Phase Space Past and Future

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Phase space

Phase space is an old subject. It played a fundamental role in the evolution of physics. While almost taken for granted today, it is relevant not only to the past, but also relevant to the future.

Enlarging the 3-D *(x, y, z)* dynamics in real space to 6-D (x, p_x, y, p_y, z, p_z) phase space, physics was significantly enriched and deepened.

$$\vec{f} = m\ddot{\vec{r}}$$

Newton was not aware of phase space

Written this way misses a lot of physics.

➔ ➔

$$\dot{\vec{r}} = \vec{p}$$
$$\dot{\vec{p}} = \vec{f}$$

A seemingly trivial step of factorization

(Hint: $\vec{p}$ has its own strong physical meaning, i.e. Nature performs Newton equation in two steps, not one step.)

➔ ➔

$$H(\vec{r}, \vec{p}, t)$$
$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}}$$
$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}}$$

The ground-shaking concept of phase space and Hamiltonian

Phase space concept guided the developments from classical to modern physics, including the fields of dynamics and electrodynamics, and in particular the accelerator physics.

But one must ask: The artificial extension from 3-D to 6-D must introduce much subtleties.

- Phase space must have some hidden internal structure after the artificial extension from 3-D to 6-D. Unlike the 3-D physical space, the 6-D phase space must have internal constraints.
- Most significant is the Liouville theorem.
- And, as we will explain, there is a necessity of chaos and chaotic layers. Chaos is not a mere nuisance!

Phase space evolution will continue to impact on accelerator physics:
the future of accelerator physics research = the deepening of phase space studies and manipulations.

Constraints in phase space

Intuitively we must expect some hidden constraints in phase space when we artificially extend from 3-D to 6-D. Up to half of its degrees of freedom must be "empty". But how to quantify this "emptiness"?

The answer is given by the condition of symplecticity:

$$\tilde{M} S M = S, \quad \text{where} \quad S = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$\leftarrow$ must be valid over the entire phase space,

M is the 6x6 matrix map if linear dynamics; Jacobi map if nonlinear dynamics.

It is truly amazing that this "degree of emptiness" can be written in such an elegant, self-contained, if-and-only-if condition like the symplecticity condition. Nature must have intended the phase space and the Hamiltonian and the symplecticity in her CDR. Even when we exclusively use Newton equation and avoid phase space and Hamiltonian, they still control the dynamics behind the scene.

Why do we take symplecticity condition so seriously?

A quick answer might read: It is needed in order to do long term dynamic aperture studies. A slightly nonsymplectic map will lead to erroneous conclusions on long term stability of the system in particle tracking simulation studies.

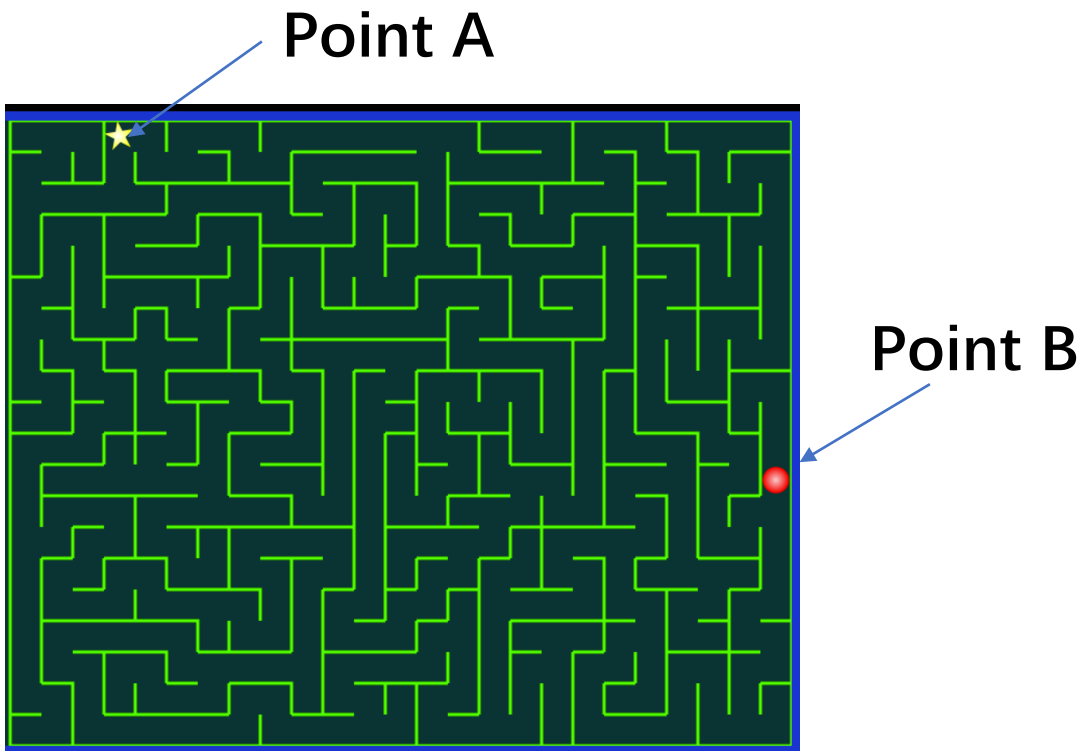
While being true, this statement trivializes the true meaning of the symplecticity condition. Satisfying symplecticity condition means the system obeys the underlying dynamical principles, i.e. “there exists a Hamiltonian of the system, even if we don’t explicitly use it.” Satisfying symplecticity assures the integrity of the system, so its subsequent analysis is trustworthy. Failing symplecticity, it may imply that the conclusions, whether analytical or numerical, cannot be trusted.

