# Geometrical Approaches to Evaluate Feynman Integrals

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**Based on: arXiv:2309.00409 & PhysRevLett.127.151601**

LTP(izza)hD

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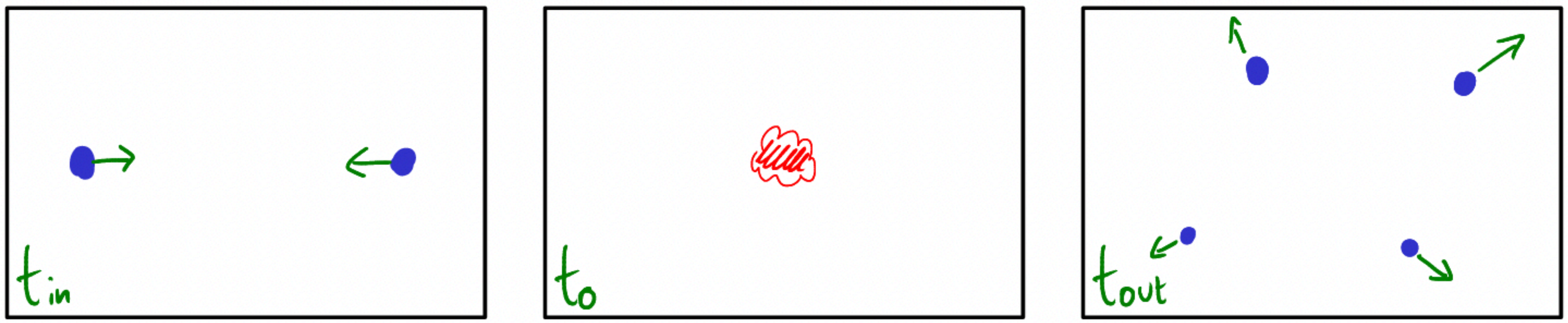
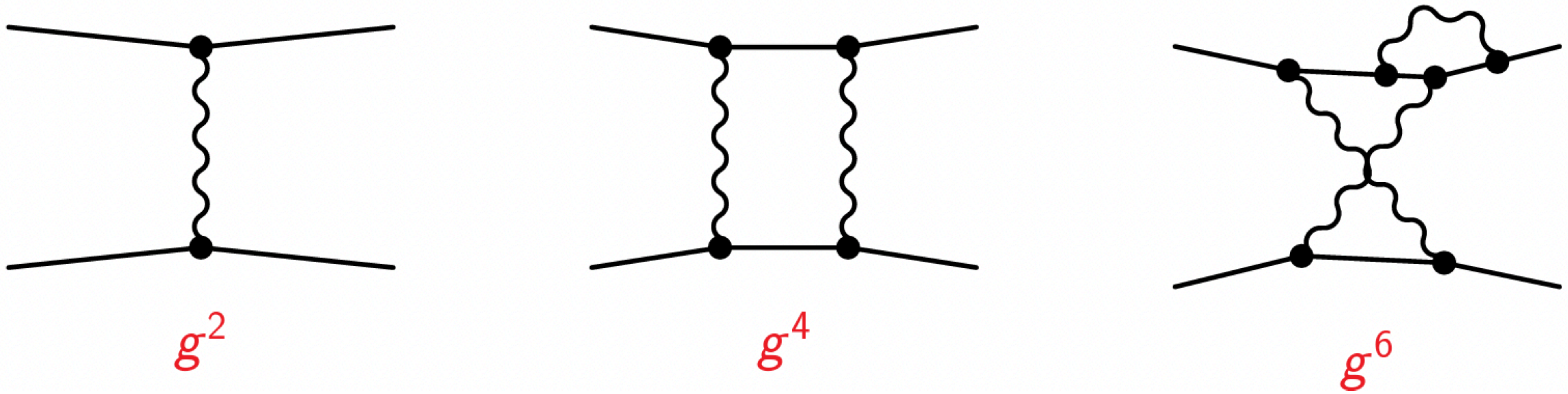
# What are Feynman Diagrams?

