Random matrix universality and modular invariance

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Bernoulli Center, 30/09/2024

2301.05698: with C. Marteau, W. Reeves, M. Rozali
2309.00611, 2309.02533: with W. Reeves, M. Rozali
work in progress: with J. Boruch, G. Di Ubaldo, A. Etkin, E. Perlmutter, M. Rozali

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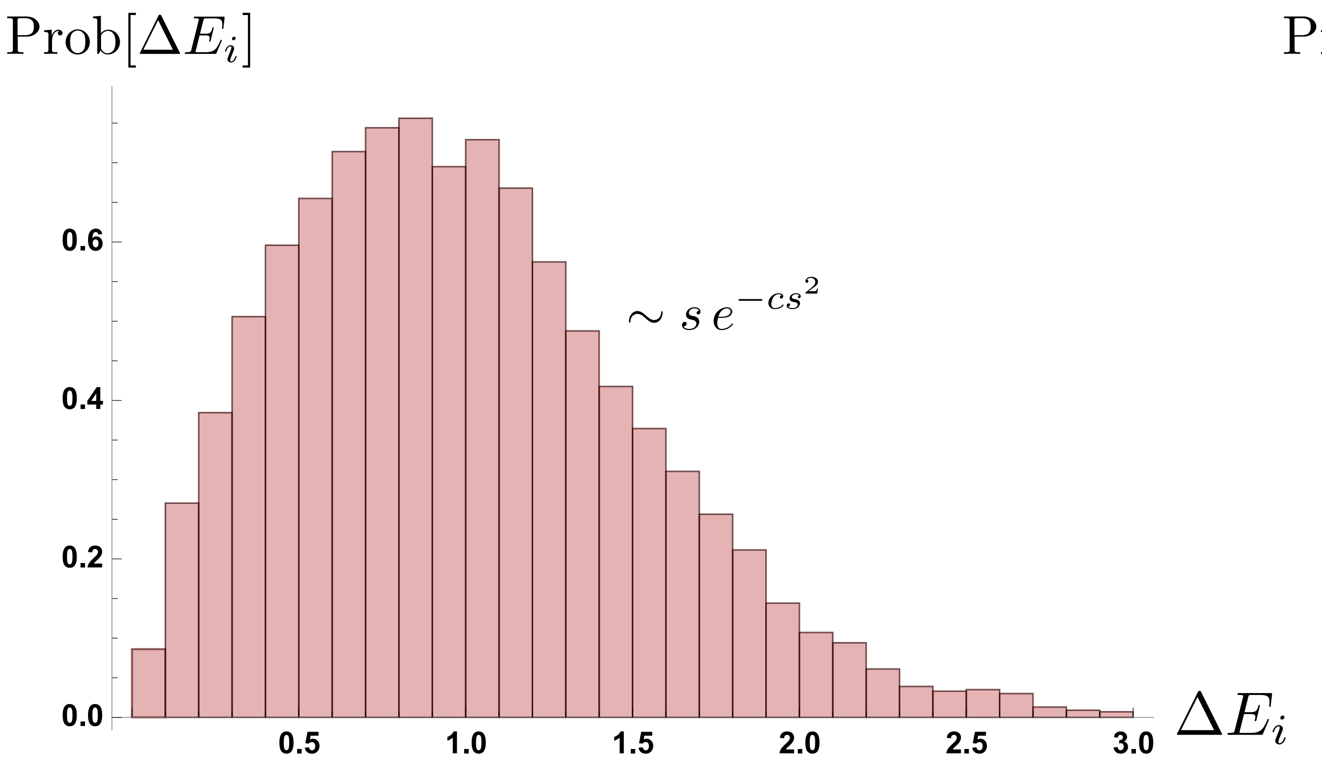
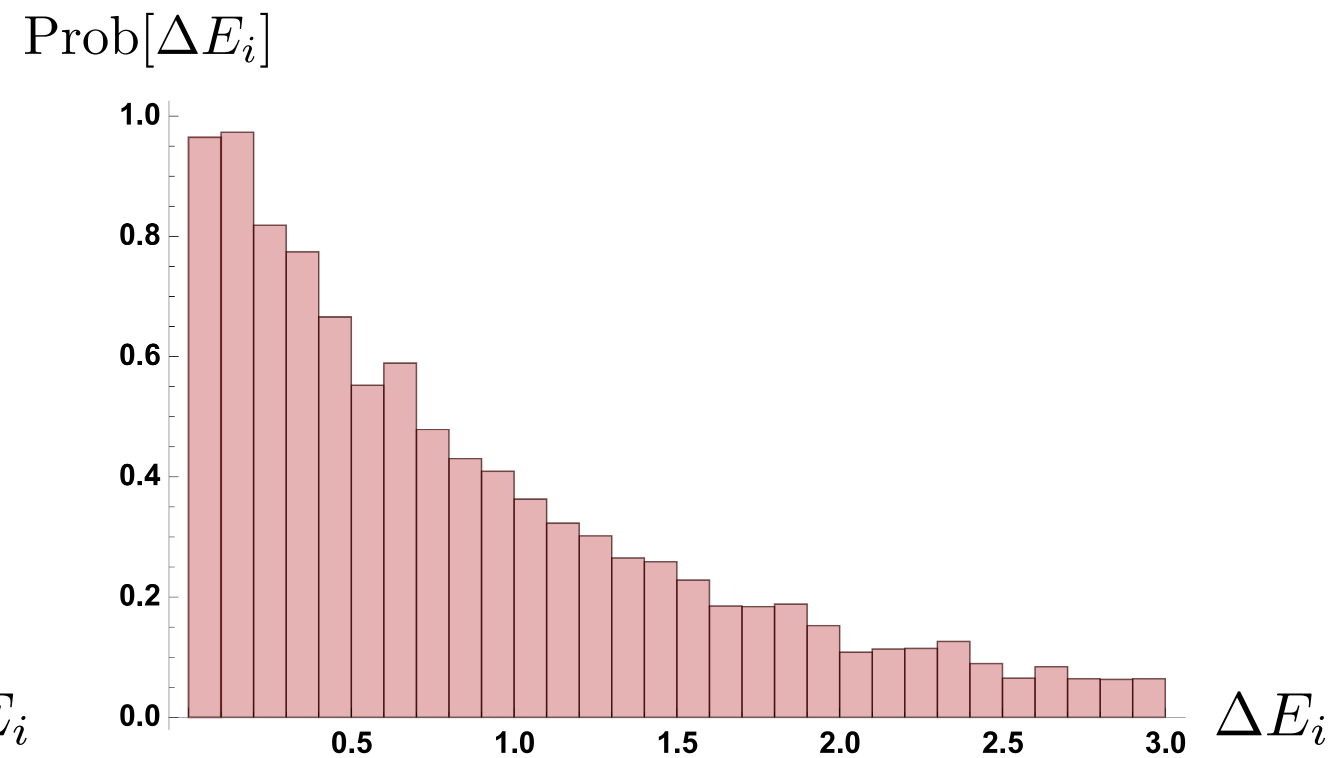
Signatures of chaos

Random matrix statistics





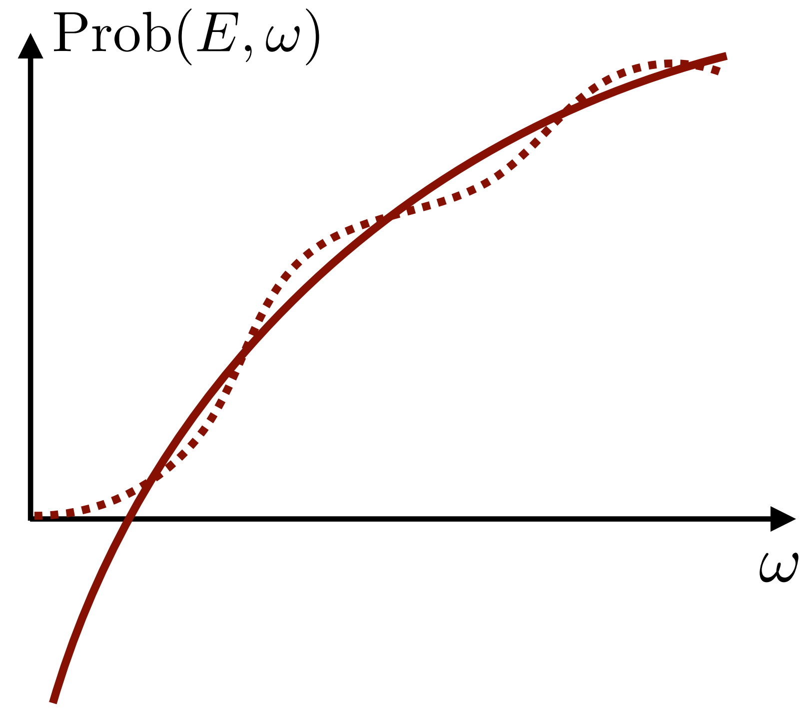
GOE matrix eigenvalues

Poisson random numbers

eigenvalue repulsion

eigenvalue attraction

- Stronger statement for $\rho(E) = \sum_i \delta(E - E_i)$:

