ETH and Random Matrix Universality

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Questions

- to what extent observables can be described by random matrices?
- how to quantify cross-correlations between matrix elements?

$$O_{ij} = O^{eth}(E)\delta_{ij} + e^{-S/2} f(E,\omega) R_{ij}$$

RMT and physics

- relevant scales and regimes?
- connection to thermalization dynamics?

Cross-correlations matter

Eigenstate Thermalization Hypothesis

$$O_{ij} = O^{eth}(E)\delta_{ij} + e^{-S/2} f(E,\omega) R_{ij}$$

- correlations matter to describe OTOC
  Foini, Kurchan '2018
  Chan, De Luca, Chalker '2018
- correlations matter to describe gravity
  Belin et al '2020-2023
  Jafferis et al '2022
- correlations matter to describe *thermalization dynamics*
  AD '2018
  Delacretaz '2020-2023

How to quantify cross-correlations?

- full (generalized) ETH, free probability theory

  Foini, Kurchan '2018, Belin et al '2020-...

  Pappalardi, Foini, Kurchan '2022 (talk by Laura)

- eigenstate correlations

  Hahn, Luitz, Chalker '2023

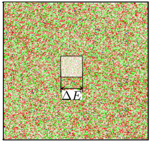
- collective properties of matrix elements probed by the spectrum of a submatrix O_{ij}

  AD, Liu '2017 AD '2018
  Richter et al '2020, Wang et al '2022-2023
  Pappalardi, Foini, Kurchan '2023, Iniguez, Srednicki '2023

Probing m. truncated operators through spectrum

- focus on spectral properties of microcanonically-truncated observables

$$O_{ij} = (POP)_{ij}, \; |E_i - E_j| \leq \Delta E$$



AD, Liu 1702.07722,
Richter et al 2007.15070

- maximal eigenvalue $\lambda(O_{ij}) = \lambda(\Delta E)$ bounds thermalization dynamics

$$\left| \frac{1}{T} \int_0^T \langle \psi | O(t) | \psi \rangle dt \right| \lesssim \lambda(2\pi/T)$$

AD 1806.04187
uncorrelated O_{ij} require $T \geq \tau\, L$
AD 1804.08626

Uncorrelated O_{ij} = hydrodynamics ends

- thermalization (Thouless) time $\tau = L^2/D$ marks saturation of the correlation functions

  classical diffusion in 1D

$$\langle O(t,x) O \rangle = \sum_n \frac{e^{2\pi i x/L - n^2 D t/L^2}}{L} = \begin{cases} \sim t^{-1/2}, & t \ll \tau, \\ (1+e^{-t/\tau})/L, & t \gtrsim \tau, \\ 1/L, & t \gg \tau, \end{cases}$$

- after thermalization time only the slowest mode survives

$$\langle \psi | O(t) | \psi \rangle \approx \sum_n c_n \, e^{-n^2 t/\tau} \sim e^{-t/\tau}, \ \text{for}\, t \gtrsim \tau$$

- hydrodynamics extends until quantum fluctuations

$$\langle \psi | O(t) | \psi \rangle \sim e^{-S/2}, \quad t \approx T \sim \tau \, S$$

AD 1804.08626, Delacretaz 2006.01139

Bound on ΔE_{GUE}

- (semi)classical dynamics

$$\langle\psi|O(t)|\psi\rangle \sim e^{-t/\tau}$$

- bound on dynamics + Gaussian RMT assumption

$$\frac{\tau}{T} = \left|\frac{1}{T}\int_0^T \langle\psi|O(t)|\psi\rangle dt\right| \lesssim \lambda(2\pi/T) = f(0)\sqrt{\frac{2\pi}{T}}$$

$$f^2(0) = \int_0^\infty dt < O(t)O >_c \sim \sqrt{\tau} \sim L$$

- bound on uncorrelated RMT scale $\Delta E_{GUE} = 2\pi/T$

$$T > \tau\, L$$

AD 1804.08626

Big picture

correlations of O_{ij} are important to describe thermalization dynamics

- uncorrelated RMT applies only *after* hydrodynamics ends and quantum fluctuations start

$$\Delta E_{GUE} \ll \Delta E_{Th}$$

- big question: how to describe semiclassical hydrodynamic behavior?

$$\langle\psi|O(t)|\psi\rangle \sim e^{-t/\tau}$$

  there is no problem to "absorb" any 2-pt function into the ETH function $f^2(E,\omega)$; real challange is to describe expectation of $O(t)$ in a typical non-trivial ψ (which accounts to multi-point functions)

RMT universality of matrix elements?