## Motivation

- Outcomes of High Energy Scattering Experiments are probabilistic




- Feynman Diagrams encodes probabilities for different outcomes

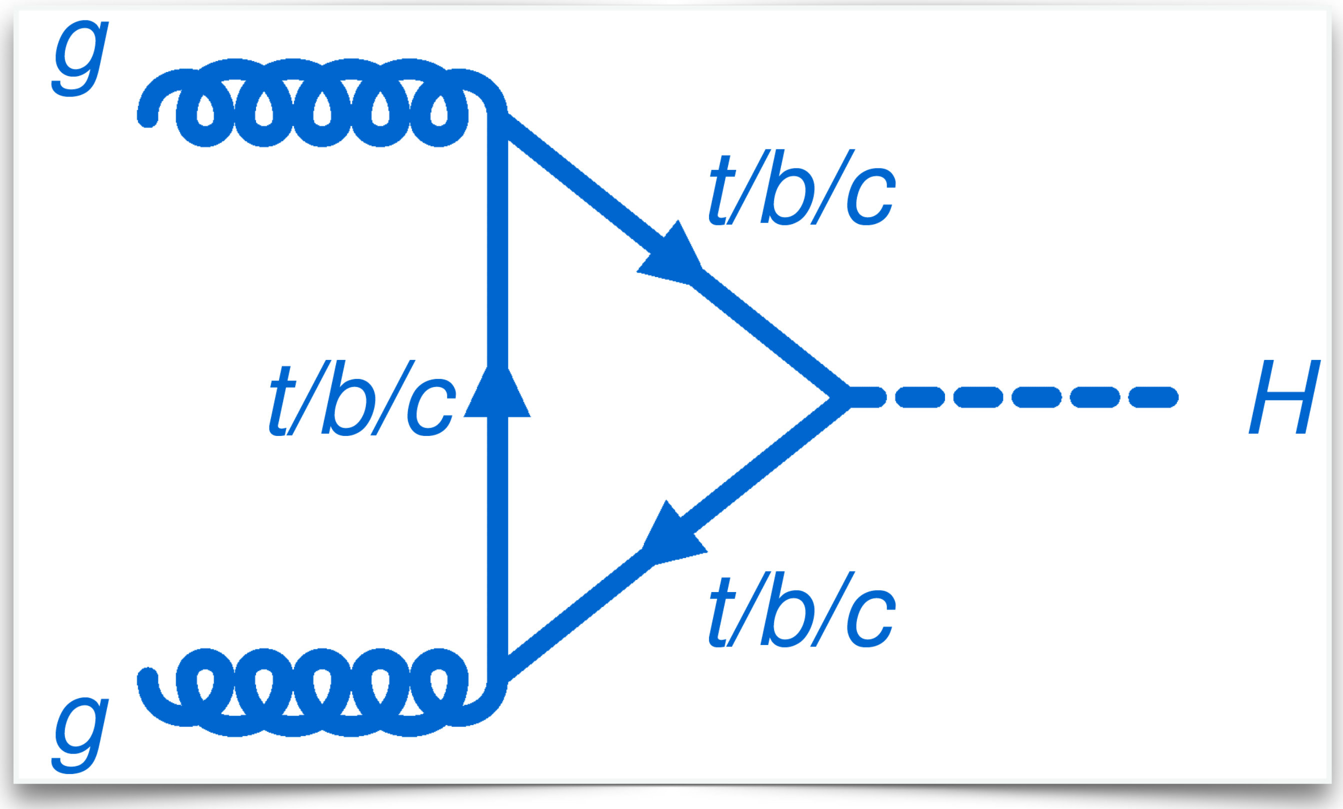



Introduced in 1948 by R. Feynman

- Feynman Diagrams crucial for precise predictions

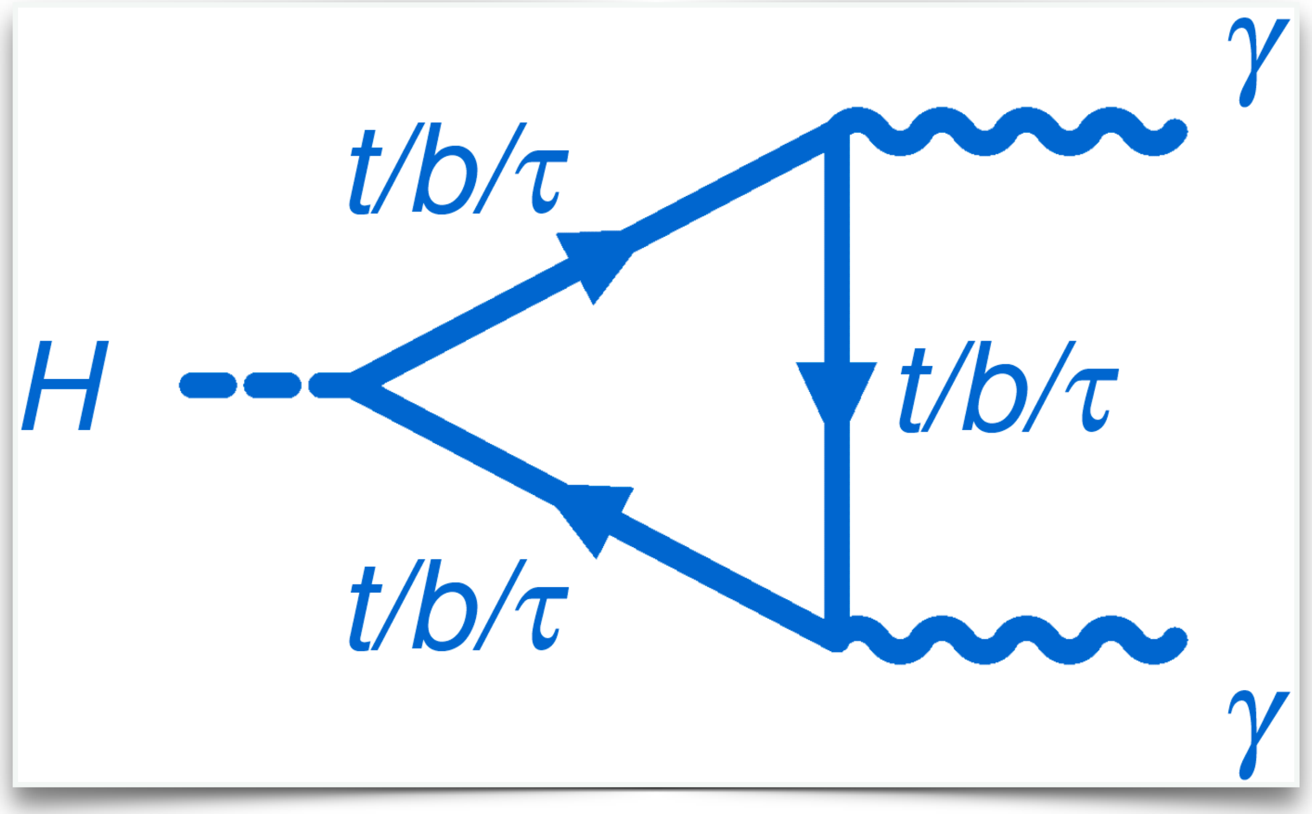
# Example Feynman Diagrams

- Dominant Feynman diagram leading to Higgs boson discovery




**Production:** $\sigma \approx 48\,pb$

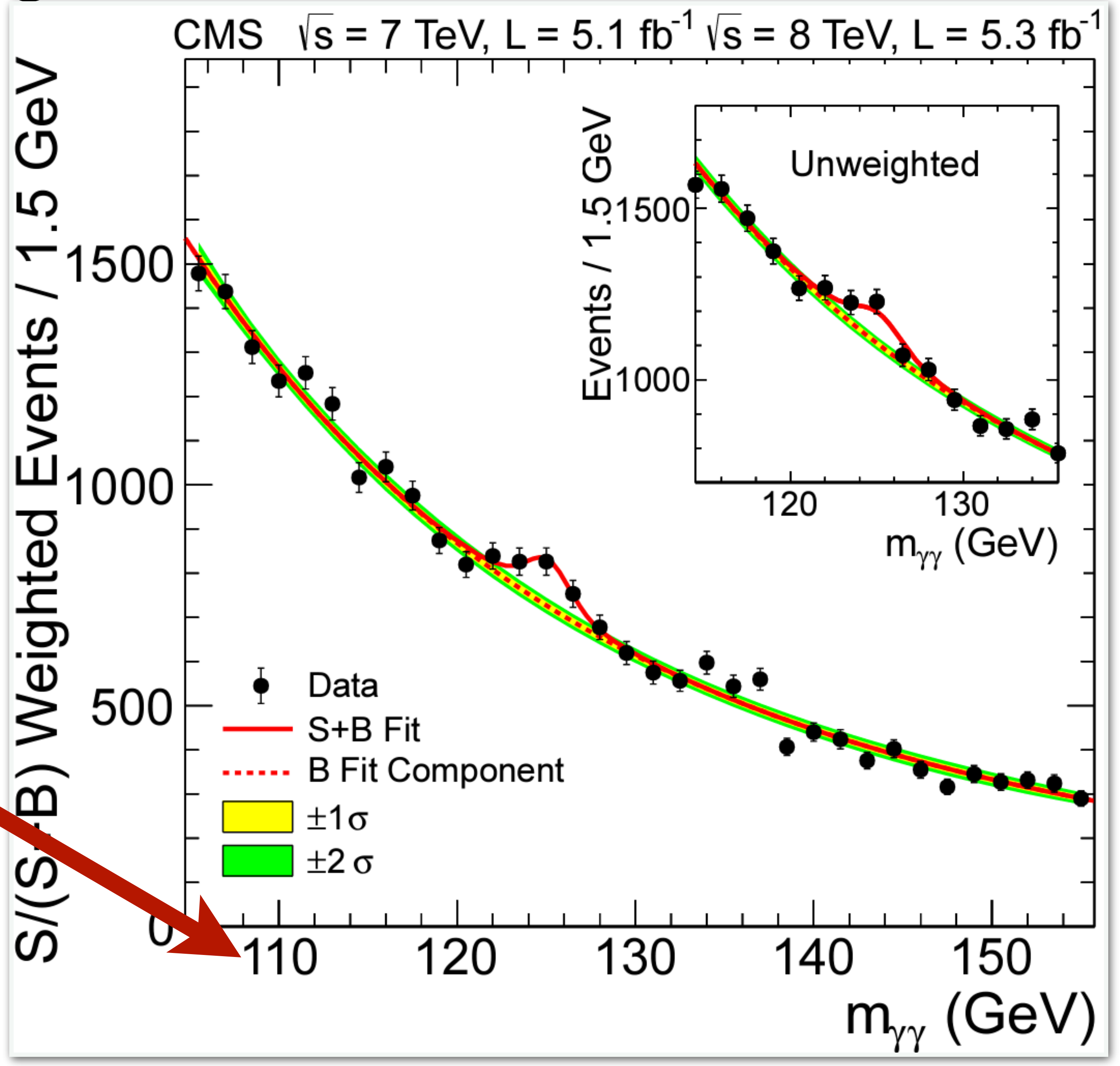
# of H produced = $\sigma \times L$




**Branching Ratio:** $\approx 0.2\,\%$

- Higgs discovered from di-photon invariant mass

$$m_{\gamma\gamma} = (p_{\gamma_1} + p_{\gamma_2})^2$$



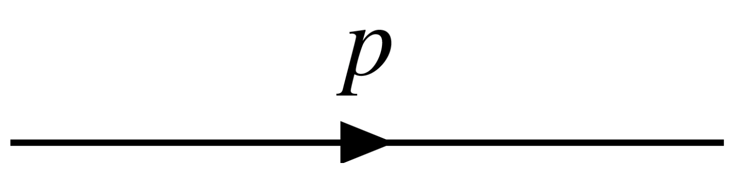

- Feynman Integrals crucial for High-Luminosity LHC

# Feynman Rules

## Scalar Theory

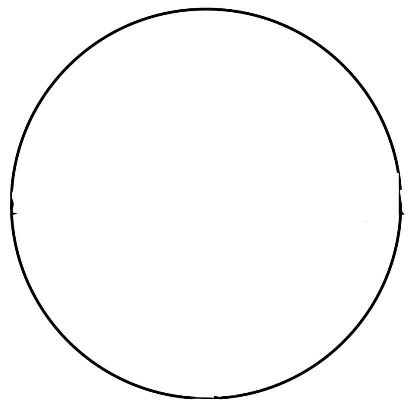
**Feynman Diagram** → Feynman Rule → **Feynman Integral**

- Propagator



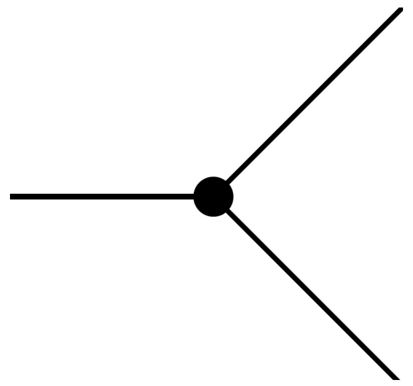

$$\frac{1}{p^2+m^2}$$

- Loop Factor



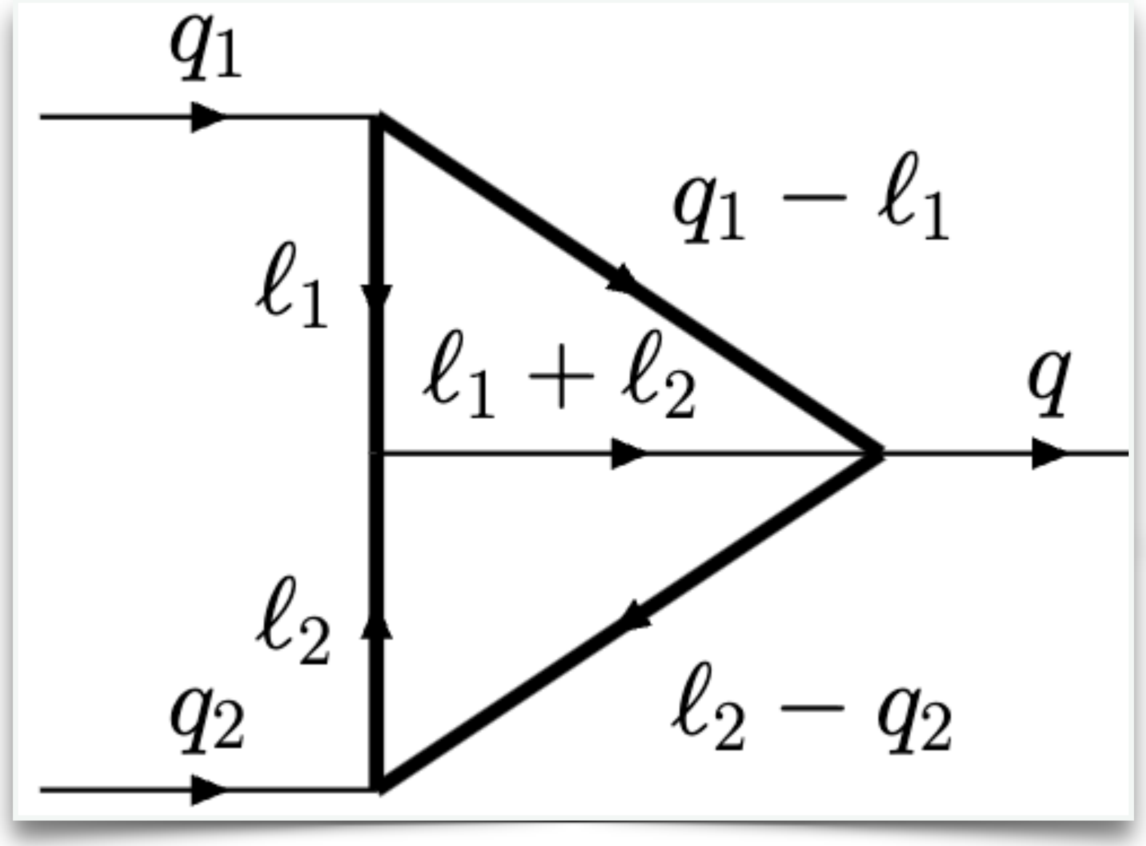
$$\int \frac{d^D l}{\pi^{D/2}}$$

- Vertex Factor



$$\lambda$$

**Example: Two-Loop Triangle**




$$I=\lambda^4\int\frac{d^Dl_1}{\pi^{D/2}}\int\frac{d^Dl_2}{\pi^{D/2}}\frac{1}{[l_1^2+m^2][l_2^2+m^2][(q_2-l_2)^2+m^2][(q_1-l_1)^2+m^2][(l_1+l_2)^2]}$$