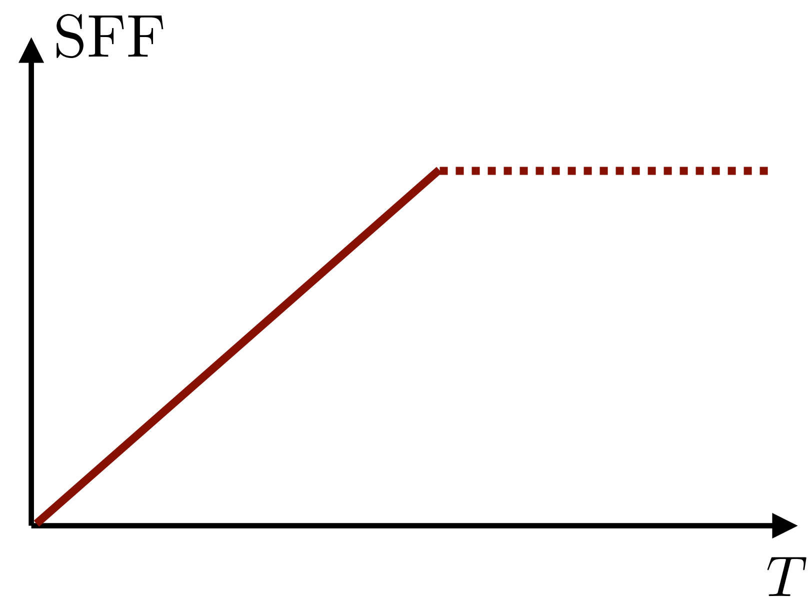


$$\langle \rho(E + \omega/2)\rho(E - \omega/2)\rangle_c \sim -\frac{1}{\pi^2\omega^2} + \dots$$

$$\mathrm{SFF}_\beta(T) \equiv |Z(\beta + iT)|^2 = \int dE d\omega \, \langle \rho(E + \omega/2)\rho(E - \omega/2)\rangle \, e^{-2\beta E} e^{-iT\omega}$$

$$\sim (\text{connected}) + \frac{T}{2\pi\beta} + \dots$$

"linear ramp"



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  - Covariance matrices of large samples [Wishart '28]

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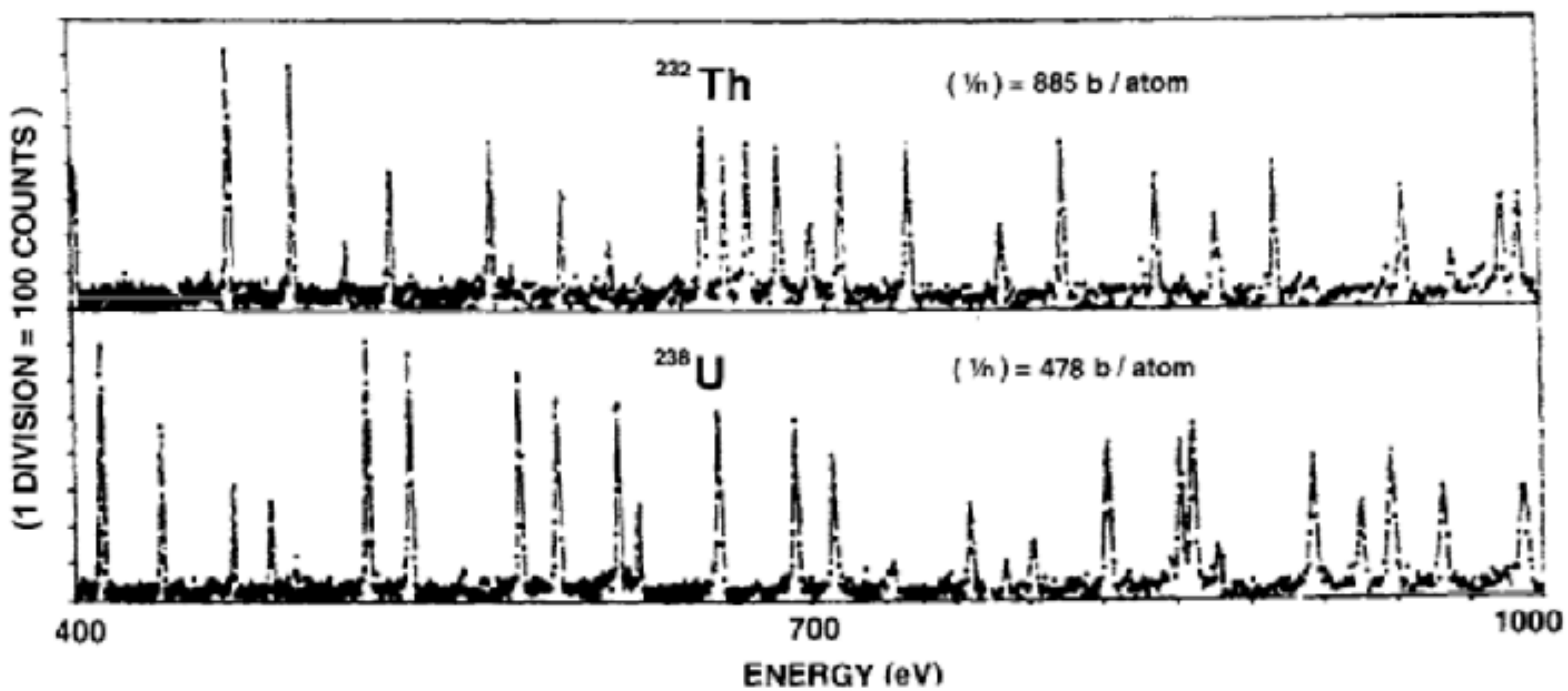

Figure 1.1. Slow neutron resonance cross-sections on thorium 232 and uranium 238 nuclei.

[Mehta "Random Matrices"]

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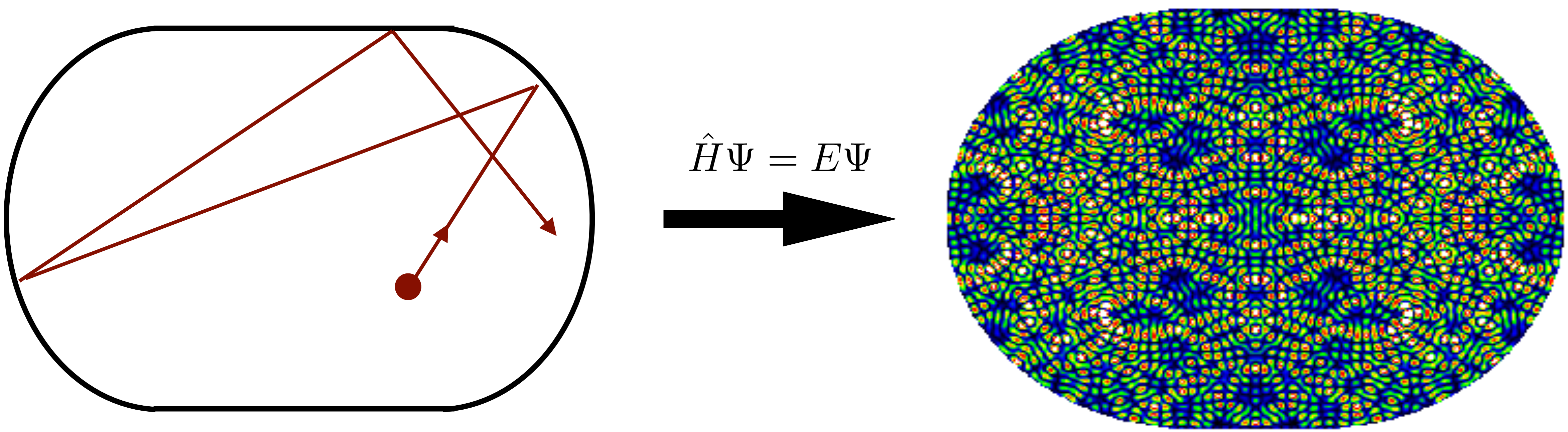
  - Quantum spectra with classically chaotic counterpart [BGS '84]



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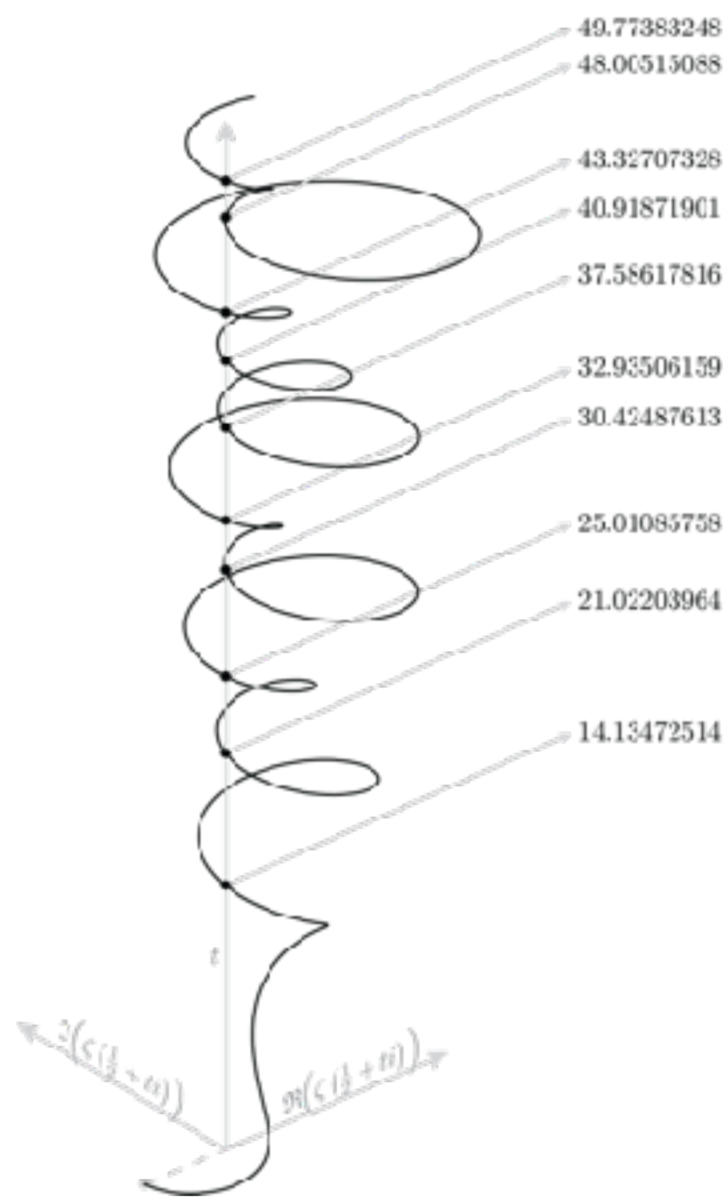
  - Distribution of zeros of Riemann zeta-function [Montgomery (Dyson) '73]



[curiosamathematica.tumblr.com]

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  - Many more….

