\gamma_\mu D_\mu + m_q)^{-1}\Phi$$

  grows for: small quark masses m_q, and large $\frac{L}{a}$

- $k = \mathrm{cond}\,(M) \;\propto\; \frac{\lambda_{max}}{\lambda_{min}}$

The cost of dynamical fermions

$$\langle O[\psi, \overline{\psi}, U] \rangle = \frac{1}{Z} \int \mathcal{D}\, U\, \mathcal{D}\, \phi\, \mathcal{D}\, \phi^\dagger\ e^{-S_G[U] - \phi^\dagger\, (D^\dagger(m_q) D(m_q))^{-\frac{N_f}{2}}\, \phi}\, O[U, \phi, \phi^\dagger]$$

$$\int \mathcal{D}\, \phi\, \mathcal{D}\, \phi^\dagger\ e^{-\phi^\dagger\, (D^\dagger(m_q) D(m_q))^{-1}\, \phi} \quad \propto \quad [det\, (\gamma_\mu D_\mu + m_q)]^2$$

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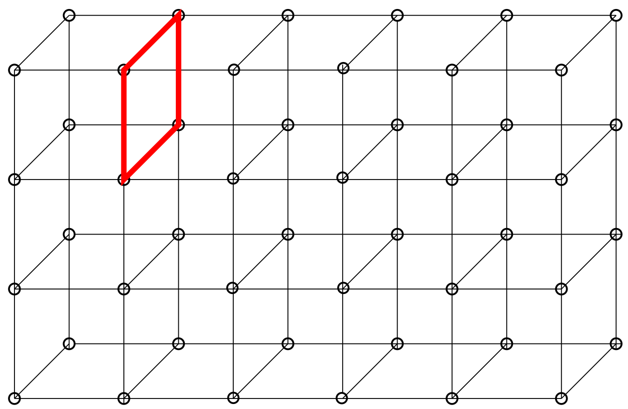
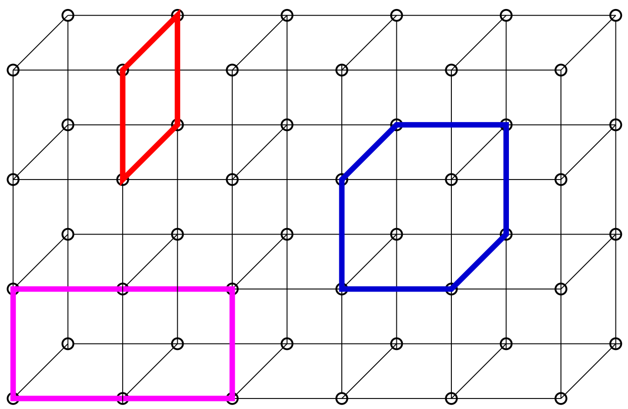
S_G Wilson