- rotational-invariant ETH

$$\mathcal{P}(O_{ij}) = \mathcal{P}((UOU^\dagger)_{ij})$$

Foini, Kurchan '19
effective energy scales?
requires exact diagonalization?

- probing unitary symmetry through moments

$$\mathcal{M}_n(\Delta E) = \frac{\mathrm{Tr}(POP)^n}{d}$$

Wang et al 2110.04085
Wang et al 2310.20264

Signature of Unitary Symmetry

- free cumulants Δ_k

$$\Delta_k = \mathcal{M}_k - \sum_{j=1}^{k-1} \Delta_j \sum_{a_1+\cdots+a_j=k-j} \mathcal{M}_{a_1} \ldots \mathcal{M}_{a_j}$$

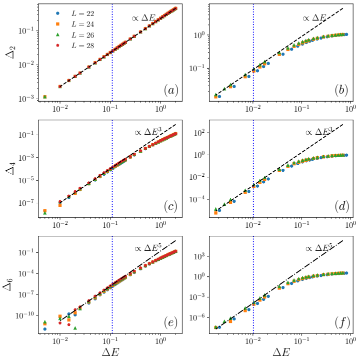
- unitary symmetry constraints Δ_k as a functions of ΔE

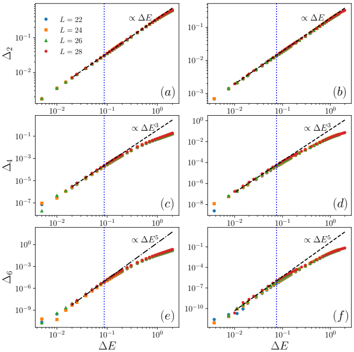
$$\Delta_k(\Delta E) \propto (\Delta E)^{k-1} \quad \text{for} \quad \Delta E \leq \Delta E_U^{(k)}$$

2310.20264

Numerical evidence

- density wave operators in non-integrable spin chains

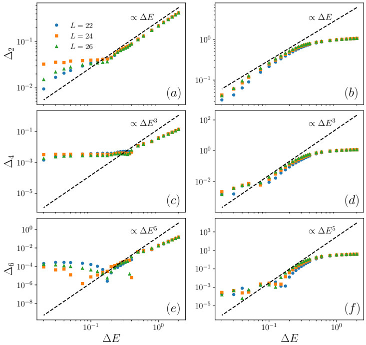




- evidence for $\ln \Delta_k = (k-1)\ln \Delta E + \text{const}$ for $\Delta E \leq \Delta E_U^{(k)}$

Integrable model – nor RMT

- density wave operators in integrable spin chain



- violation of $\ln \Delta_k = (k-1)\ln \Delta E + \text{const}$ behavior for small ΔE

Connection to full ETH

- $\Delta_k(\Delta E) \propto \Delta E^k$ = short frequency plateau of f_k

$$\overline{O_{i_1 i_2} \dots O_{i_k i_1}} = e^{-(k-1)S(\bar{E})} f_k(\vec{\omega})$$

  f_k – cumulant functions of full ETH

  Pappalardi, Foini, Kurchan '2022

- $\Delta E_U^{(k)}$ marks onset of "k-invariance," relation to late-time chaos?

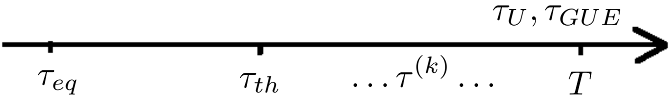
  Cotler, Hunter-Jones, Liub, Yoshida'2017
  Fava, Kurchan, Pappalardi'2023

Emerging picture

- Gaussian RMT $\Delta E_{GUE} \gtrsim 1/(\tau_{th}\, S) \sim 1/L^3$
- unitary RMT $\Delta E_U \gtrsim \Delta E_{GUE}$

  full RMT emerges when thermalization ends and quantum fluctuations start, $t = T = \tau_{th} S$

- Thouless/thermalization energy
  $\Delta E_{th} = \Delta E_U^{(2)} \sim 1/L^2 \ggg \Delta E_U \equiv \Delta E_U^{\infty}$
- a cascade of "k-design" scales from $\Delta E_U^{(2)}$ to ΔE_U?



Open questions

- understand the cascade of $\Delta E_U^{(k)}$
- connect the RMT picture with late time hydrodynamics
- "RMT" theory of ETH (free probability theory?)

- bound on ΔE_{RMT}, scales $\Delta E_U^{(k)}$ in SYK/JT and 2d CFT
  bound on ΔE_{RMT} from thermalization dynamics in holographic 2d CFTs