- Universality: statistics of eigenvalues "locally" independent of exact probability distribution, i.e., only depends on few general properties of the ensemble. —> can be modelled by RMT

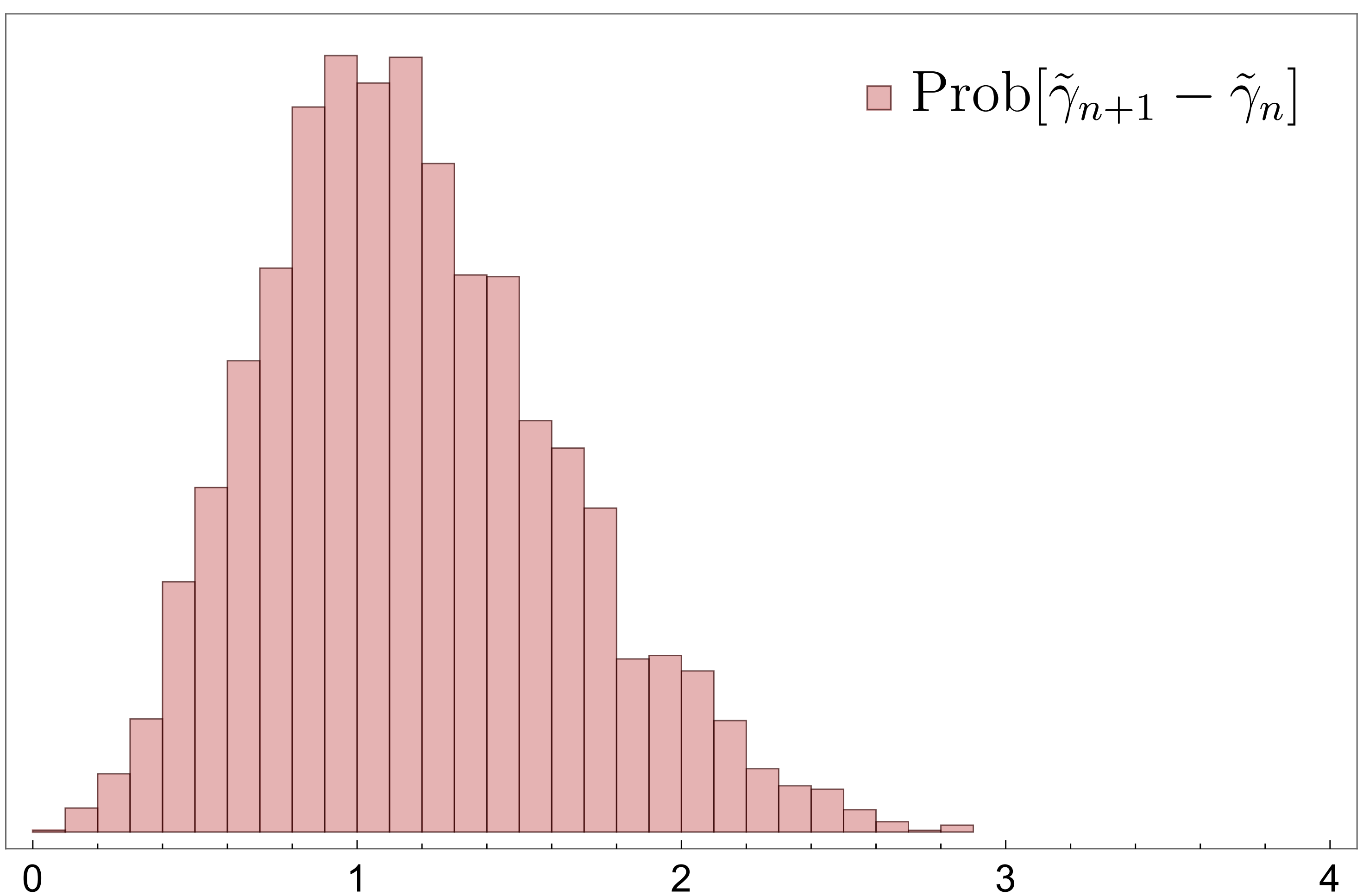
"What is here required is a new kind of statistical mechanics,
in which we renounce exact knowledge
not of the state of the system but of the system itself."

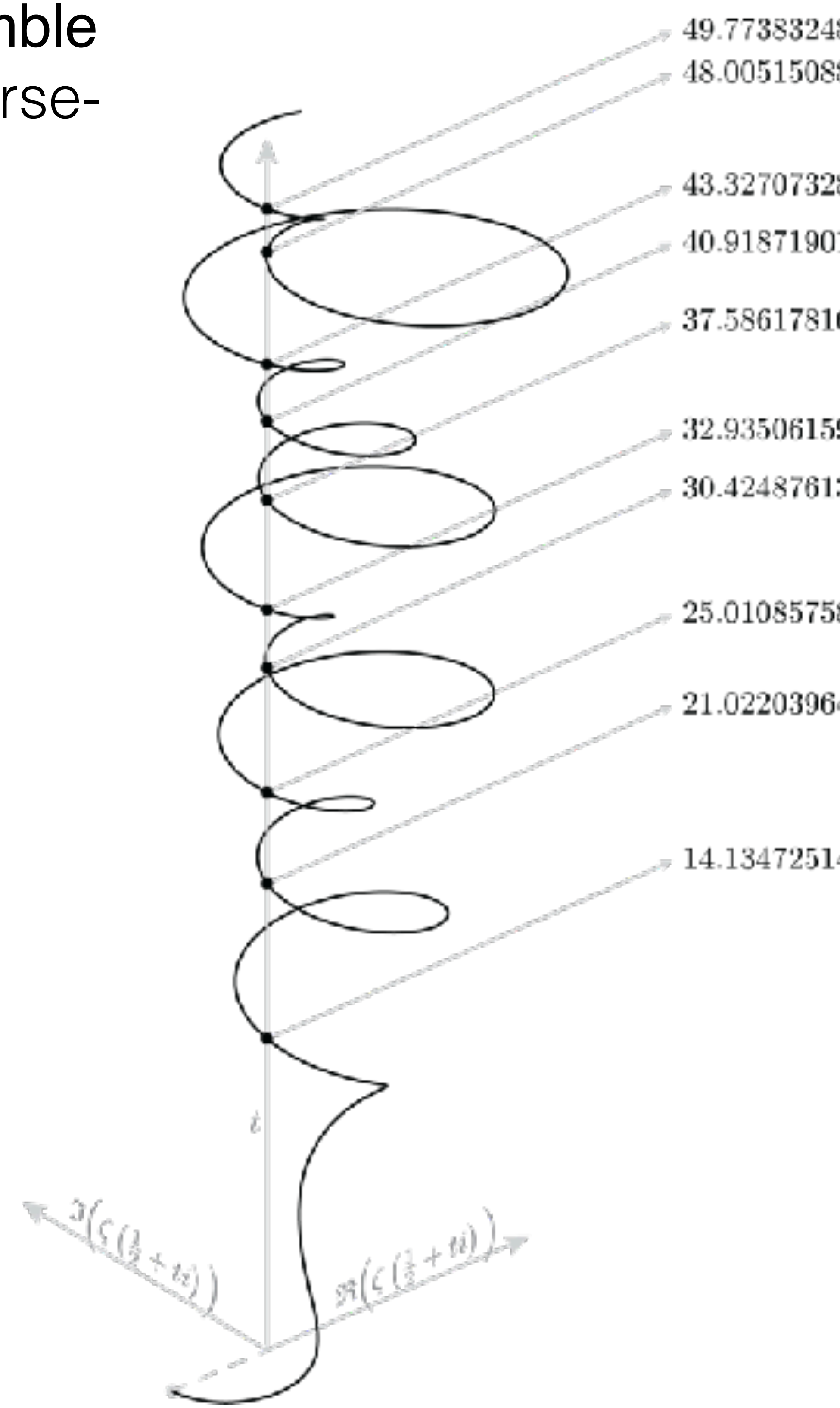
[Dyson]

Comment 1

- Random matrix universality can be seen in **ensemble averages,** but also in **individual systems** after coarse-graining
  - E.g.: distribution of zeros of the Riemann zeta-function

$$\zeta\left(\frac{1}{2} + i\gamma_n\right) = 0 \qquad \tilde{\gamma}_n \equiv \frac{\gamma_n}{\text{average spacing near } \gamma_n}$$