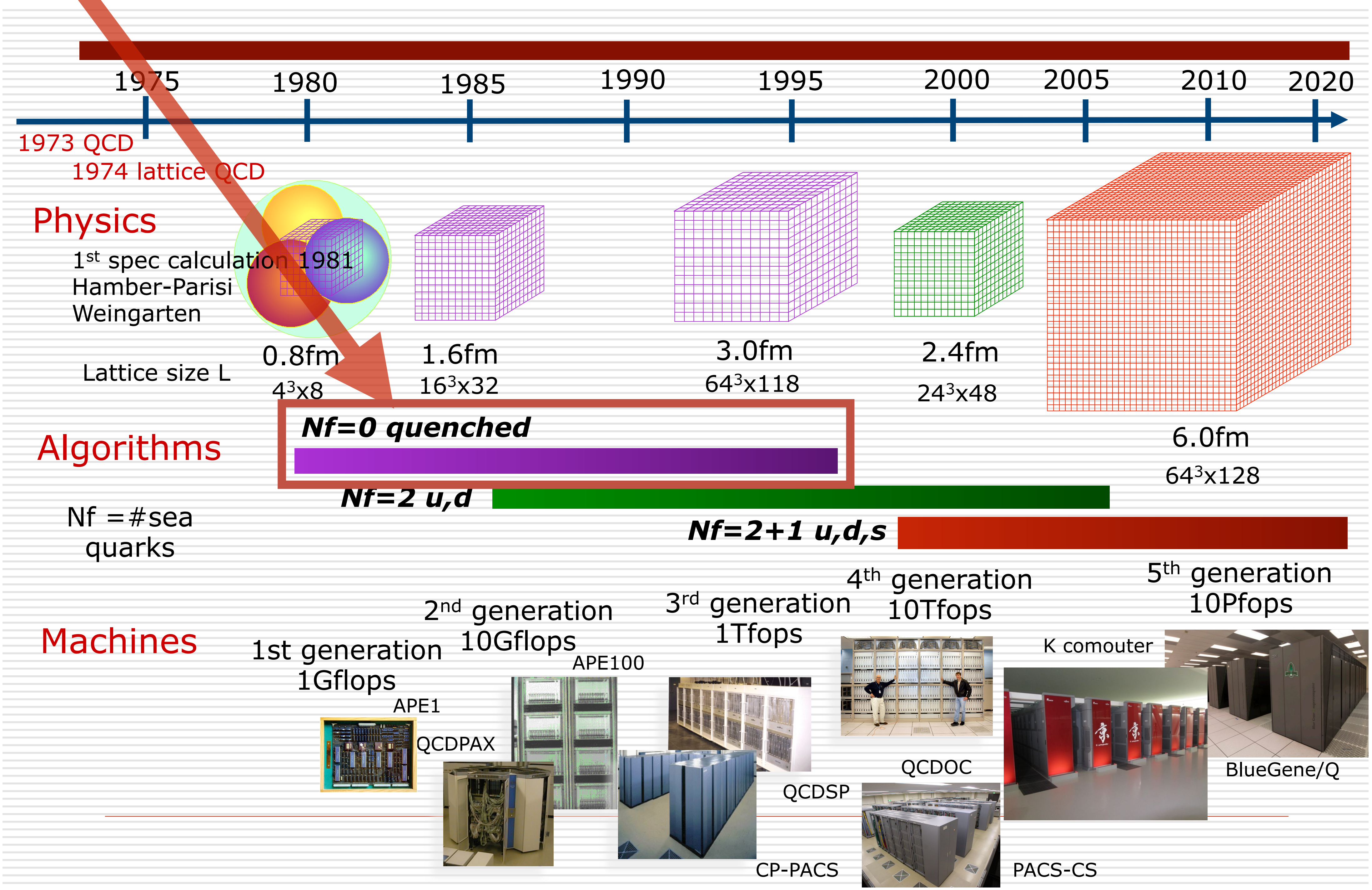
Luscher-Weisz



- Local objects: plaquettes, extended plaquettes (rectangles, chairs …)
- Rate of convergence towards continuum limit: $O(a), O(a^2), \dots$
- Monte Carlo algorithms with local updates sufficient in the approximation: $\det\,(\gamma_\mu D_\mu + m_q) \equiv 1$

Development of Lattice QCD

$$\det(\gamma_\mu D_\mu + m_q) \equiv 1$$



[Credit: A. Ukawa, HPC Summer School (2013)]