**Hard to compute!**

# Outline

- Multiple Mellin-Barnes Representation
- Analytical Evaluation
    - Conic Hull Approach
    - Triangulation Approach
- Applications
- Conclusion & Outlook

# Multiple MB Representation

## Motivation

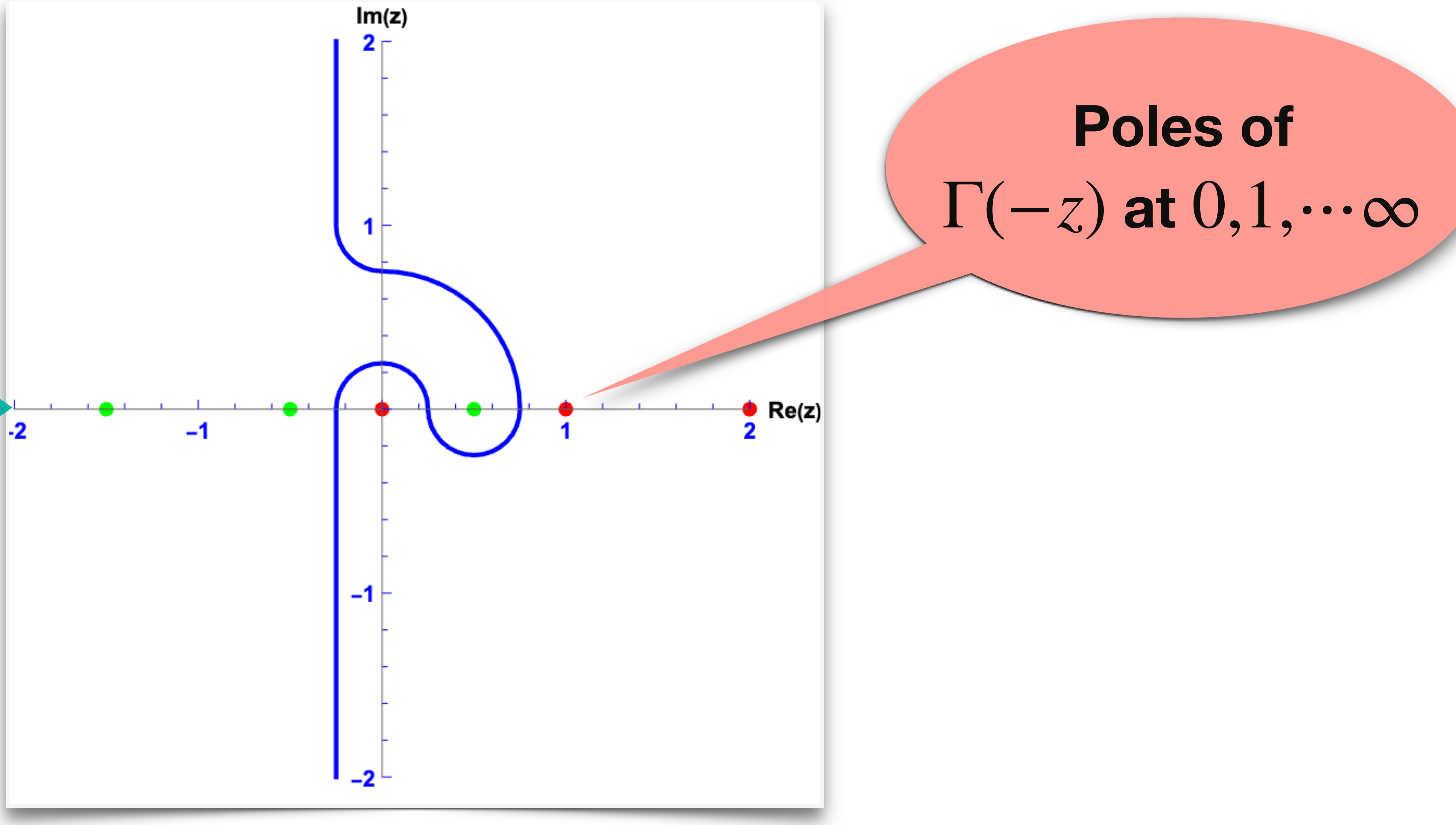
Overview: [arXiv: 2211.13733]

- Feynman Integrals can be evaluated using Mellin-Barnes (MB) Representation

$$\frac{1}{(A+B)^{\alpha}} = \frac{1}{\Gamma(\alpha)} \int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \Gamma(-z)\Gamma(\alpha+z) A^{-\alpha-z} B^{z}$$

- Contour separates poles of $\Gamma(-z)$ from $\Gamma(-1/2+z)$

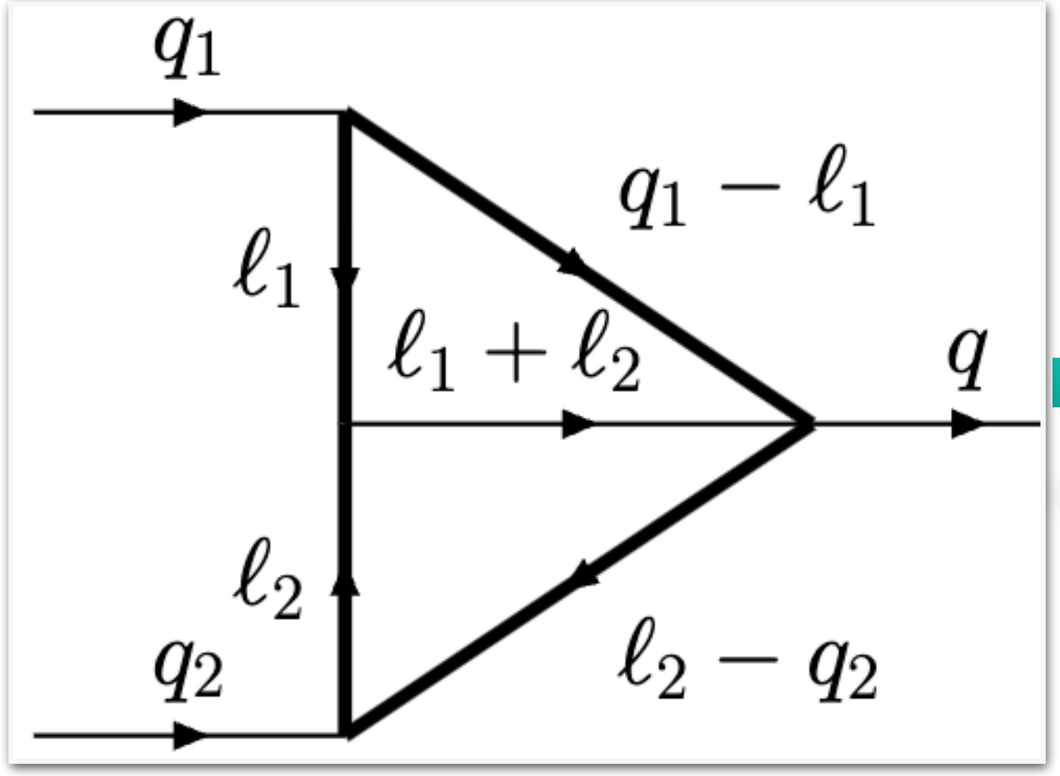
$$\int_{-i\infty}^{+i\infty} \frac{dz}{2\pi i} \Gamma(-z)\Gamma(-1/2+z)(-x)^{z}$$



# Multiple MB Representation

## Motivation

- Example: Two-Loop Feynman Diagram




$$I = \lambda^4 \int \frac{d^D l_1}{\pi^{D/2}} \int \frac{d^D l_2}{\pi^{D/2}} \frac{1}{[l_1^2 + m^2][l_2^2 + m^2][(q_2 - l_2)^2 + m^2][(q_1 - l_1)^2 + m^2][(l_1 + l_2)^2]}$$

**MB Representation**

Fold

Scale

$$\int_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi \mathrm{i}} \left(\frac{s}{4m^2}\right)^z \frac{\Gamma(-z)\Gamma^3(1+z)\Gamma(1+z+\epsilon)\Gamma(1+z+2\epsilon)}{\Gamma^2(2+z)\Gamma(2+z-\epsilon)\Gamma\left(\frac{3}{2}+z+\epsilon\right)}$$

**Series Solution**
**or**
**Numerical Integration**

# Multiple MB Representation

## Motivation

- All Hypergeometric Functions have MB

Useful for deriving analytic continuations

$$_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \frac{\Gamma(-z_1)\Gamma(a+z_1)\Gamma(b+z_1)}{\Gamma(c+z_1)} (-x)^{z_1}$$

- All Multiple Polylogs special class of MB

# of numerator $\Gamma(\cdots)$ $\propto$ weights

$$\mathrm{Li}_{m_1,m_2}(x_1,x_2) = x_1 x_2^2 \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{dz_3}{2\pi i} \Gamma(-z_1)\Gamma(-z_2)\Gamma(1+z_1)\Gamma(1+z_2) \frac{\Gamma^{m_1}(1+z_1)\Gamma^{m_2}(2+z_{12})}{\Gamma^{m_1}(2+z_1)\Gamma^{m_2}(3+z_{12})} (-x_1 x_2)^{z_1} (-x_2)^{z_2}$$

- MB appear in optional pricing, electromagnetic wave propagation, etc

# Multiple MB Representation

## N-Fold Case

- N-fold MB Representation

$$\int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \cdots \int_{-i\infty}^{+i\infty} \frac{dz_N}{2\pi i} \frac{\prod_{i=1}^{k} \Gamma^{a_i}(\mathbf{e}_i \cdot \mathbf{z} + g_i)}{\prod_{j=1}^{l} \Gamma^{b_j}(\mathbf{f}_j \cdot \mathbf{z} + h_j)} x_1^{z_1} \cdots x_N^{z_N}$$

$\mathbf{e}_i$ & $\mathbf{f}_j$ $N$-dimensional

$\mathbf{z} = \{z_1, \cdots z_N\}$

**Analytic Evaluation**

This talk

- MBConichulls.wl
- MBsums.m [arXiv: 1511.01323]