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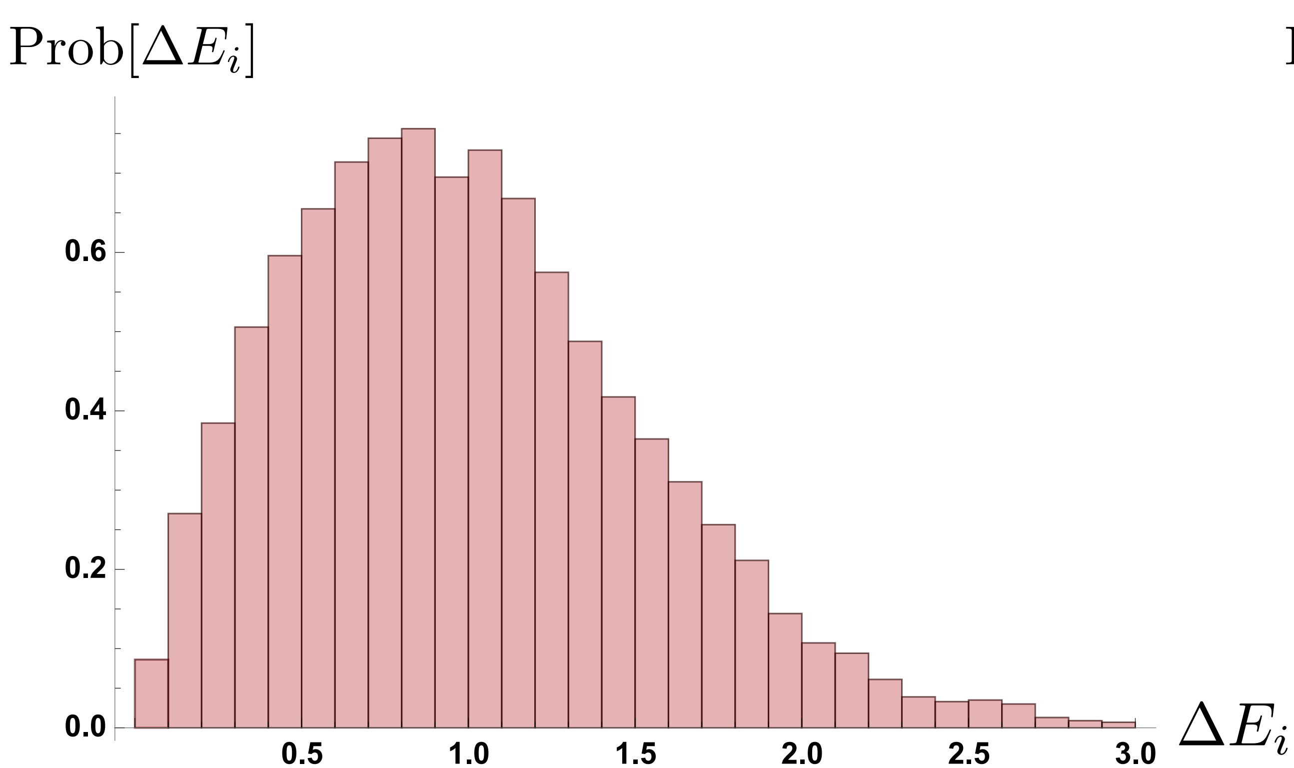
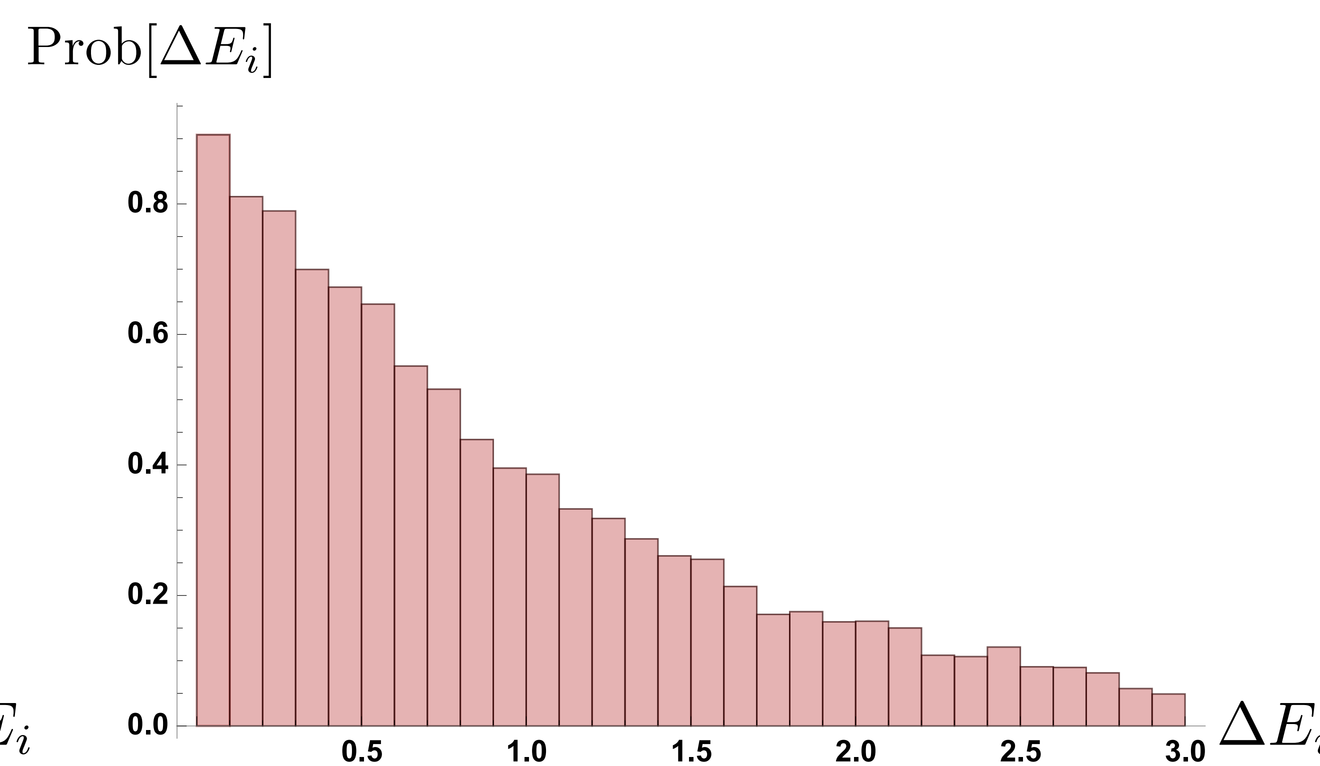
Comment 2

- Symmetries obscure RMT statistics



single GOE matrix
—> RMT

$$\begin{pmatrix} & & \\ & [\mathrm{GOE}]_{N\times N} & \\ & & \end{pmatrix}$$



10 GOE blocks
—> Poisson

$$\begin{pmatrix} [\mathrm{GOE}]_{\frac{N}{10}\times\frac{N}{10}} & & & \\ & [\mathrm{GOE}]_{\frac{N}{10}\times\frac{N}{10}} & & \\ & & \ddots & \\ & & & [\mathrm{GOE}]_{\frac{N}{10}\times\frac{N}{10}} \end{pmatrix}$$

- With symmetries: focus on fixed “charge sector” to see chaos

- Goal of this talk:
  - Propose formalism for studying random matrix statistics in systems of interest in holography: 2d conformal field theories

- Main points:
  - RMT universality can be uplifted to 2d CFTs
  - Interesting interplay: RMT universality <—> rigid symmetry structure
  - Discover “gravity from chaos”

Outline

- Holography
- Quantum chaos in 2d CFT
- Spinning operators and arithmetic chaos
- Topological expansion