The Metropolis algorithm

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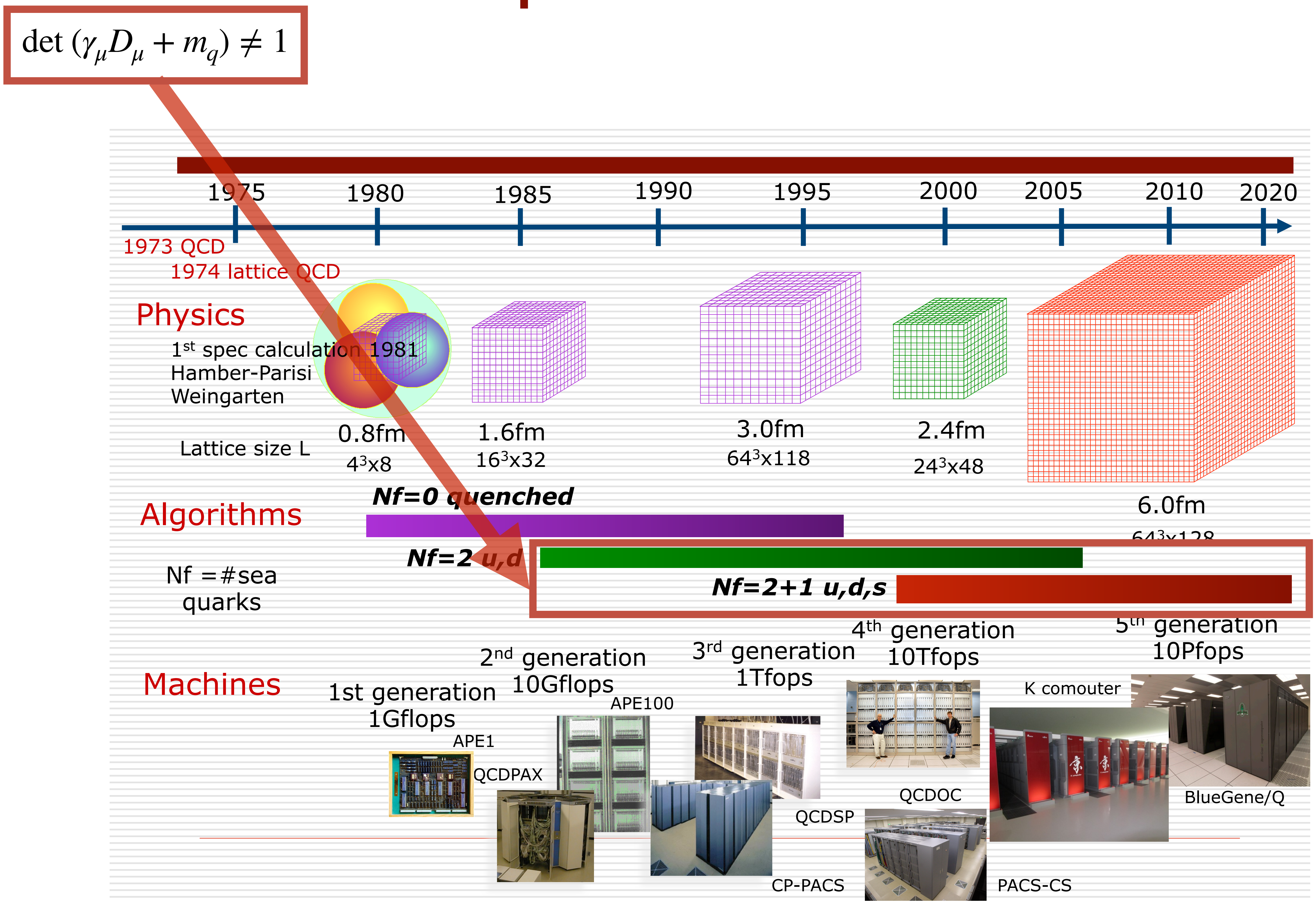


[Credit: A. Ukawa, HPC Summer School (2013)]

Hybrid Monte Carlo algorithm for QCD

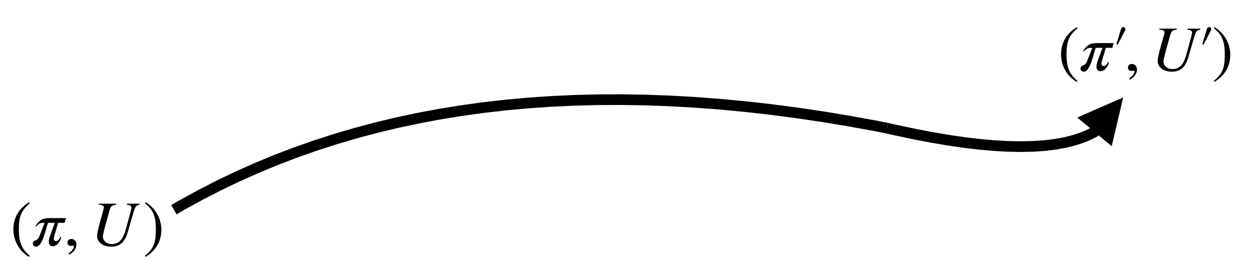
- Most widely used exact method for lattice QCD [Duane, Kennedy, Pendleton, Roweth, Phys. Lett. B, 195 (1987)]

- Introduce momenta $\pi_\mu(n)$ conjugate to fundamental fields $U_\mu(n)$ and the Hamiltonian

$$\mathscr{H} = \frac{1}{2}\sum_{n,\mu}\pi^2_{n,\mu} + S[U]$$

- **Momentum Heat-bath**: refresh momenta π (Gaussian random numbers)

- **Molecular Dynamics (MD)** evolution of π and U
  - ➡ numerically integrating the corresponding equations of motion (fictitious time τ):