Clarification: The "emptiness" mentioned above does not mean there are holes in the 6-D phase space (like Swiss cheese). All points, A or B, in the phase space can still be occupied by particles (accelerator physics). "Emptiness" means not all paths from A to B are allowed. What is empty is the degrees of freedom, not the phase space itself.

For a linear 6-D particle motion, there are 36 degrees of freedom to move from A to B. But only 21 of them are allowed. The degree of emptiness of degrees of freedom is 21/36.

The situation is like a 6-D maze.

- A particle can be allowed at any state, e.g. A or B;
- But not all paths from A to B are allowed;
- Only symplectic paths are allowed;
- If we intend to move a volume around A to another volume around B, then the two volumes must be equal.



The job of accelerator physicists is to manipulate a particle (a point in phase space) or a beam (a phase space volume) from a given state (point A) to another state (point B). Once A and B are given, the job is to find the path from A to B, but the path must be symplectic. The job of accelerator physicists is to solve the phase space maze game.

Actually it is an area around A to an area around B, because we are speaking of a "beam".

In the past, a typical maze game would be 2-D --- the job is called "lattice matching". In the future, the required phase space manipulations become increasingly challenging. The job can be 4-D or even 6-D. The future of accelerator physics will need to find more demanding tricks

Thanks to the Liouville theorem, these increasingly intricate phase space tricks of the future will stay preserved during the process pf manipulation.

Liouville theorem

One of the consequences of symplecticity is Liouville theorem.

Symplecticity ➔ Liouville (but not the reverse).

Phase space volume is conserved --- and conserved strictly and down to finest details without compromise. It follows several important consequences, e.g.

- All manipulations to the beam's phase space distribution --- even if micro-scaled and intricate --- will be safely and faithfully remembered without smearing.

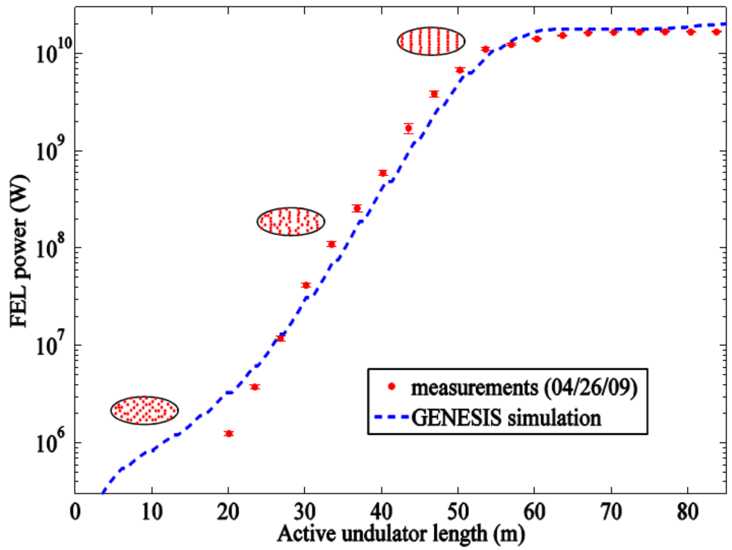
- Single particle simulations are meaningful. The validity of computer tracking to examine the single-particle nonlinear dynamics, such as long-term dynamic aperture, is based on the validity of Liouville theorem. The fact that we use point-particles in the simulations and the corresponding phase space manipulations means we trust these simulations down to details as fine as a point in the phase space. This trust is based on Liouville theorem.