**Numerical Evaluation**

- MB.m
- MBresolve.m
- MBnumerics.m

[arXiv: 2211.00009]

- Century-old problem in Mathematics

# Analytic Evaluation

## Evaluating Appell $F_1$ using Conic Hulls

$^5C_2 = 10$ possible 2-combinations

- Appell $F_1$ MB Representation:

$$\int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i}(-u_1)^{z_1}(-u_2)^{z_2}\frac{\overset{1}{\Gamma}(-z_1)\overset{2}{\Gamma}(-z_2)\overset{3}{\Gamma}\left(a+z_1+z_2\right)\overset{4}{\Gamma}\left(b_1+z_1\right)\overset{5}{\Gamma}\left(b_2+z_2\right)}{\Gamma\left(c+z_1+z_2\right)}$$

- 2-Combinations of Numerator Gamma Functions

1. $\{\overset{1}{\Gamma}(-z_1), \overset{2}{\Gamma}(-z_2)\}$
2. $\{\overset{1}{\Gamma}(-z_1), \overset{3}{\Gamma}(a+z_1+z_2)\}$
3. $\{\overset{3}{\Gamma}(a+z_1+z_2), \overset{5}{\Gamma}(b_2+z_2)\}$
4. $\{\overset{4}{\Gamma}(b_1+z_1), \overset{5}{\Gamma}(b_2+z_2)\}$
5. $\{\overset{2}{\Gamma}(-z_2), \overset{4}{\Gamma}(b_1+z_1)\}$
6. $\{\overset{3}{\Gamma}(a+z_1+z_2), \overset{4}{\Gamma}(b_1+z_1)\}$
7. $\{\overset{2}{\Gamma}(-z_2), \overset{3}{\Gamma}(a+z_1+z_2)\}$
8. $\{\overset{1}{\Gamma}(-z_1), \overset{5}{\Gamma}(b_2+z_2)\}$

- Singular 2-Combinations Omitted: ✖ $\{\Gamma(-z_1), \Gamma(b_1+z_1)\}$ ✖ $\{\Gamma(-z_2), \Gamma(b_2+z_2)\}$

# Analytic Evaluation

## Evaluating Appell $F_1$ using Conic Hulls

- 8 Associated Building Blocks

**1**

$$\{\overset{1}{\Gamma}(-z_1), \overset{2}{\Gamma}(-z_2)\}$$

→

$B_{1,2}$

$$\sum_{n_1,n_2=0}^{\infty} \frac{\Gamma(a+n_1+n_2)\Gamma(b_1+n_1)\Gamma(b_2+n_2)}{\Gamma(c+n_1+n_2)} \frac{u_1^{n_1} v_2^{n_2}}{n_1!n_2!}$$

**Residues of poles of $\{\Gamma(-z_1), \Gamma(-z_2)\}$ at $(z_1, z_2) = (n_1, n_2)$**

- 8 Associated Conic Hulls

**1**

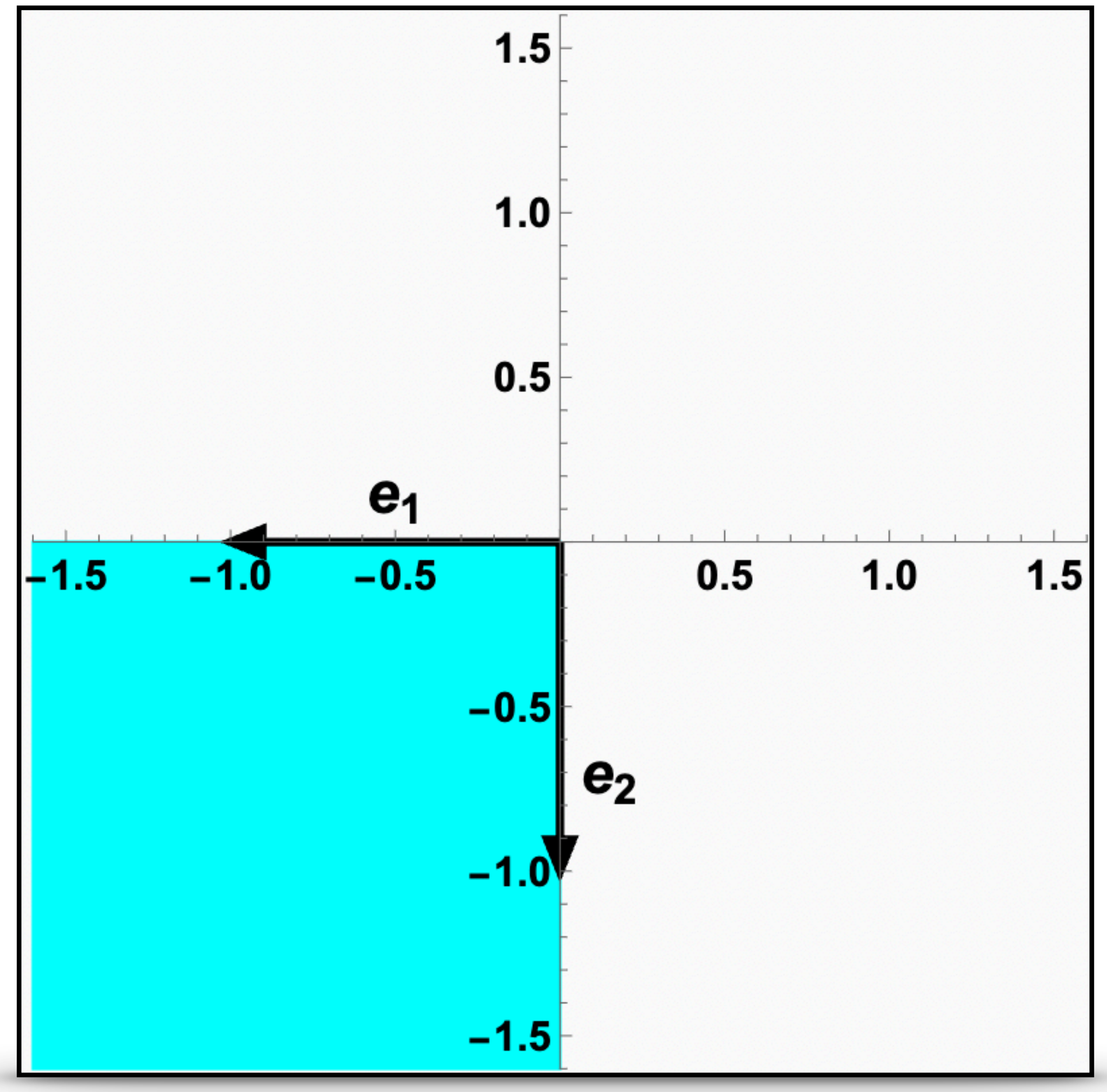
$$\{\overset{1}{\Gamma}(-z_1), \overset{2}{\Gamma}(-z_2)\}$$