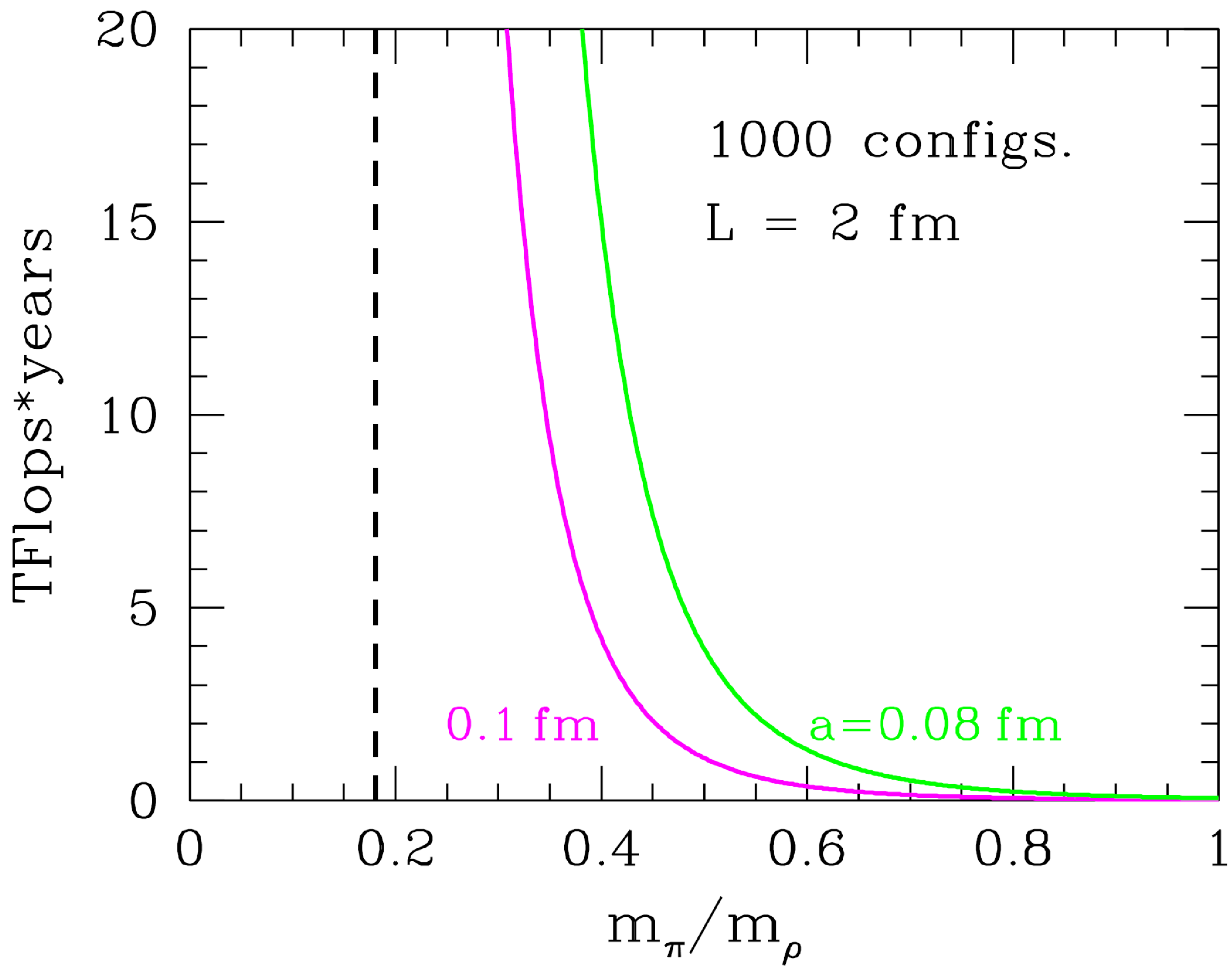
reversible, area-preserving scheme such as leap-frog, Omelyan…

- **Metropolis accept/reject step**
  - ➡ correcting for discretization errors of the numerical integration

$$P_{acc} = min\{1, e^{-(\mathscr{H}(\pi',U') - \mathscr{H}(\pi,U))}\}$$

"Berlin Wall"