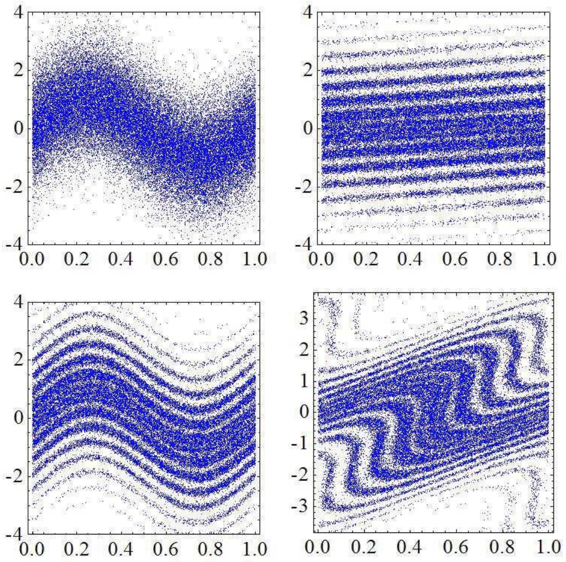
- The combination of phase space + Liouville theorem is the basis of a large number of accelerator applications:

  Courant-Snyder dynamics
  emittance preservation
  RF gymnastics
  phase space displacement acceleration
  KAM theorem
  Chaos
  6-D phase space gymnastics
  emittance exchange
  echoes
  free electron lasers + various FEL manipulations and seeding all work confidently
  various harmonic generation schemes
  steady state microbunching

This is an awesome list. We have come a very long way with single-particle dynamics, and the field is expanding.

- It is meaningful to microbunch a beam. Microbunching property is surprisingly robust. FEL is one example.





- Microbunching can be hidden in 6-D phase space and preserved for a long time, e.g. in various echo effects. EEHG is one example.

- Quantum mechanics puts a limit of how detailed may we trust the phase space. But phase space element volumes typically in accelerator applications >> h^3, far from this quantum limit.

Phase space gymnastics

Early well known applications of phase space gymnastics included the Courant-Snyder formalism in the *x* and *y* phase space, and the RF bucket dynamics in the *z* phase space.

Traditionally the phase space gymnastics mostly refer to longitudinal phase space. The transverse and longitudinal phase spaces are adiabatically decoupled, assuming slow synchrotron oscillation.

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This rich area accumulated much accelerator physics expertise in the past.

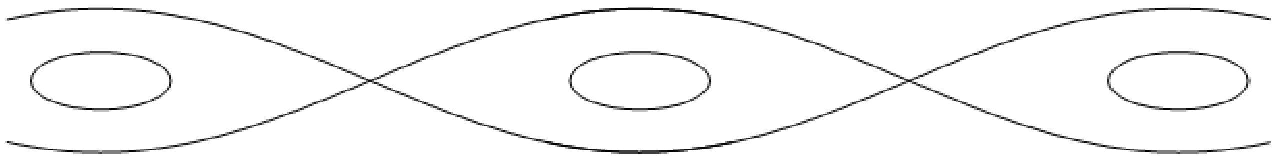
- Bunch compression, rotation, stretching
- injection, adiabatic trapping in RF buckets
- nonlinear alpha buckets
- staggered buckets
- phase space displacement acceleration

Example 1. Adiabatic trapping

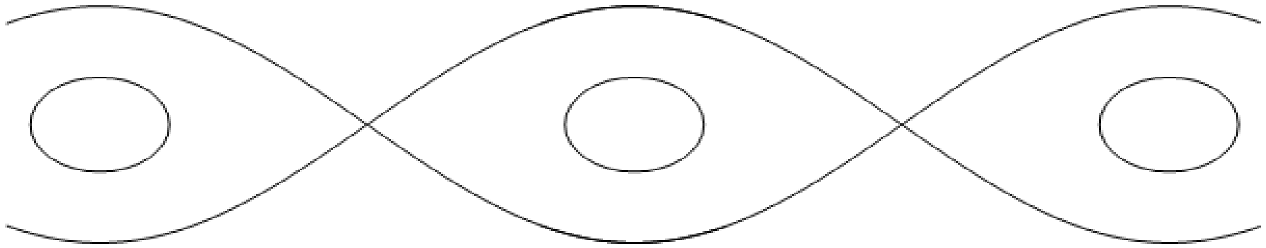
Consider the case when the RF voltage is slowly increased ➔ the bucket area slightly increases.