$\vec{e}_1 = (-1,0)$

$\vec{e}_2 = (0,-1)$

→

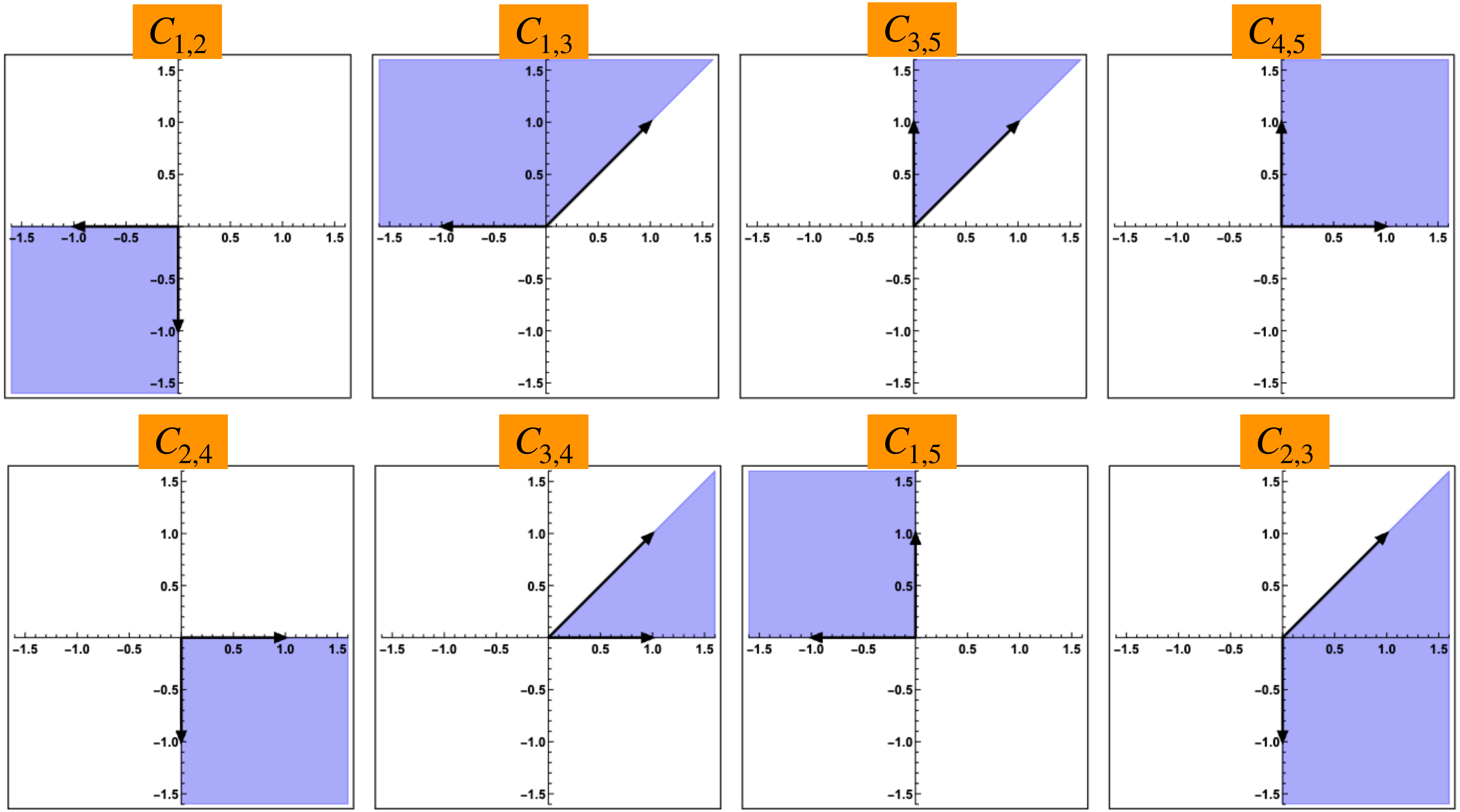
$C_{1,2}$



# Analytic Evaluation

## Appell $F_1$ Solutions

- All 8 Conic Hulls



# Analytic Evaluation

## Appell $F_1$ Solutions

Master Conic Hull

- 5 Largest Subsets → 5 Series Solutions

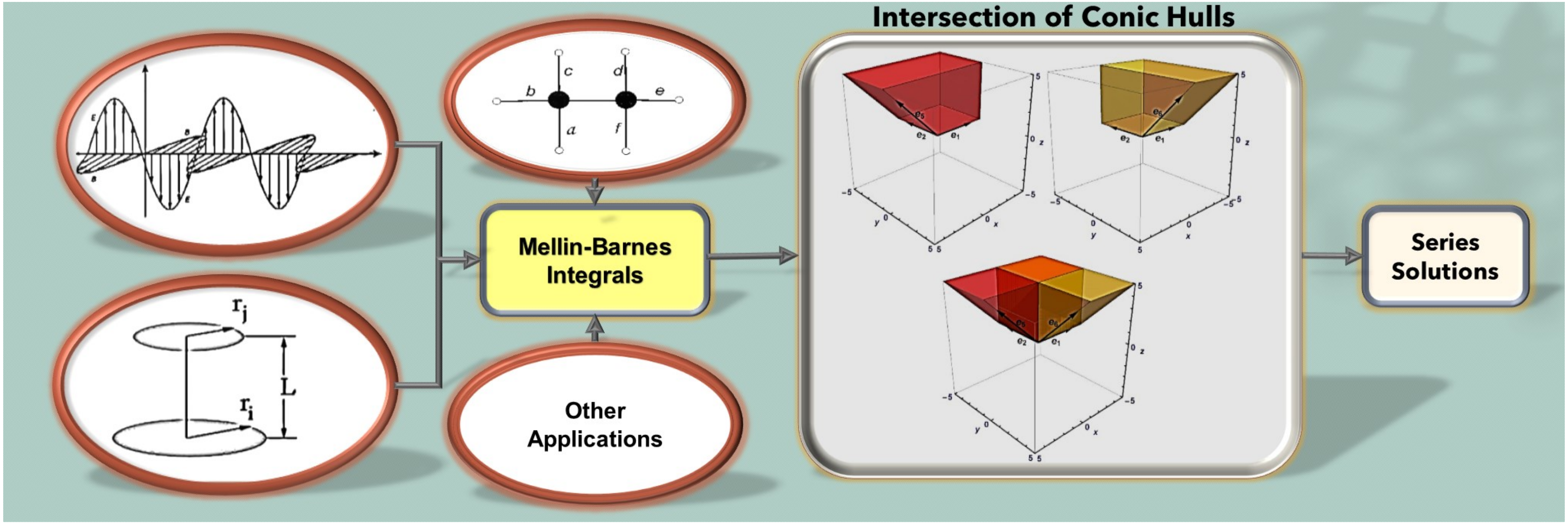

Largest Intersecting Subset

- Solutions:

1. $B_{1,2}$
2. $B_{1,3} + B_{3,5} + B_{4,5}$
3. $B_{1,3} + B_{1,5}$
4. $B_{2,3} + B_{2,4}$
5. $B_{2,3} + B_{3,4} + B_{4,5}$

# Analytic Evaluation

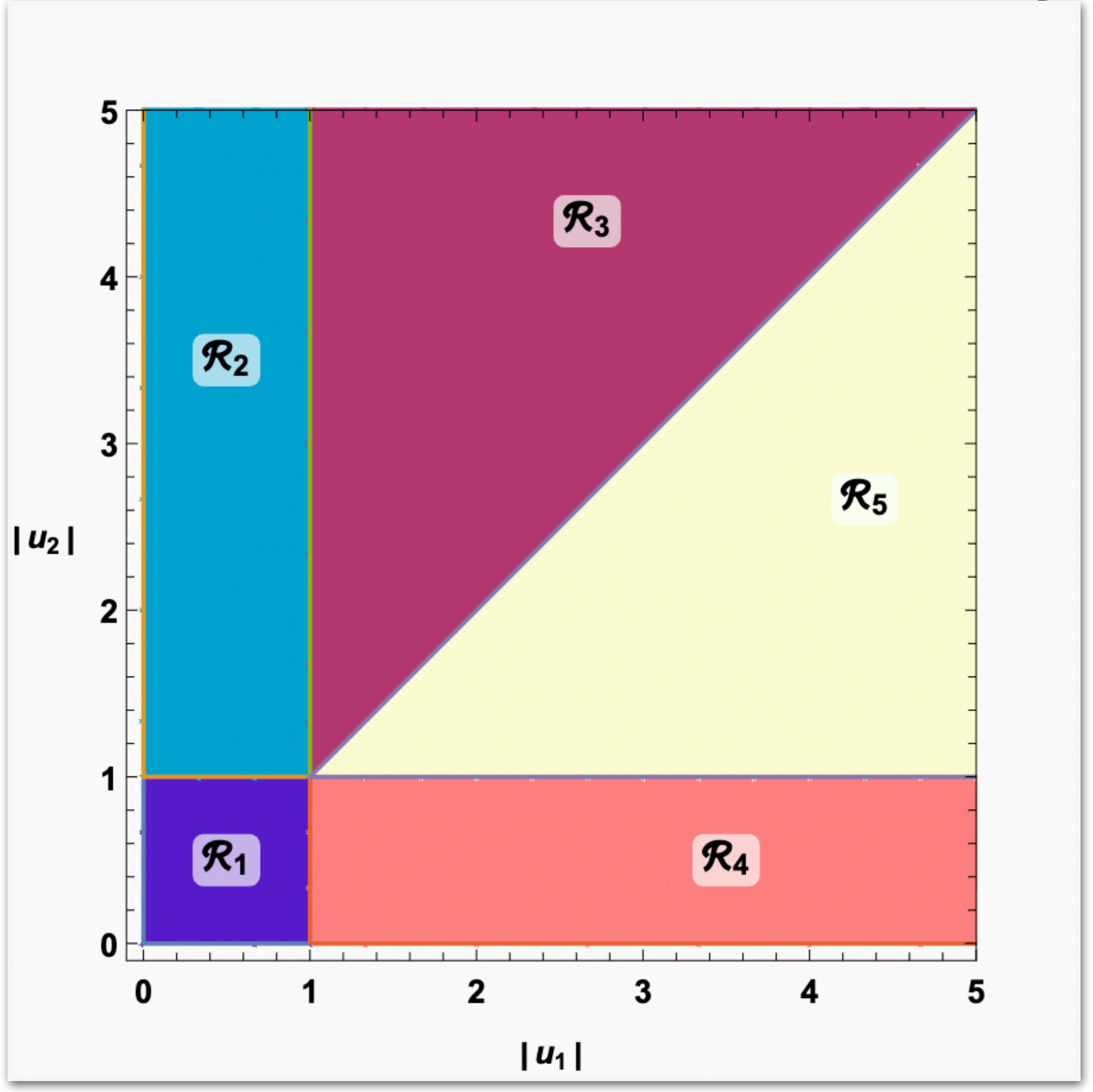
## Conic Hull Approach Overview



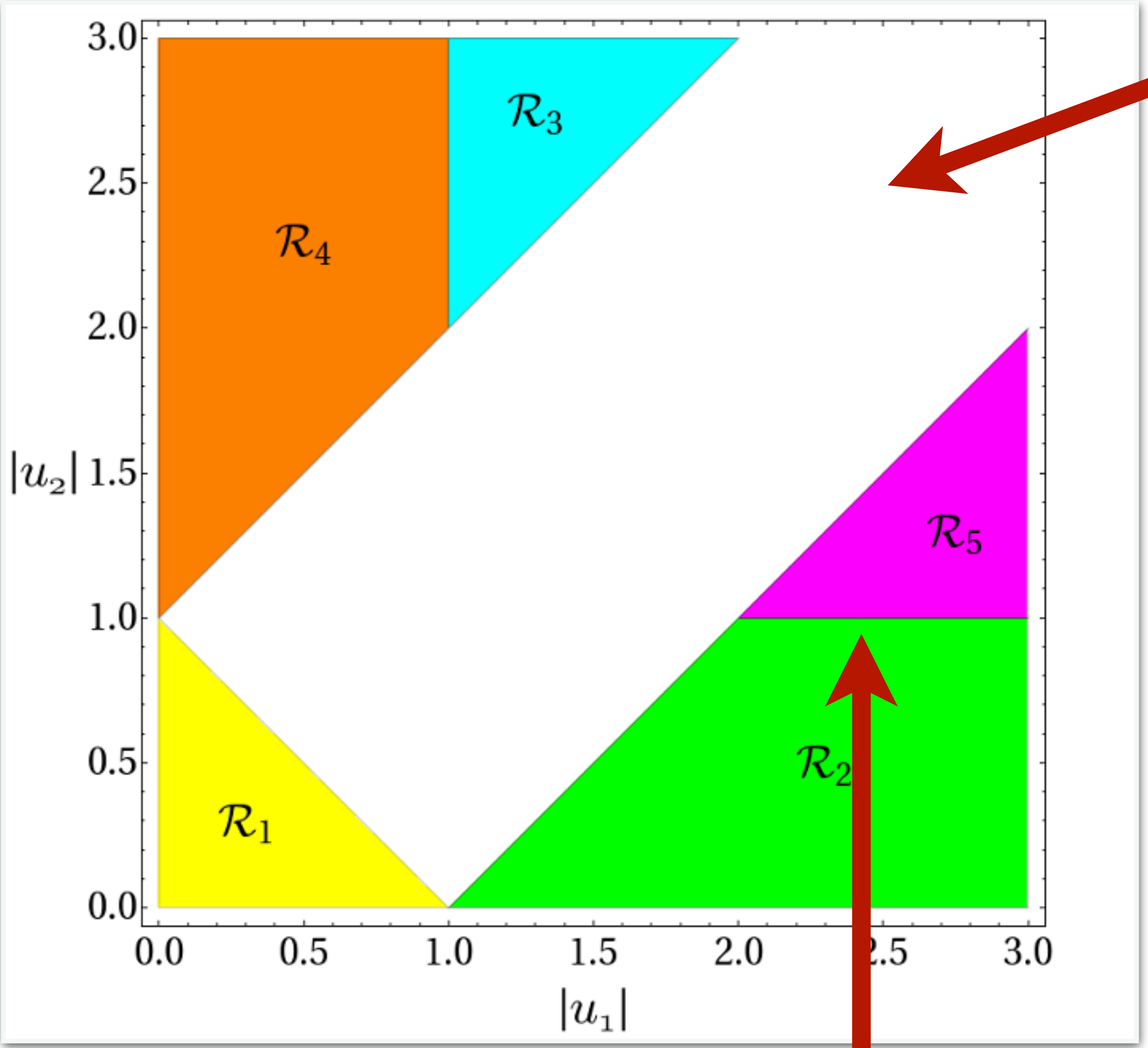
# Analytic Evaluation

## Challenges in Conic Hull Approach

- Convergent Solutions if # of Scales= # of Folds
- Full set of solutions may not always converge for all values (White Zone)