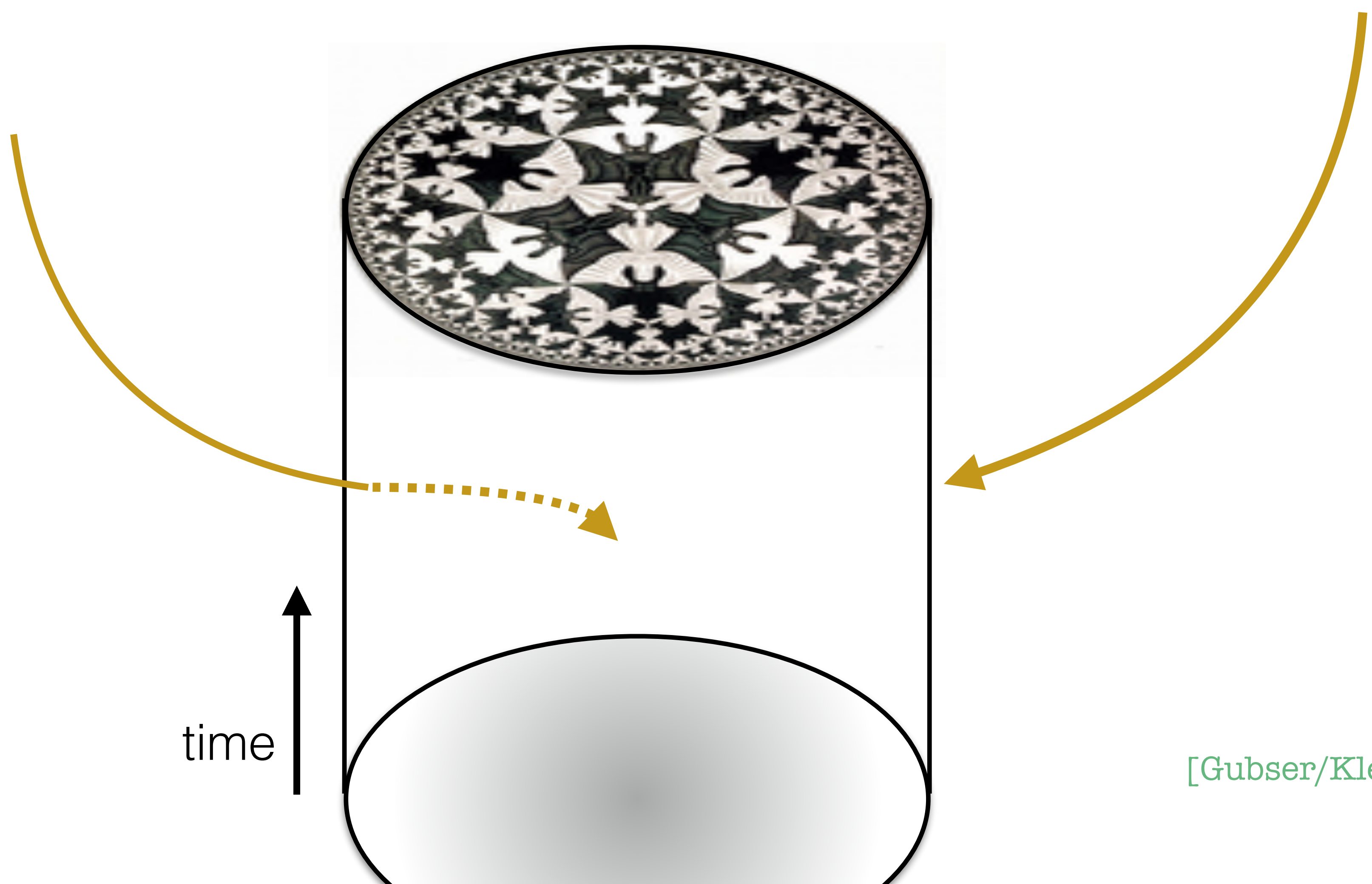
Holography

AdS/CFT duality

- Some (quantum) gravitational systems are dual to quantum many body systems:

Quantum gravity in d+1 dimensions
(negatively curved spacetime: asymptotically anti-de Sitter)

QFT in d dimensions
(with conformal symmetry)



[Maldacena ’97]
[Gubser/Klebanov/Polyakov ’98]
[Witten ’98]

Quantum gravity in d+1 dimensions

(negatively curved spacetime: asymptotically anti-de Sitter)

↔

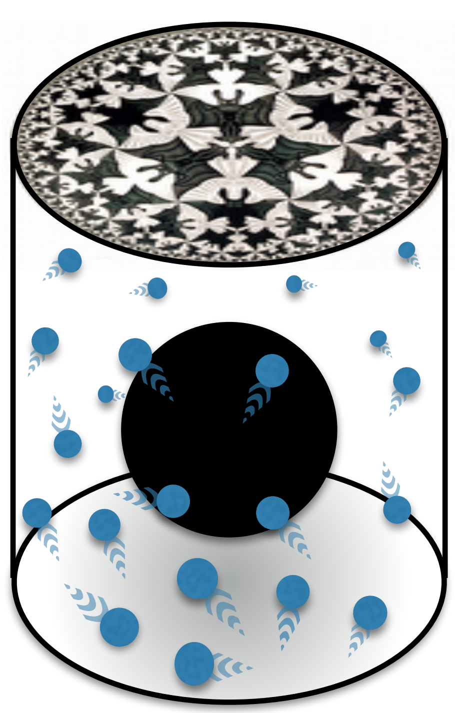
QFT in d dimensions

(with conformal symmetry)

weakly coupled gravity ↔ strongly coupled CFT

black hole geometry ↔ CFT thermal state

black hole entropy ↔ thermal entropy of microstates

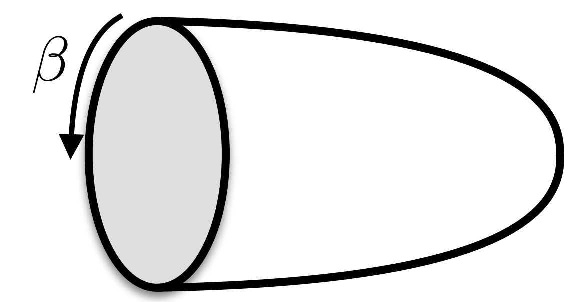


A laboratory for quantum gravity

Factorization puzzle

- Thermal CFT partition function:

$$Z_{\mathrm{CFT}}(\beta) = e^{-S_{\mathrm{grav}}(\beta)}$$



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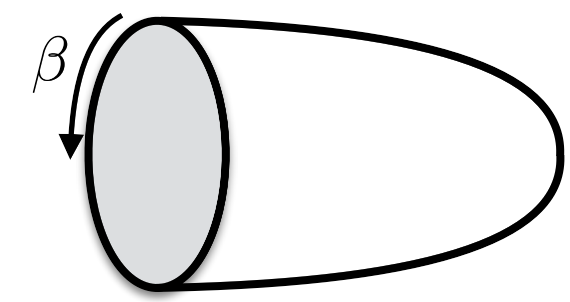
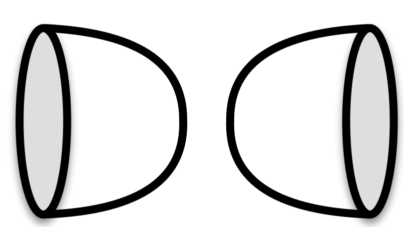


- What if the boundary conditions consist of two CFTs?

$$Z_{\rm CFT,1}(\beta_1)\, Z_{\rm CFT,2}(\beta_2) \quad =$$



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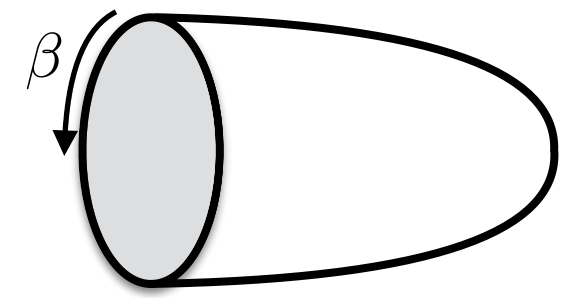

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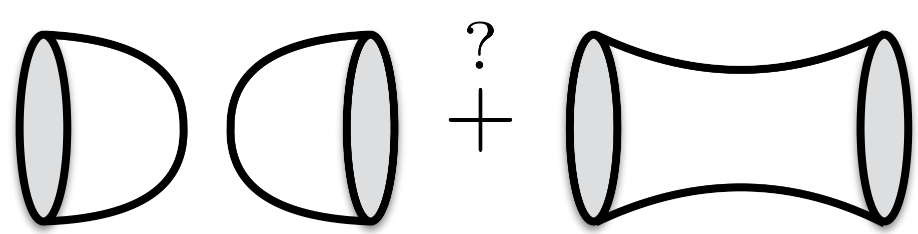

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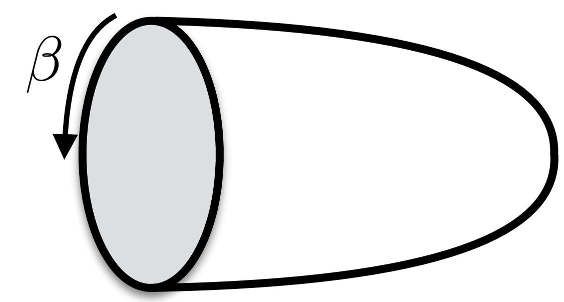

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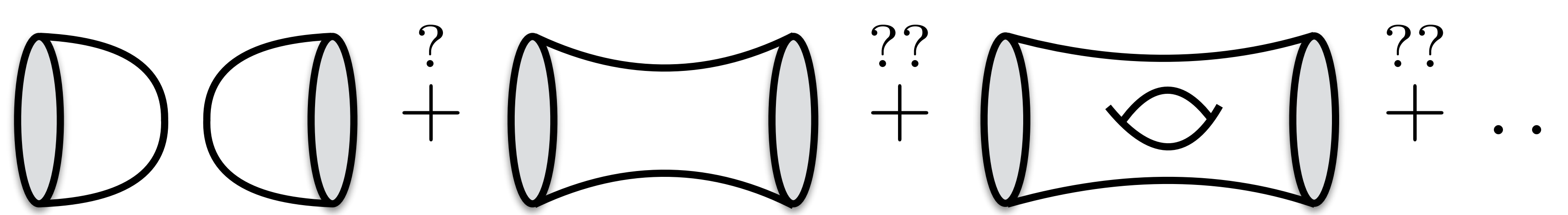

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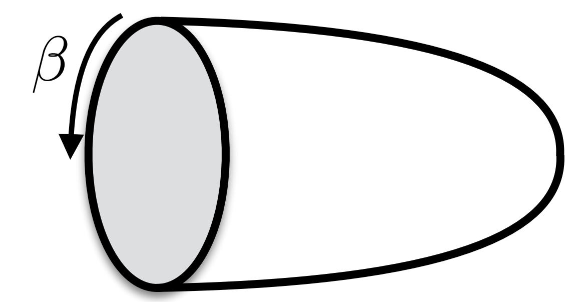