- Changing bucket area, no matter how slowly/adiabatically, immediately threatens Liouville theorem.
- Due to Liouville, the increased bucket area must come from somewhere else. But the bucket boundary prevents any trespassing.
- This is an internal self-inconsistency problem that Nature has to resolve. In order to conform to the Liouville theorem, something that Nature seems to want to insist, nonlinear dynamics invents a drastic solution, i.e. the chaotic layers!

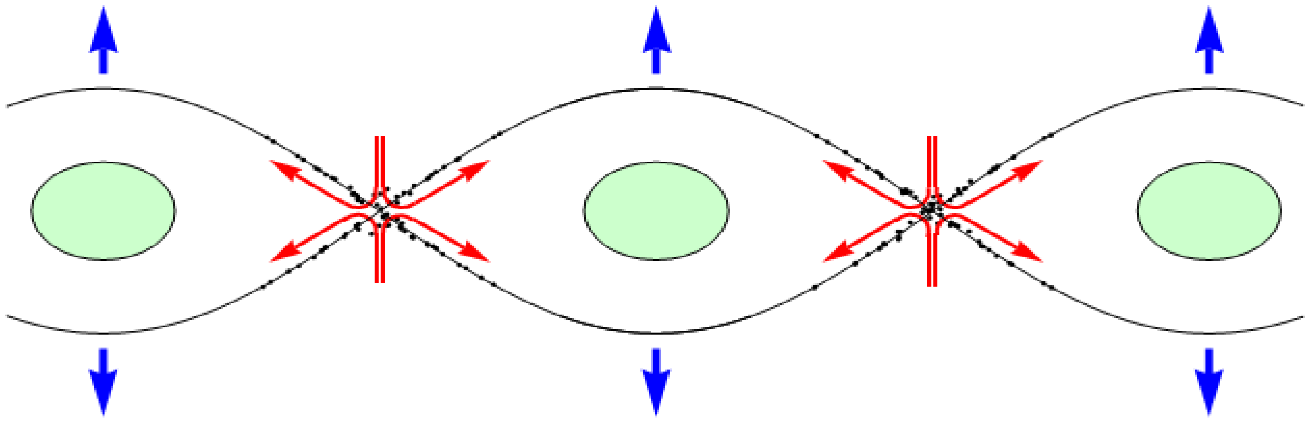
Adiabatic approximation (lower voltage)



Adiabatic approximation (higher voltage)



Without adiabatic approximation



- The solution chosen by Nature: Linear dynamics can be kept intact but compromise is unavoidable with nonlinear dynamics.

- The green regions, being mostly linear, are intact from the enlarging buckets. The fresh phase space will invade only the outer region of the bucket. The bucket core region is not being diluted, allowing RF trapping during injection.

- The way Nature does it is to allow phase space to negotiate through the otherwise forbidden crossing of phase space boundaries with an intricate system of chaotic layers around the unstable fixed points.

- The intricacy is easily seen: (a) The opening of the chaotic layer must exactly match the amount needed to increase the bucket area, which in turn depends on the speed the RF voltage is being increased; (b) No particles are allowed to occupy the same phase space position at the same time. All particles must take turns and negotiate the timings of their crossings extremely orderly in spite of the appearance of chaos.

- Chaos is not just an annoyance. It is a necessary tool of Nature. It is amazing that Nature chooses to invoke such an elaborate mechanism just to save symplecticity.