Appell $F_1$




$R_{-1}(u_1, u_2)$

- Final solutions may converge slowly near boundaries
- Slow for high-fold MB

# Analytic Evaluation

## Evaluating Appell $F_1$ using Triangulation

- Appell $F_1$ MB Representation:

$$\int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i}\frac{dz_2}{2\pi i}(-u_1)^{z_1}(-u_2)^{z_2}\frac{\overset{1}{\Gamma}(-z_1)\overset{2}{\Gamma}(-z_2)\overset{3}{\Gamma}\left(a+z_1+z_2\right)\overset{4}{\Gamma}\left(b_1+z_1\right)\overset{5}{\Gamma}\left(b_2+z_2\right)}{\Gamma\left(c+z_1+z_2\right)}$$

- Point Configuration: # of points = # of numerator $\Gamma(\cdots)$

$$P_1=\begin{pmatrix}1\\1\\0\end{pmatrix}\quad P_2=\begin{pmatrix}1\\0\\1\end{pmatrix}\quad P_3=\begin{pmatrix}1\\0\\0\end{pmatrix}\quad P_4=\begin{pmatrix}0\\1\\0\end{pmatrix}\quad P_5=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$z_1$ coefficients of non-trivial gamma

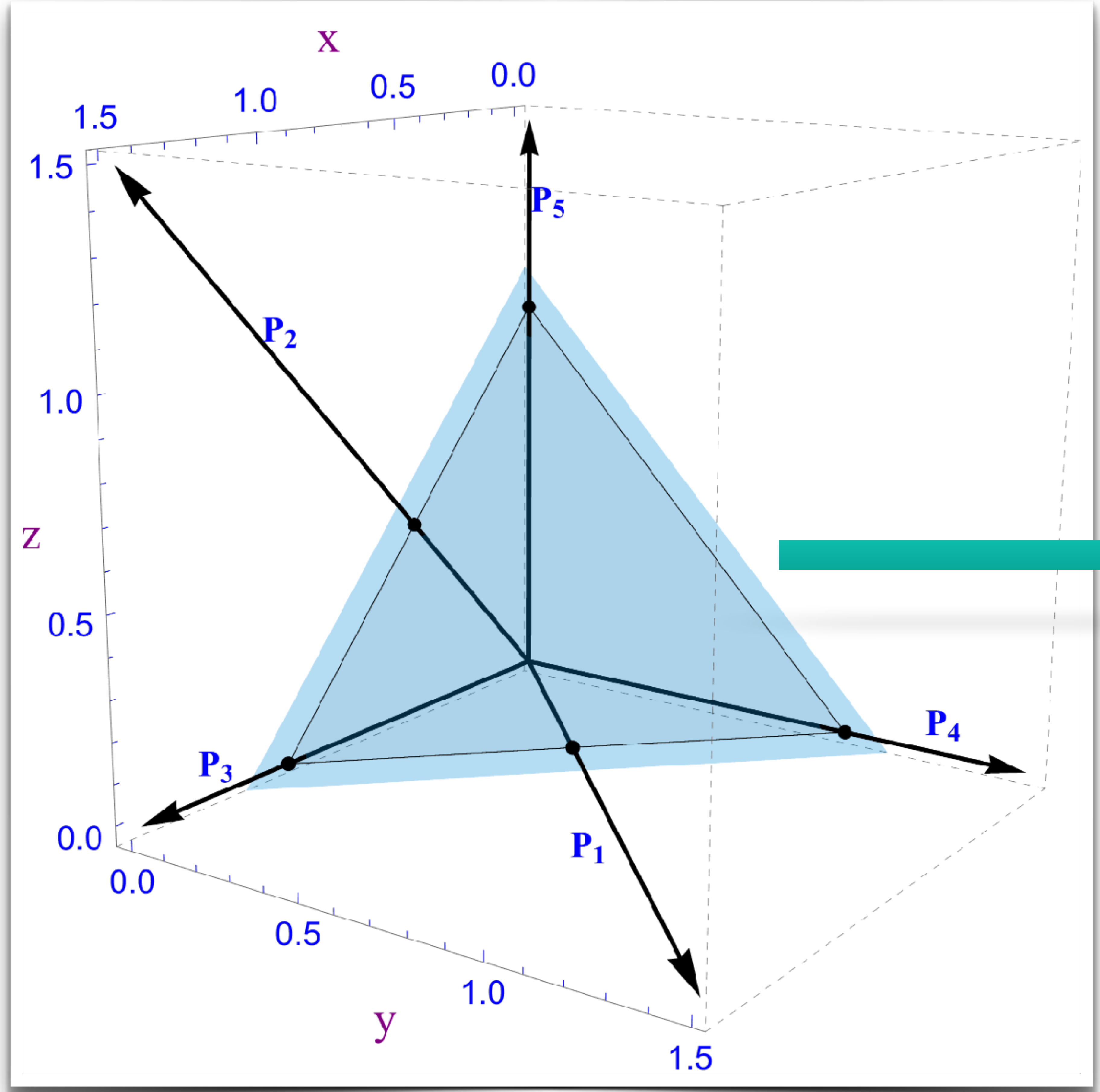