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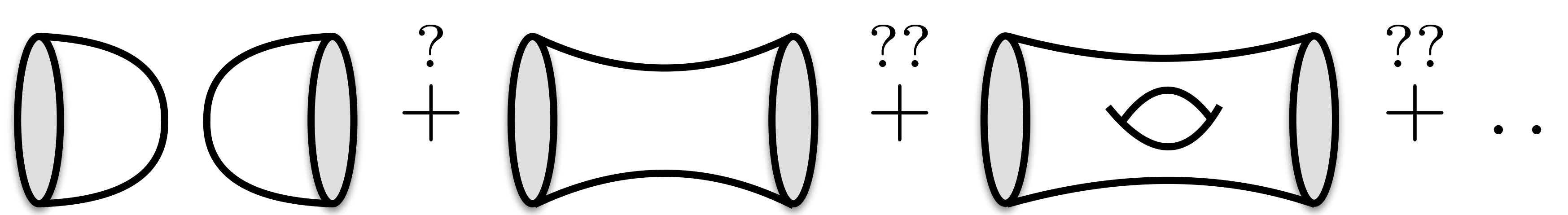

factorized

not factorized

[Van Raamsdonk '10]

[Saad/Shenker/Stanford '19]

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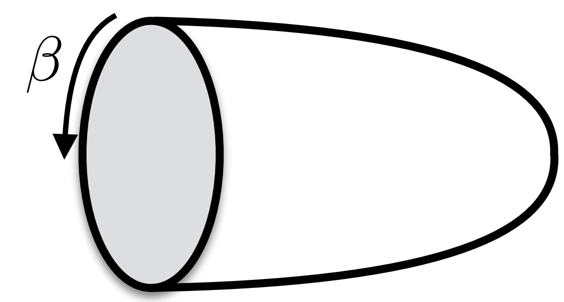

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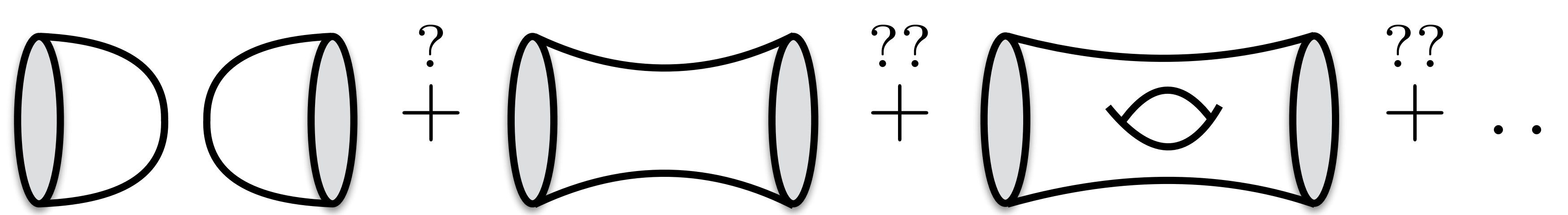

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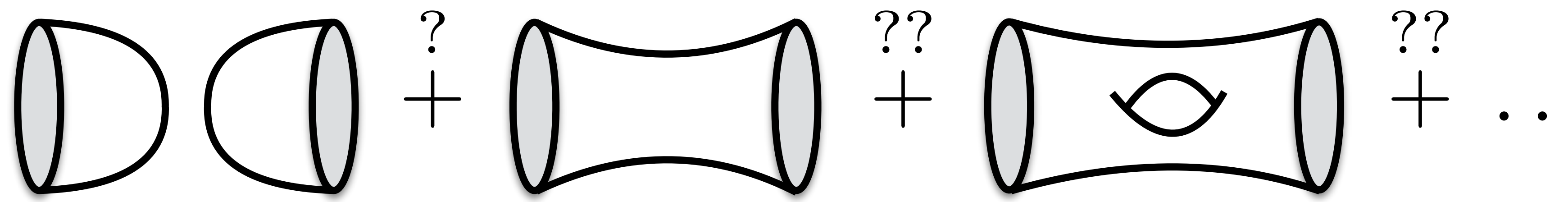

- For 2d gravity: dual to random matrix theory
  - Average over random matrices [Saad/Shenker/Stanford '19] ...

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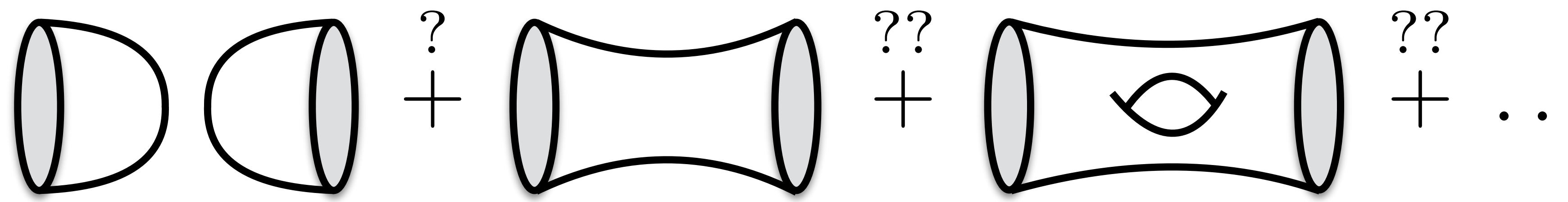

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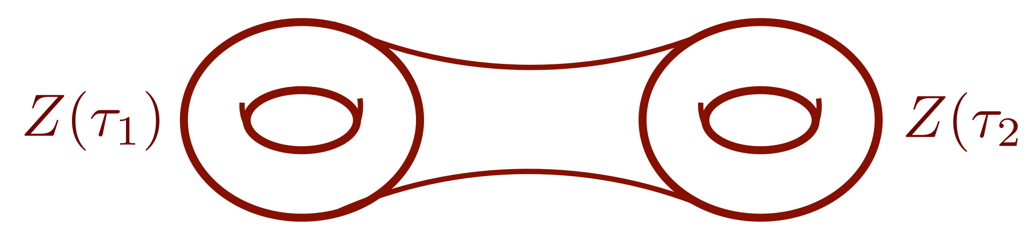

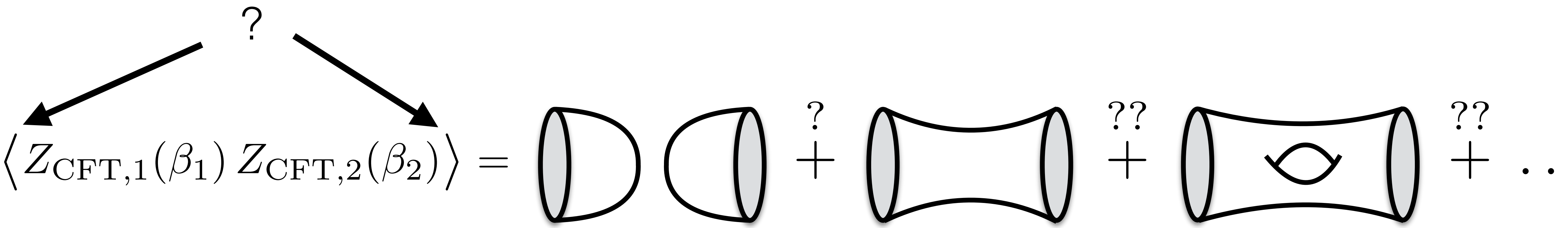

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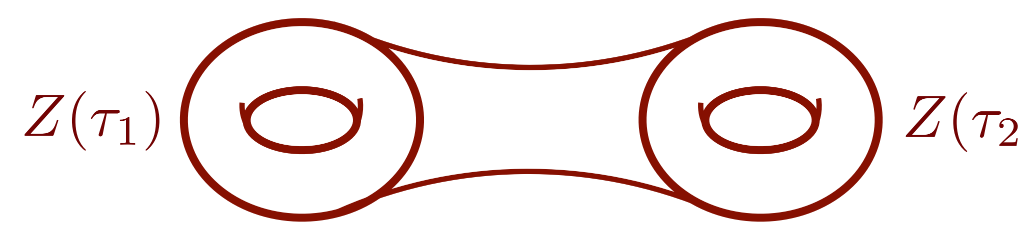
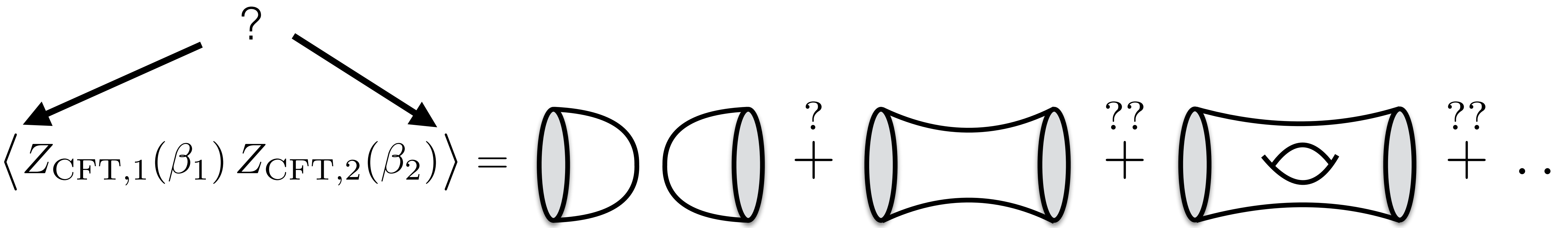


  - One idea: **average over ensemble of approximate CFTs**

    [Belin/de Boer '20] [+ Anous, Jafferis, Liska, Nayak, Sonner] ... [Chandra/Collier/Hartman/Maloney '22] ...