Example 2. Phase space displacement acceleration

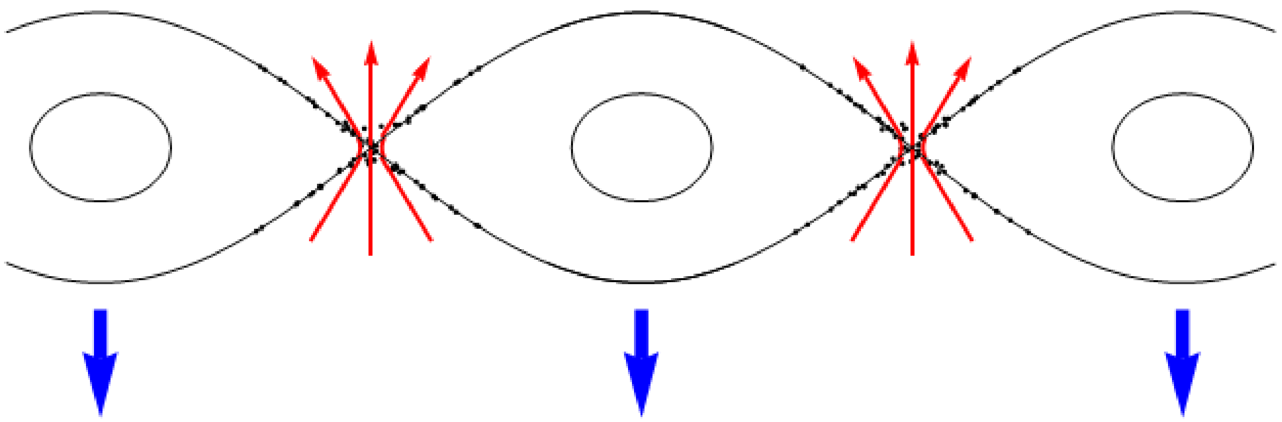
This is arguably the most striking demonstration of Liouville theorem and existence of chaos.
It is a tremendous accomplishment in accelerator physics.

In Example 1, the bucket is enlarged. Chaotic layers must appear so as to maintain the Liouville theorem.

In this Example 2, the buckets keep a fixed size, but they are made to move, e.g. to move downward as shown. Phase space is displaced, again by permeating through the unstable fixed points.