$z_2$ coefficients of non-trivial gamma

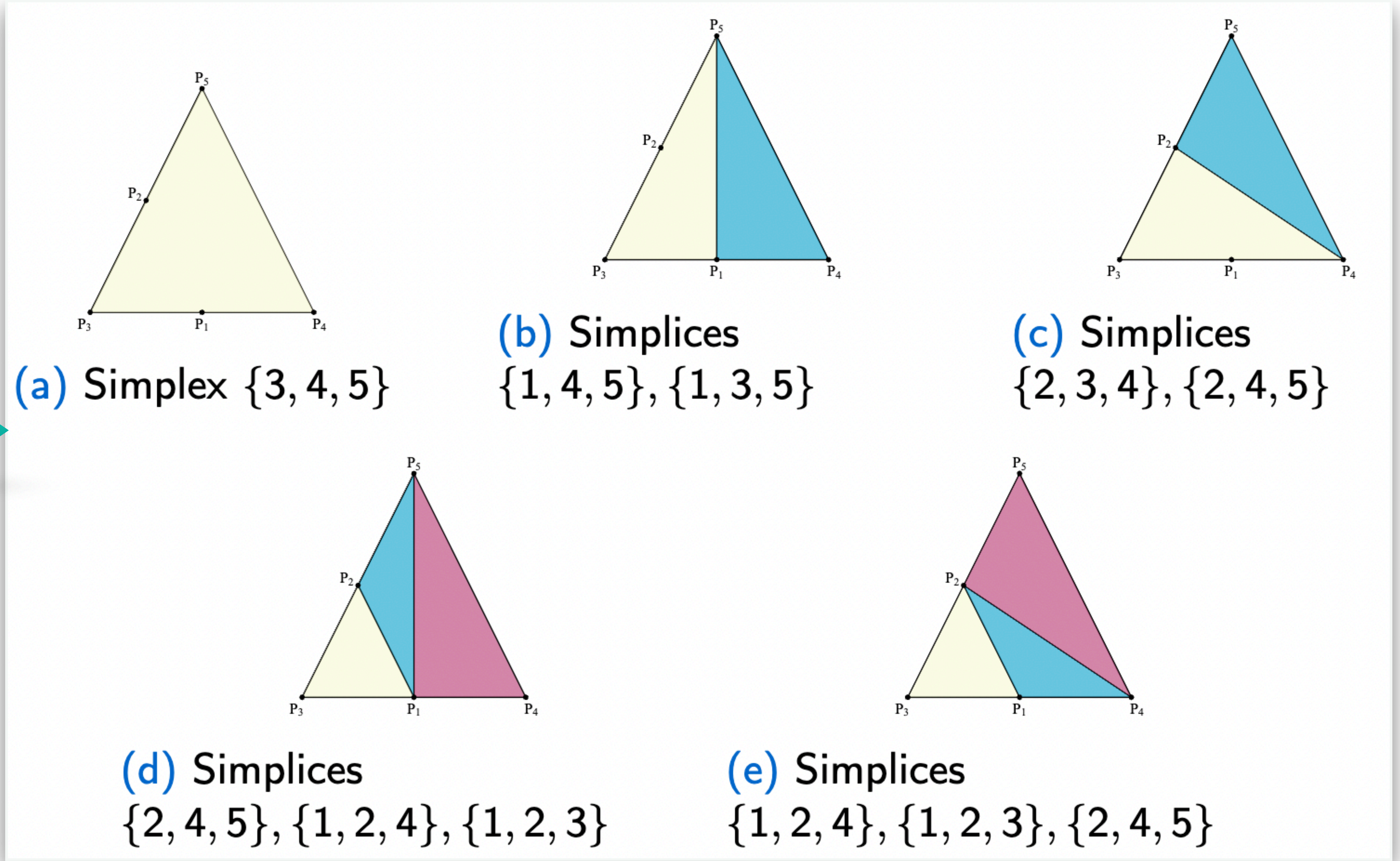
Unit Vectors

# Analytic Evaluation

## Evaluating Appell $F_1$ using Triangulation

- Point Configuration: $P = \{P_1, P_2, P_3, P_4, P_5\}$

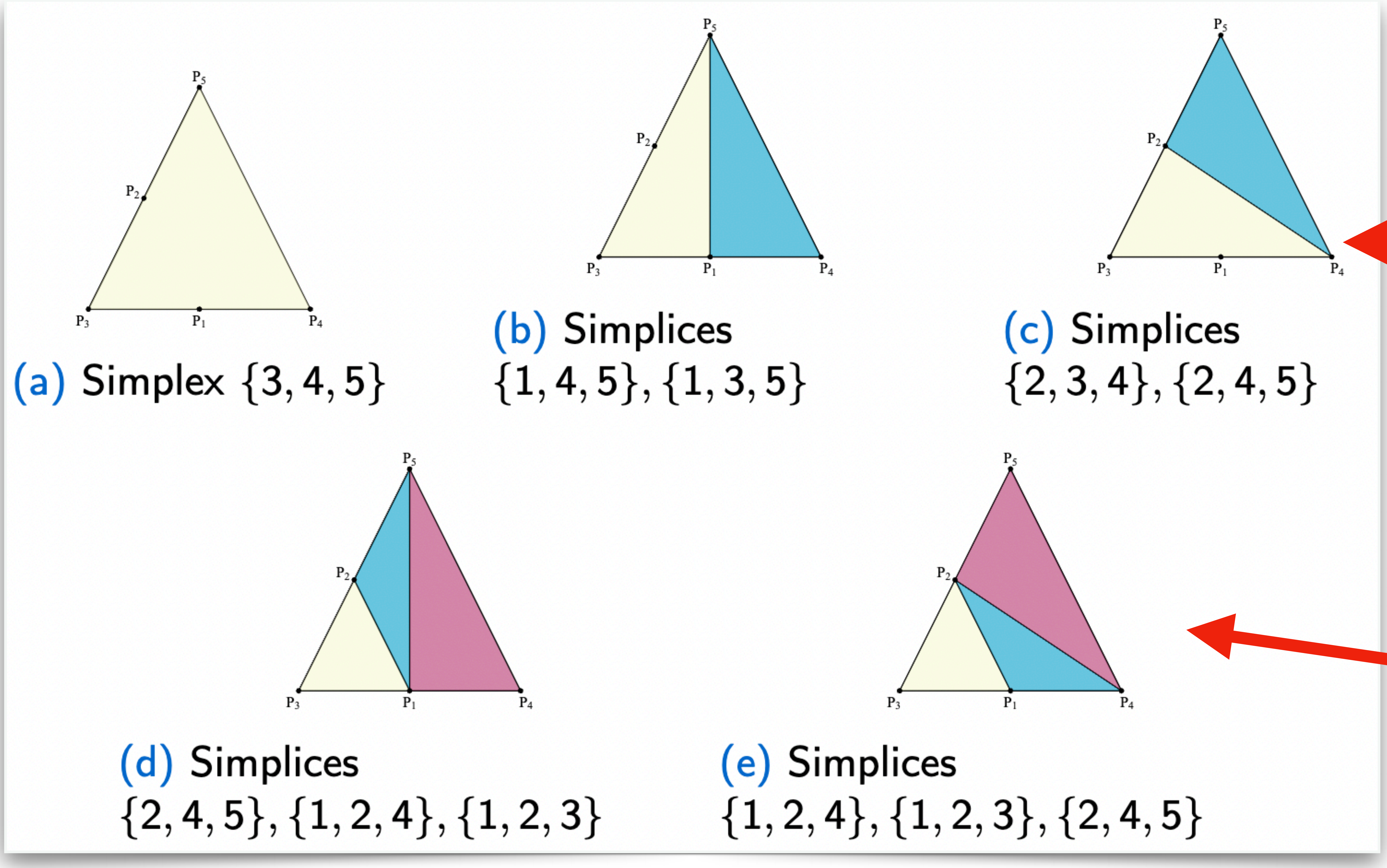


# Analytic Evaluation

## Evaluating Appell $F_1$ using Triangulation

- **Five** Possible Triangulations




(a) Simplex $\{3,4,5\}$

(b) Simplices $\{1,4,5\},\{1,3,5\}$

(c) Simplices $\{2,3,4\},\{2,4,5\}$

(d) Simplices $\{2,4,5\},\{1,2,4\},\{1,2,3\}$

(e) Simplices $\{1,2,4\},\{1,2,3\},\{2,4,5\}$

1. $\{C_{1,2}\}$
2. $\{C_{1,3}, C_{1,5}\}$
3. $\{C_{2,3}, C_{2,4}\}$
4. $\{C_{1,3}, C_{3,5}, C_{4,5}\}$
5. $\{C_{2,3}, C_{3,4}, C_{4,5}\}$

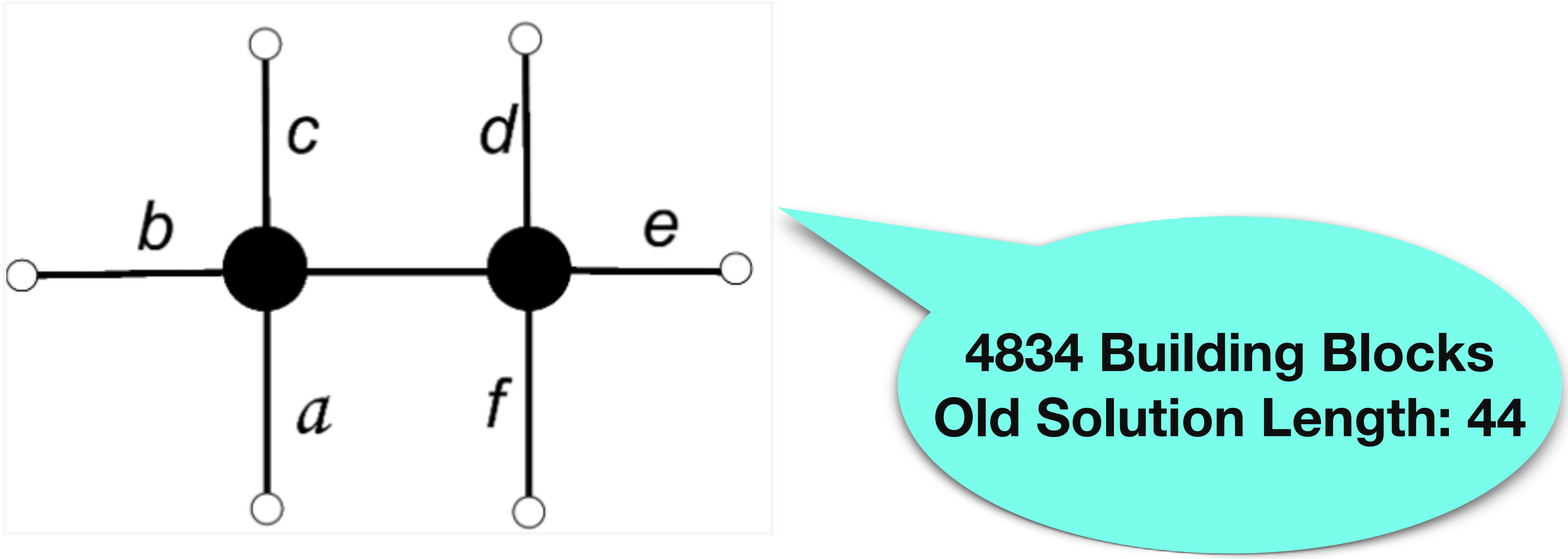
**Same solution as conic hull approach**

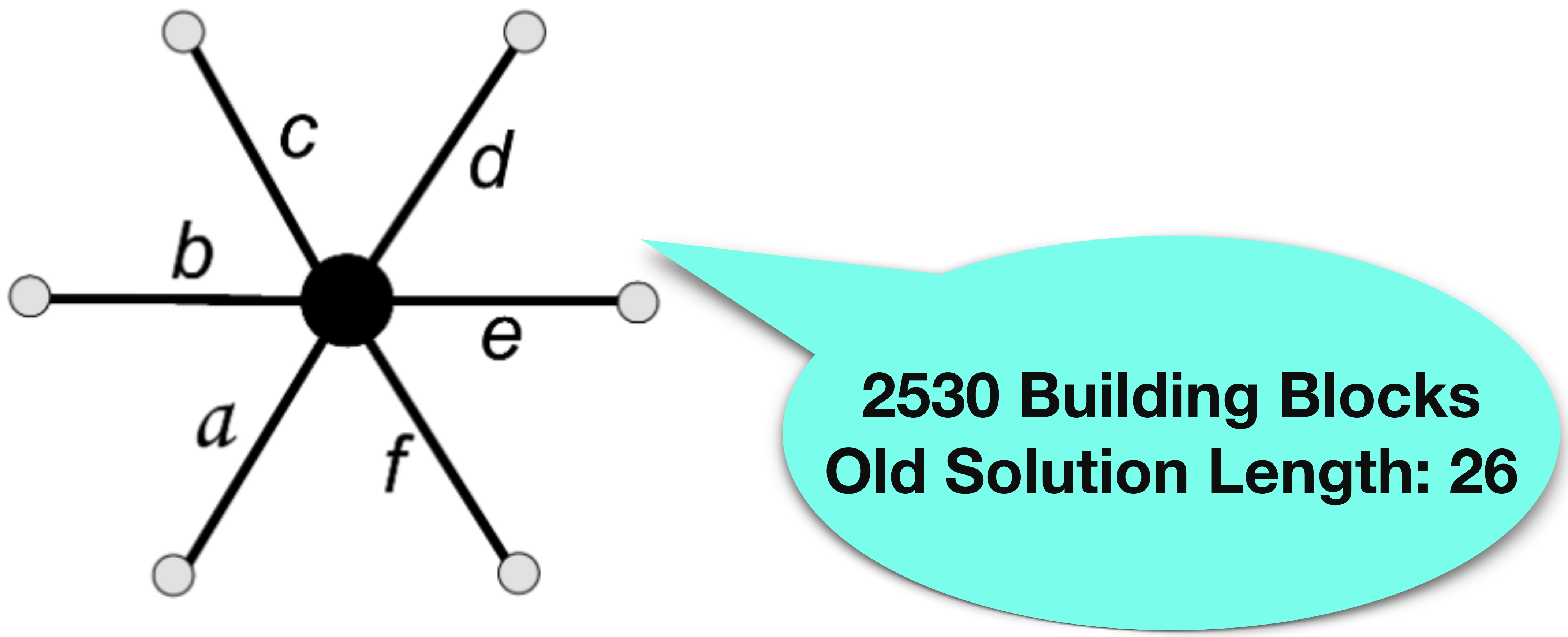
- Take complement of $\{1,2,\cdots,5\}$ with each simplex in the triangulation