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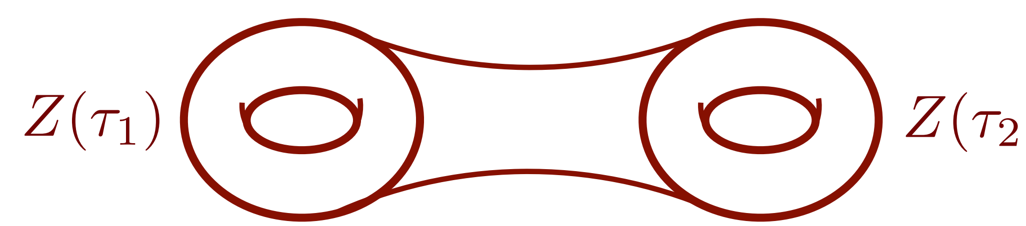

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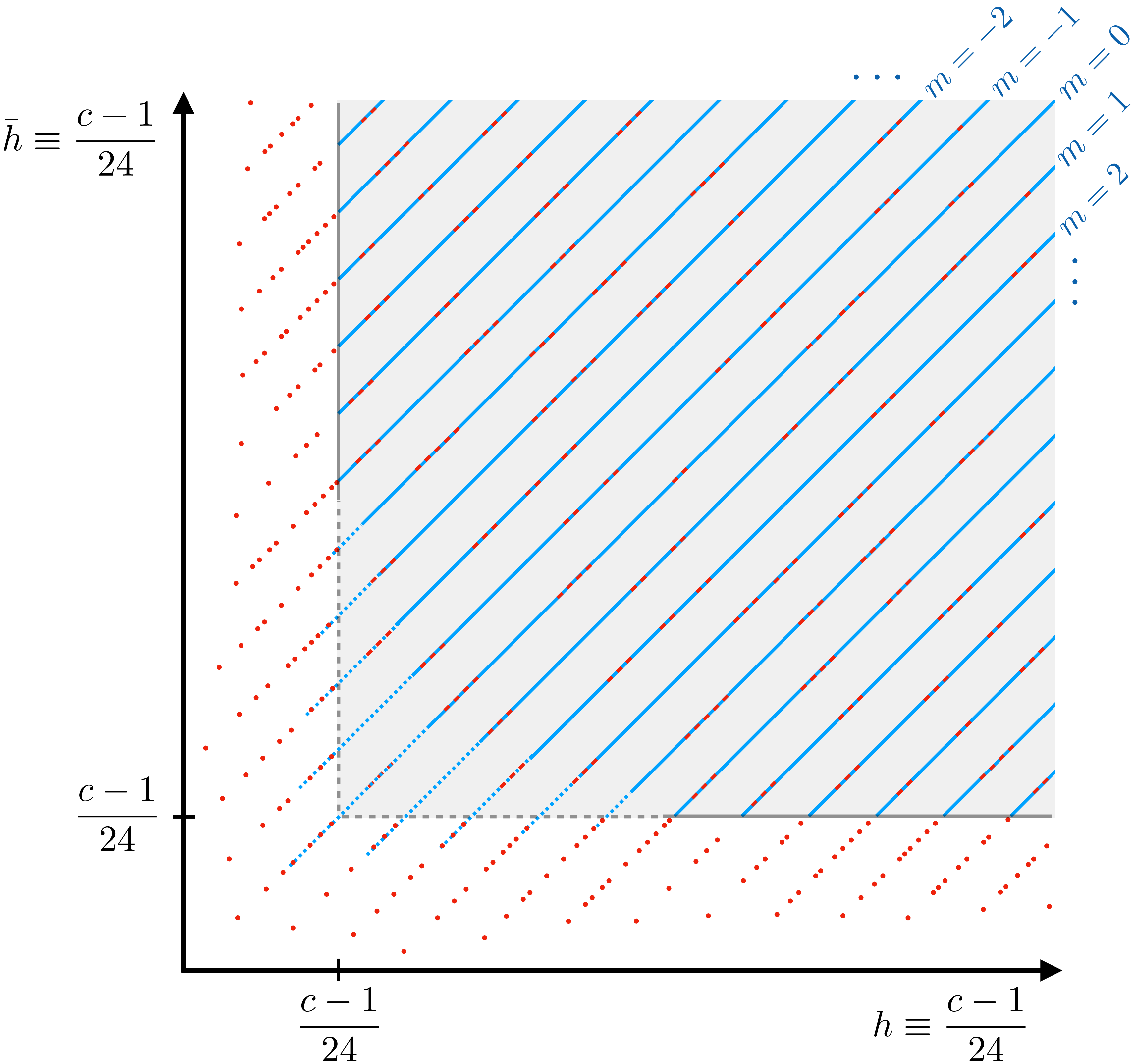
    [Belin/de Boer '20] [+ Anous, Jafferis, Liska, Nayak, Sonner] ... [Chandra/Collier/Hartman/Maloney '22] ...
  - Focus of this talk: (i) average microcanonically over a single CFT
    [Pollack/Rozali/Sully/Wakeham '20] ...
    [**FH**/Marteau/Reeves/Rozali '23] [Di Ubaldo/Perlmutter '23] ...

    (ii) uplift structure of SSS matrix model to 2d: "RMT₂"
    [Boruch/Di Ubaldo/**FH**/Perlmutter/Rozali (to appear)]

Chaos in 2d CFT

- Cartoon of holographic 2d CFT spectrum:



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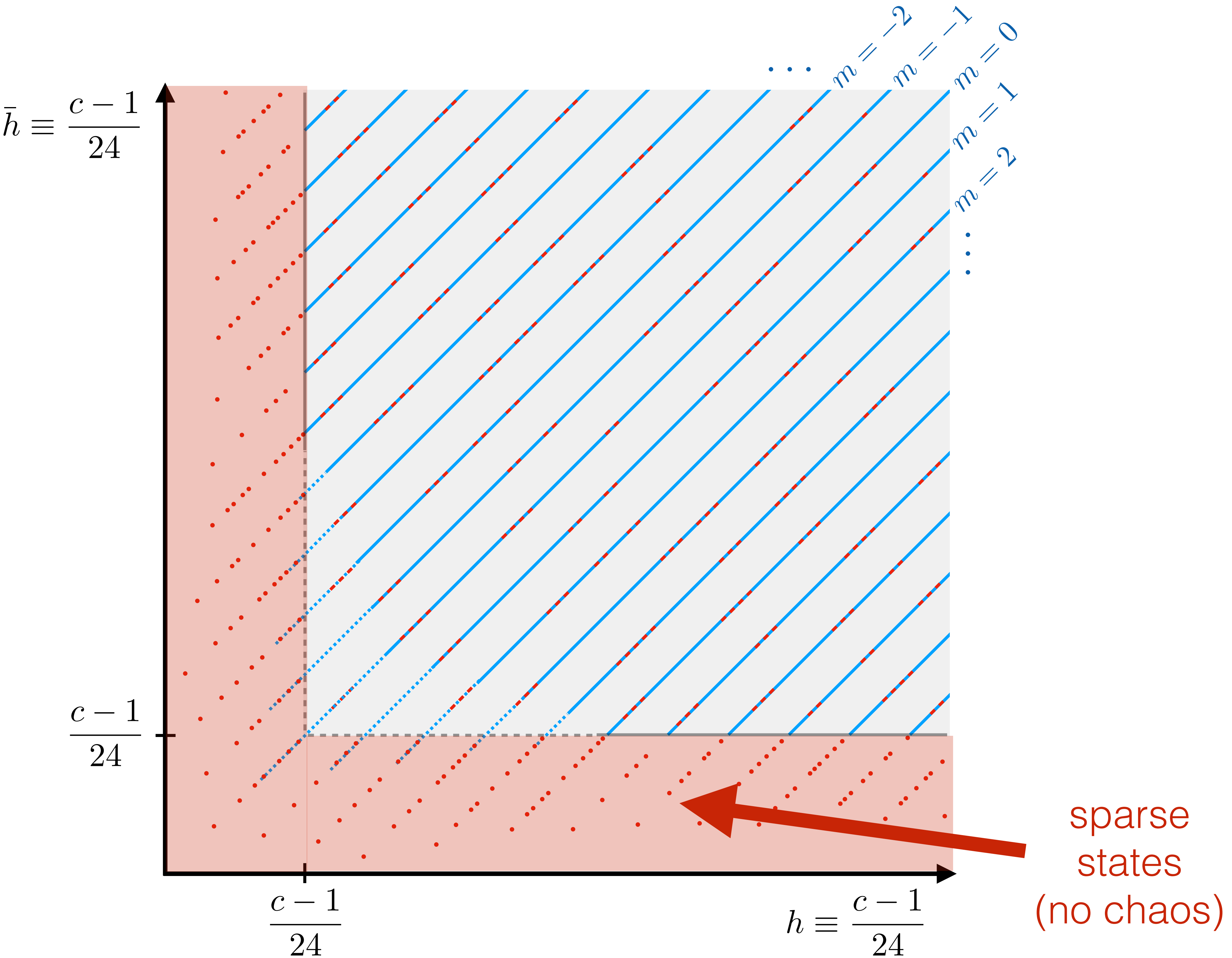