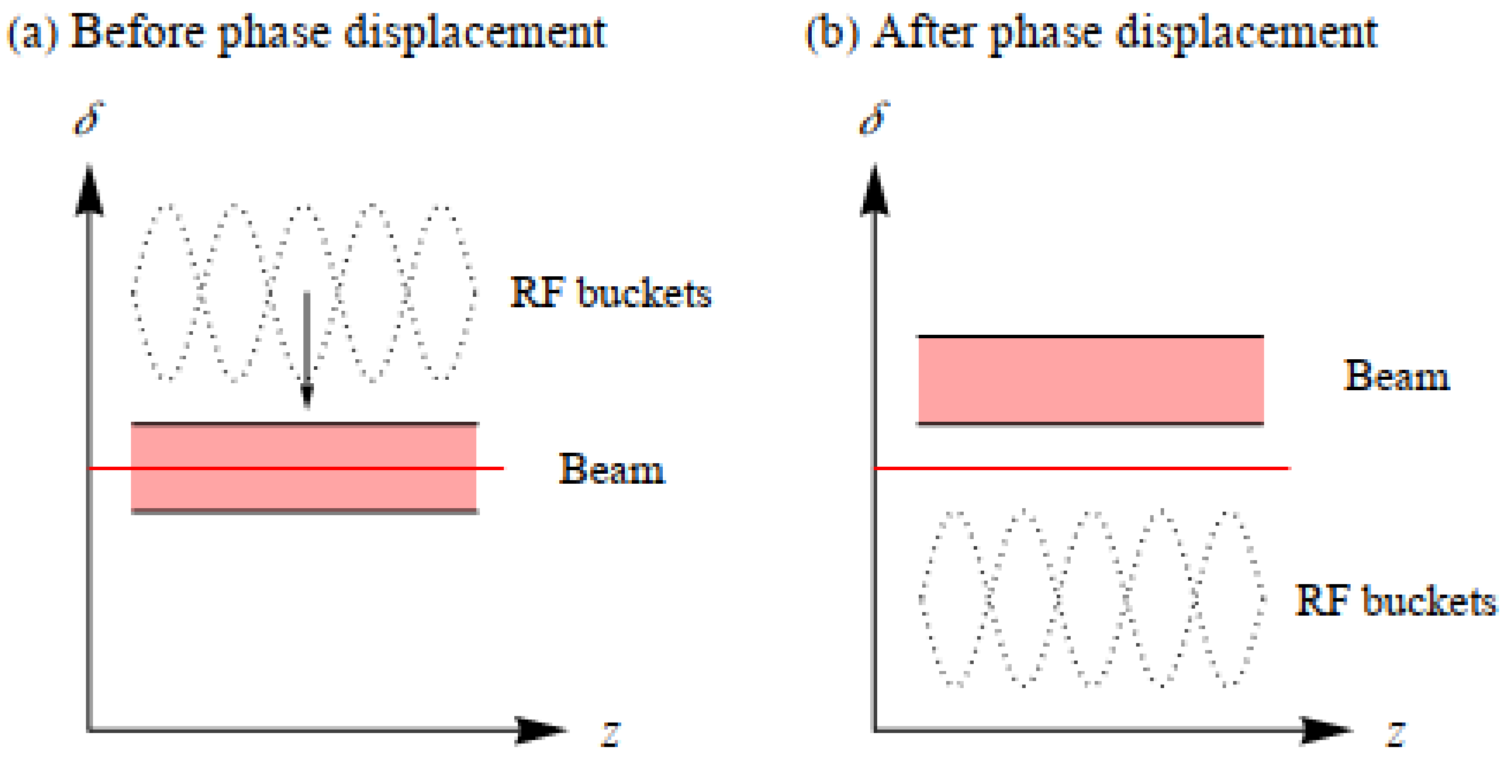
Phase space displacement

Move the buckets downward (time dependent)
➔
Chaos layers appear around the buckets.



- A string of RF buckets is formed above the beam and moved downward.
- Beam particles navigate through the buckets via the chaotic layers near the unstable fixed points.
- The beam gets accelerated by repeating the procedure. Acceleration is slow.
- All known acceleration mechanisms require synchronized electric fields. But this mechanism accelerates by chaotic fields in the chaotic layers. It is not even clear where the acceleration electric fields came from.

This mechanism can be used to accelerated beams.
[K.N. Henrichsen, M.J. de Jonge, CERN-ISR-RF-MA/74-21 (1974).]



Future light sources

So far, the phase space gymnastics focuses on either purely transverse (diffraction limited rings) or purely longitudinal (RF gymnastics), as mentioned above.

Future light sources, however, are likely to invoke extensions:

- The current efforts focus on transverse diffraction limited storage rings. Its target is to reduce the transverse emittances. Once emittances are reduced, the goal is accomplished.

- The next attention is turning to diffraction limit in the longitudinal bunching factors at designated wavelengths, $B(\lambda)$. A much wider open field lies ahead beyond transverse diffraction limit.

- Very short bunch length (isochronous ring) is one way to proceed, but not the only way.

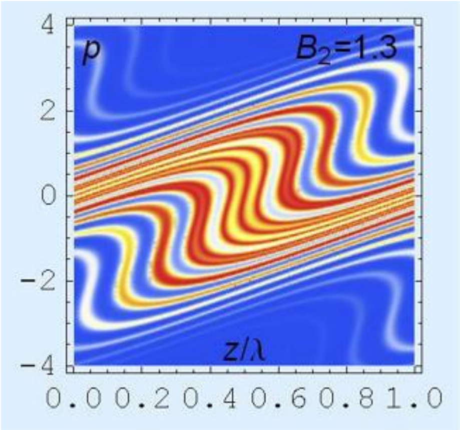
- FEL is a pioneering example of $B(\lambda)$ microbunching in longitudinal phase space. FEL points an important revolution in the advance of accelerator physics.

- Further beyond this stage, transverse-longitudinal coupled systems are still wide open field. One such example is the Steady State Microbunching (SSMB) storage ring.

More on phase space gymnastics

A modern round of phase space gymnastics started with Derbenev’s immersed gun and “phase space adaptor” ideas.
[Y. Derbenev, Michigan Univ. report 91-2 (1991), 93-20 (1993), 98-04 (1998)]

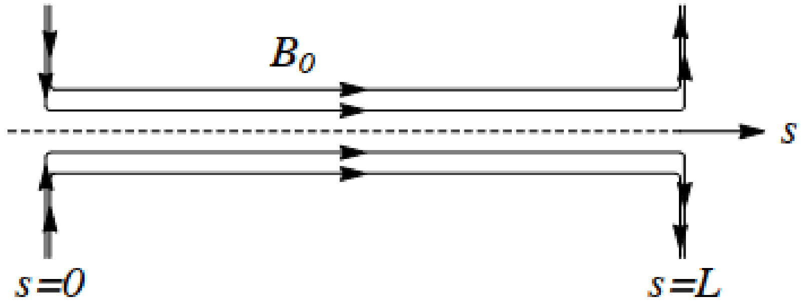
We start to enter the 6-D maze game. Phase space becomes the platform of 6-D gymnastics. All gymnastics are highly skillful and exquisitely beautiful.



Example: Immersed gun in a solenoid

- The motion is symplectic in the (x, p_x, y, p_y) phase space.
- But it is not symplecticmin the (x, x', y, y') modified phase space inside the solenoid.
- Physically, this can be attributed to end fields of a solenoid.
- Opened up the clever idea of using the nonsymplecic end fields to do x-y phase space manipulation.