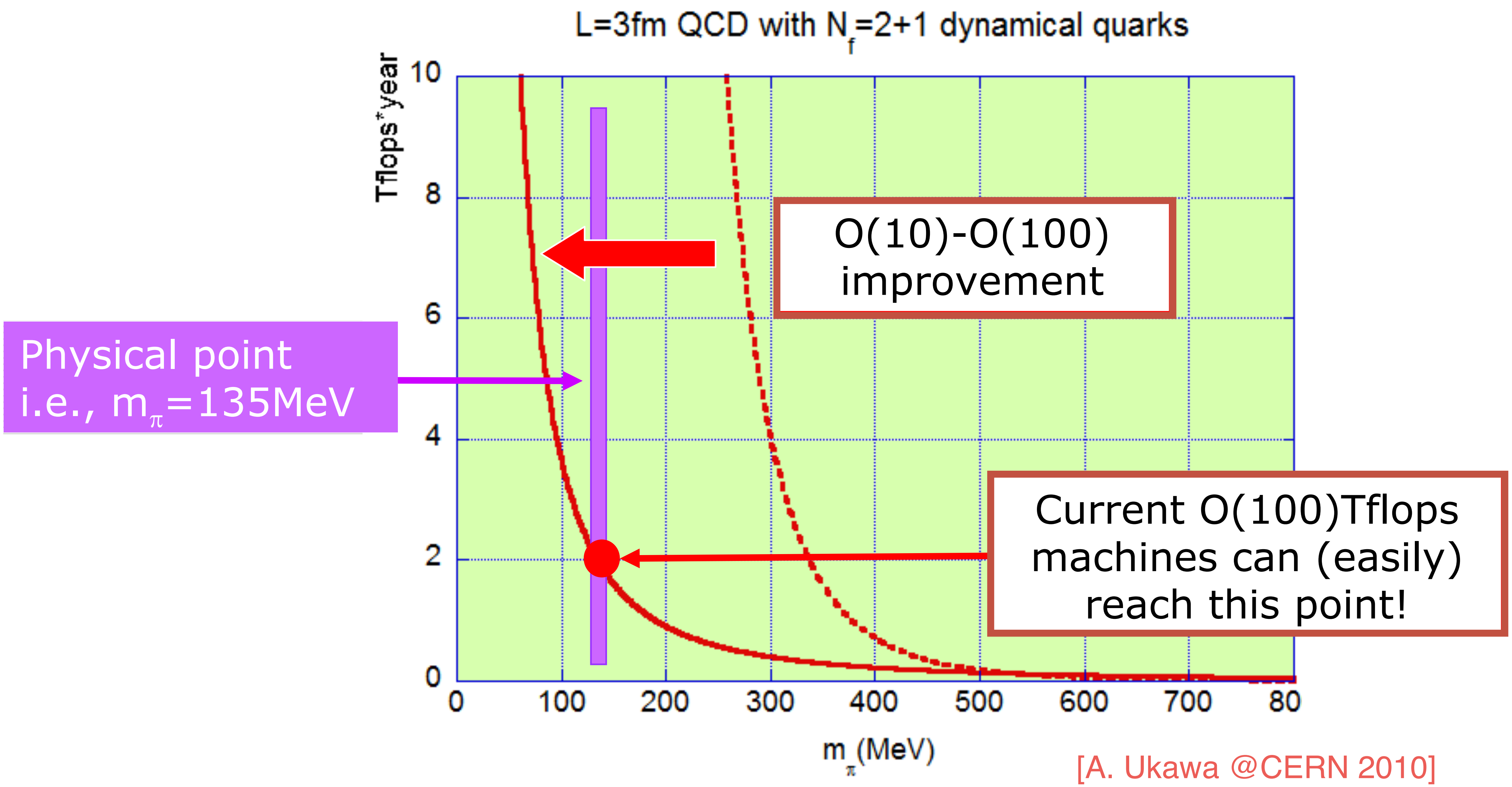
[Lattice 2001 (Berlin)]



- Status in 2001: [A. Ukawa (2001)]
  - ➡ quarks 16x heavier than in nature; coarse lattices $a \approx 0.1$ fm (typical length scale is 1fm)
  - ➡ Cost of a simulation: $C\left[\frac{\#conf}{1000}\right]\left[\frac{m_q}{16m_{phys}}\right]^{-3}\left[\frac{L}{3\text{fm}}\right]^{5}\left[\frac{a}{0.1\text{fm}}\right]^{-7}; \quad C \approx 2.8$

Berlin Wall update

[2010, start of my PhD]



[A. Ukawa @CERN 2010]

Simulating physical quark masses becomes reality!

In the second lecture:

- QCD can be formulated on a Euclidean space time lattice

- Quantization amounts to summing over all gauge configurations; this can again be computed by Monte Carlo methods

- Different discretizations give different lattice artefacts

  ➡ universal in continuum limit!

- Simulations with dynamical fermions computationally costly; many tricks in algorithms need to be applied

  ➡ precise predictions of hadron spectrum, non-pert. renormalization, muon g-2, CKM matrix elements, QCD phase diagram etc.

Thank you!