# Double Box and Hexagon

## New Solutions

- Recomputed conformal off-shell double box and hexagon
- 9-fold MB representation




- Simpler solution of length 25 found using triangulation approach

# Multiple Polylogarithms

## Analytic Continutions

- MPLs have a MB representation

$$\text{Li}_m(x) = x \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \Gamma(-z_1)\Gamma(1+z_1) \frac{\Gamma^m(1+z_1)}{\Gamma^m(2+z_1)} (-x)^{z_1}$$

- General Expressions:

[arXiv: 1311.1425]

$$\text{Li}_{m_1,\ldots,m_N}(x_1,\cdots,x_N) = \left(x_1 x_2^2 \cdots x_N^N\right) \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \cdots \int_{-i\infty}^{+i\infty} \frac{dz_N}{2\pi i} \Gamma(-z_1)\cdots\Gamma(-z_N)\Gamma(1+z_1)\cdots\Gamma(1+z_N)$$
$$\times \frac{\Gamma^{m_1}(1+z_1)\Gamma^{m_2}(2+z_{12})}{\Gamma^{m_1}(2+z_1)\Gamma^{m_2}(3+z_{12})} \cdots \frac{\Gamma^{m_N}(N+z_{1\ldots N})}{\Gamma^{m_N}(N+1+z_{1\ldots N})} (-x_1 x_2 \cdots x_N)^{z_1} (-x_2 \cdots x_N)^{z_2} (-x_N)^{z_N}$$

- Degenerate MB and no white zones

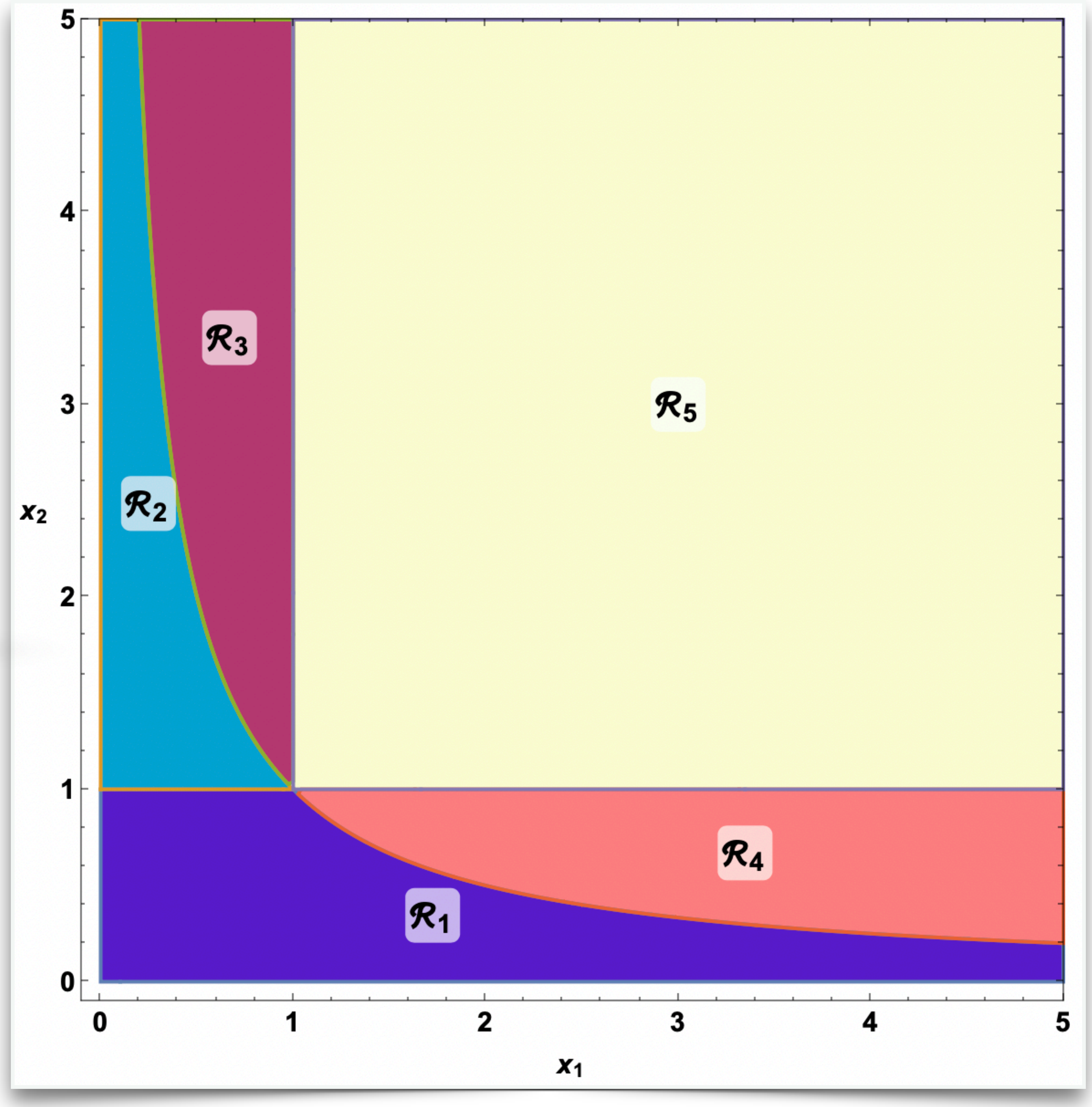
# Multiple Polylogarithms

## Analytic Continutions

- Example: $\text{Li}_{1,1}(x_1, x_2)$

$$\text{Li}_{1,1}(x_1,x_2) = x_1 x_2^2 \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{dz_3}{2\pi i} \Gamma(-z_1)\Gamma(-z_2)\Gamma(1+z_1)\Gamma(1+z_2) \frac{\Gamma(1+z_1)\Gamma(2+z_{12})}{\Gamma(2+z_1)\Gamma(3+z_{12})} (-x_1 x_2)^{z_1} (-x_2)^{z_2}$$

- 11 conic hulls associated; 6 points for triangulations
- 5 series solutions with no white zones



# Conclusion & Summary

- MB integrals can be solved using conic hulls and triangulations
- MBConicHulls.wl for automated evaluation
- Simpler solution of conformal double box and hexagon diagram
- MPLs special class of MB with no white zones
- Boundary conditions for Heavy-to-Light Form Factor. [arXiv: 2308.12169]
- Reducing fold of MB integrals a key challenge

**Thank you for your attention!**

# Backup

# Analytic Evaluation

## Triangulation Approach

- Find All Possible N-Combinations of Numerator Gamma functions

- Associate Series (Building Block) with each N-Combination

- Associate Point Configuration

- Find all Possible Regular Triangulations

- Final Solution = Sum of Building Blocks associated with each Triangulation

Triangles in each triangulation mapped to building blocks

# of solutions = # of triangulations

# Analytic Evaluation

## Conic Hull Approach

- Find All Possible N-Combinations of Numerator Gamma functions
- Associate Series (Building Block) with each N-Combination
- Associate Conic Hull with each N-Combination
- Find Largest Subsets of Intersecting Conic Hulls
- Final Solution = Sum of Building Blocks associated with each Largest Subset

Intersecting Region (Master Conic Hull)

Several Solutions possible for a given MB

# Analytic Evaluation

## Speed Comparison

- MBConicHulls.wl for automated evaluation

TOPCOM interfaced with Mathematica

- Triangulation approach much faster than the conic hull approach

| Feynman integral | MB folds | Total solution number | Conic hulls method | | Triangulation method | |
|---|---|---|---|---|---|---|
| | | | One solution | All solutions | One solution | All solutions |
| Conformal triangle | 3 | 14 | 0.186 sec. | 1.44 sec. | 0.543 sec. | 0.483 sec. |
| Massless pentagon | 5 | 70 | 1.276 sec. | 1.25 h. | 0.318 sec. | 2.78 sec. |
| Conformal hexagon | 9 | 194160 | 1 min. | - | 0.489 sec. | 40 min. |
| Conformal double-box | 9 | 243186 | 1.9 min. | - | 0.635 sec. | 1.8 h. |
| Hard diagram | 8 | 1471926 | 6 min. | - | 1.4 sec. | - |

# Double Box and Hexagon

## Numerical Comparison

| Numerical Comparison for Hexagon | | |
|---|---|---|
| Upper Sum Limit | Series Representation (Time) | Feynman Parametrization (Time) |
| 2 | 636.76884 (14 sec) | 636.76882 (9 Hours) |

Table: Computed for $u_1 = 1, u_2 = 10^{12}, u_3 = 1/10^{12}, u_4 = 1, u_5 = 1, u_6 = 100, u_7 = 1/100, u_8 = 10000, u_9 = 1/10^8$ for propagator powers $a = 42/100, b = 11/100, c = 15/100, d = 32/100, e = 59/100, f = 55/100$

# Multiple Polylogarithms

## Analytic Continutions

- Numerical comparison with GiNaC for $\mathrm{Li}_{1,1,1}(x_1, x_2, x_3)$ [Link]

```
In[87]:= Sub = {x1 → 0.03, x2 → 0.02, x3 → 10};
Lim = 80;
SumAllSeries[PolyLogWMBSeriesOut, Sub, Lim, RunInParallel → True];

Numerical Result: -0.000409347 - 0.0000191085 i

Time Taken 2.63637 seconds

In[90]:= Sub = {x1 → 0.03, x2 → 0.02, x3 → 10};
SubVal = XVars /. Sub;
Ginsh[Li[WeightList, SubVal], {}]

Out[92]= -0.00040934678637866050813551964754949320020 - 0.000019108543116203301158737782799149320410 i
```