$\nabla \cdot \vec{B} = 0$ fields cannot stop in midair. ➔ End fields must be taken into account.

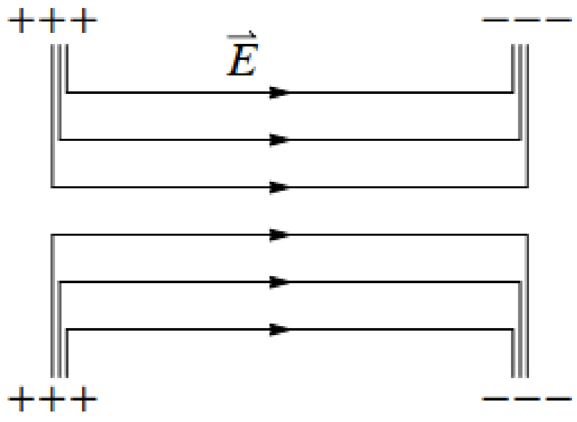


However, electrons emitting from an electron gun are produced in the (x, x', y, y') space, not in the (x, p_x, y, p_y) phase space! This phase space mismatch can be used in a clever idea of immersed gun.

- Install the electron gun inside the solenoid.
- The nonsymplectic map bringing the electrons out of the solenoid's end fields can produce exchange of (x,y) apparent emittances, yielding a very flat beam even when the electron gun is cylindrical.
- This concept triggered a new wave of accelerator physics research.

Another immersed gun variant is Electrostatic accelerator, or Wien filter. It can be used to do *x-z* or *y-z* manipulations.

$$\nabla \cdot \vec{E} = 0$$



Hard edge model

Using electric field, however, means it works only for nonrelativistic beams.

There are four types of end fields. Each of the 4 Maxwell equations demands one type of end fields. Maybe there can be 4 types of immersed guns.

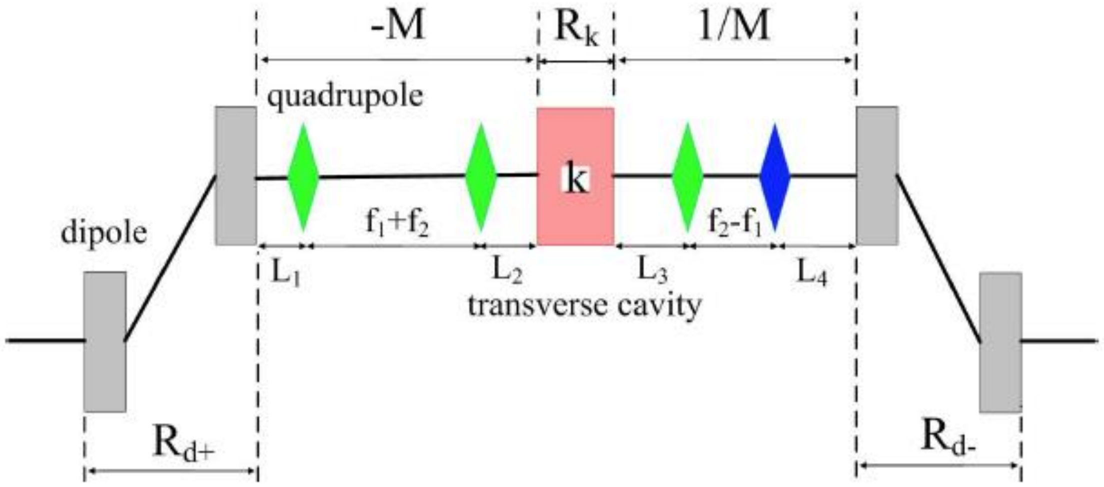
Device	Condition	End field
Dipole	$\nabla \times \vec{B} = \vec{0}$	$B_z \propto y$
Solenoid	$\nabla \cdot \vec{B} = 0$	$B_r \propto r$
Accel. sec.	$\nabla \cdot \vec{E} = 0$	$E_r \propto r$
- Wien filter - electric bend	$\nabla \times \vec{E} = \vec{0}$	$E_z \propto x$

Emittance exchange (EEX)

More advanced ideas invoke 6-D phase space by negotiating the symplectic maze from point A to point B.

Started with Cornacchia-Emma's Emittance exchange concepts.
[M. Cornacchia, P. Emma, Phys. Rev. Special Topics -- Accel. & Beams, 5, 084001 (2002)]

An example that combines three gadgets (an EEX, a $-I$ map, and a telescope) is shown below. The telescope reduces the strength of the transverse cavity by factor *M*.

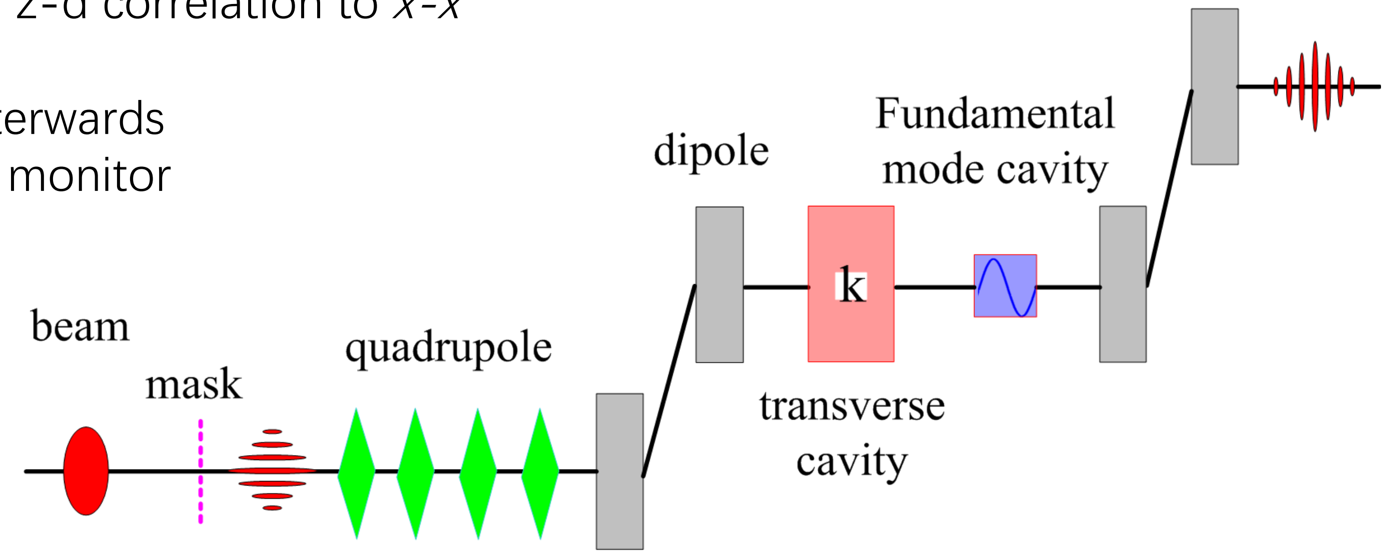


Many potential applications

[D. Xiang, Phys. Rev. Special Topics Accel. & Beams, 14, 114001 (2011)]

- When $\varepsilon_z << \varepsilon_x$, EEX allows small ε_x for FEL
- When $\varepsilon_z >> \varepsilon_x$, EEX allows bunch compression
- Observing *z*-distribution by an *x*-profile monitor
- Tailoring *z*-distribution by an *x*-scraper
- Measuring slice energy spread by an *x*-profile monitor
- Cleaning the *z*- and δ-tails by *x*-scraper
- Observing beam microbunching in *z* by an *x*-profile monitor
- Generate *z*-microbunching by modulating the *x*-profile of a beam
- Generating *z*-double bunches by *x*-wire scraper
- Longitudinal phase space linearizer by a sextupole
- Study CSR effect by converting CSR-induced z-d correlation to *x*-*x'* correlation
- Suppressing CSR by $\varepsilon_z > \varepsilon_x$ and exchange afterwards
- Observing curvature of *z(y)* by an *x*-*y* profile monitor
- Bunch compression without energy chirp
- etc.

x-z EEX experiment
x mask with 0.8 mm slits
➔ beam with 0.8 mm microbunches in *z*,
[Y.-E Sun, et al., Phys. Rev. Lett. 105, 234801 (2010).]



Not included in the above are the further nonsymplectic extensions of EEX:

- emittance partitioning
- emittance thermal equilibrium
- emittance cooling
- crystalline beams

Also not included is the active area of advanced ideas in FEL physics:

- Ultrafast
- advanced seeding concepts
- Superradiant sources
- Combined with rings

Summary

- Phase space is the underlying foundation of all classical and modern physics.
- The evolution of phase space applications has guided accelerator physics in the past and will continue to do so in the future.
- The future of accelerator physics along the path of phase space is a rich and wide open field.