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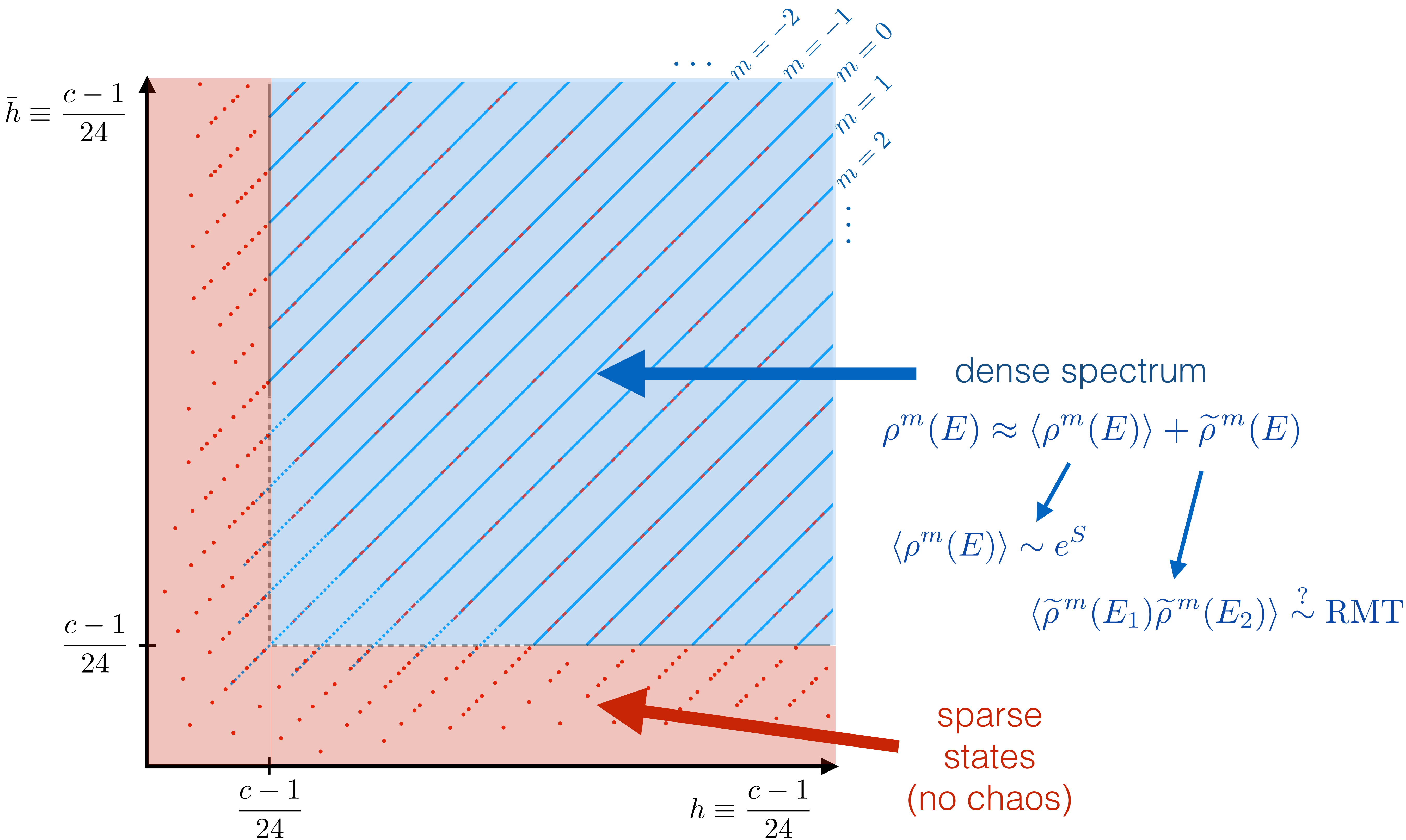
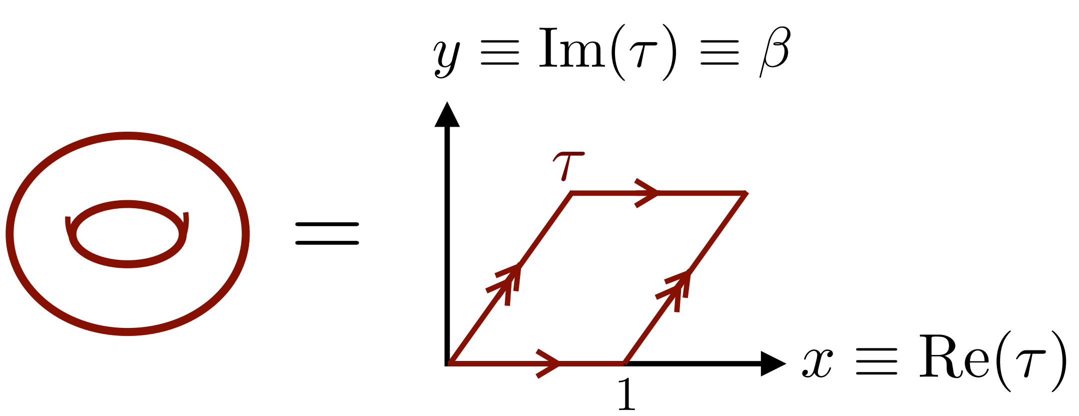


- 2d CFTs are modular invariant:

$$Z(\tau) = Z(\gamma \cdot \tau) \qquad \gamma \in \mathrm{SL}(2,\mathbb{Z})$$

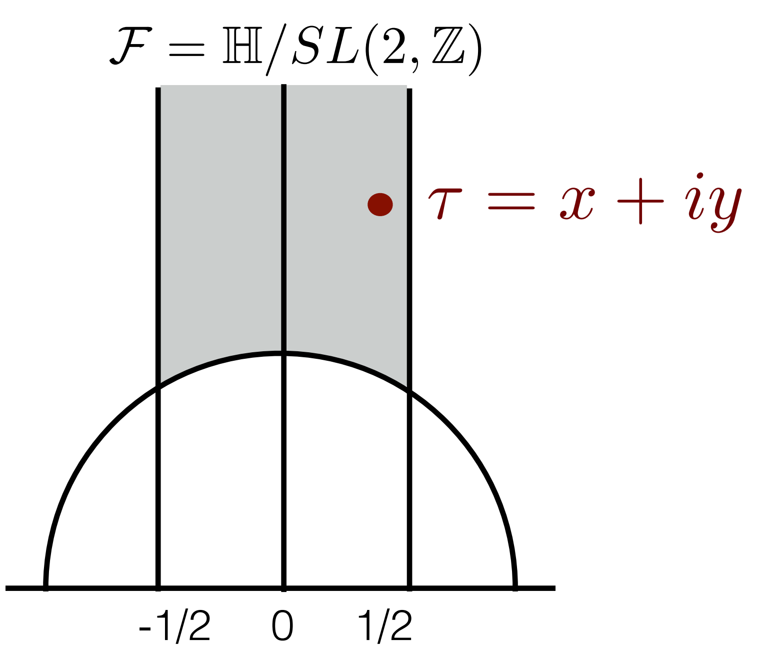


$$\gamma_T : \qquad \tau \;\mapsto\; \tau + 1$$

(implies spin quantization)

$$\gamma_S : \qquad \tau \;\mapsto\; -1/\tau$$

(relates high energy to low energy spectrum)



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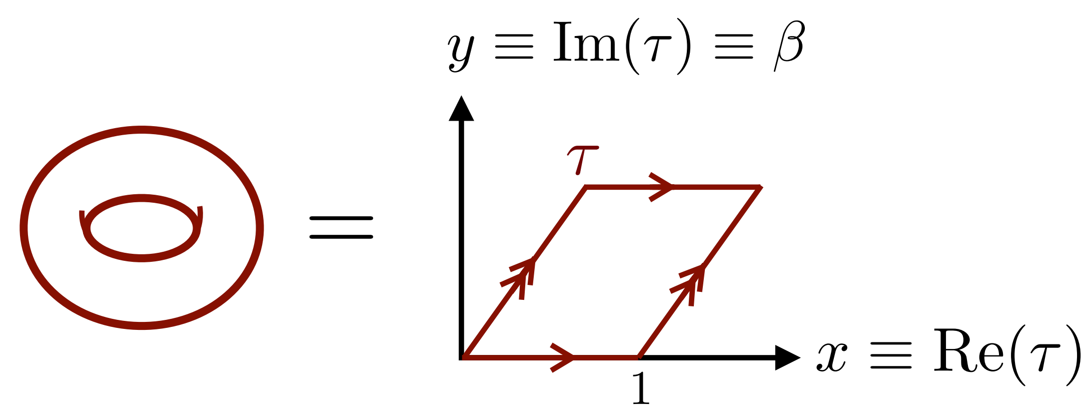

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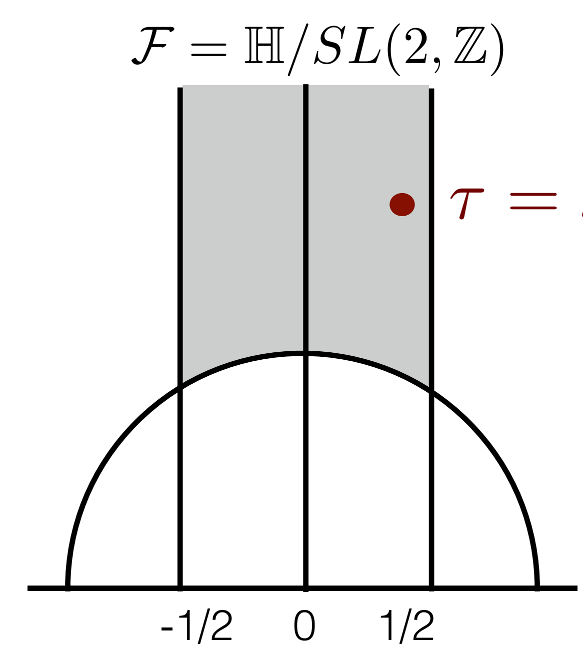
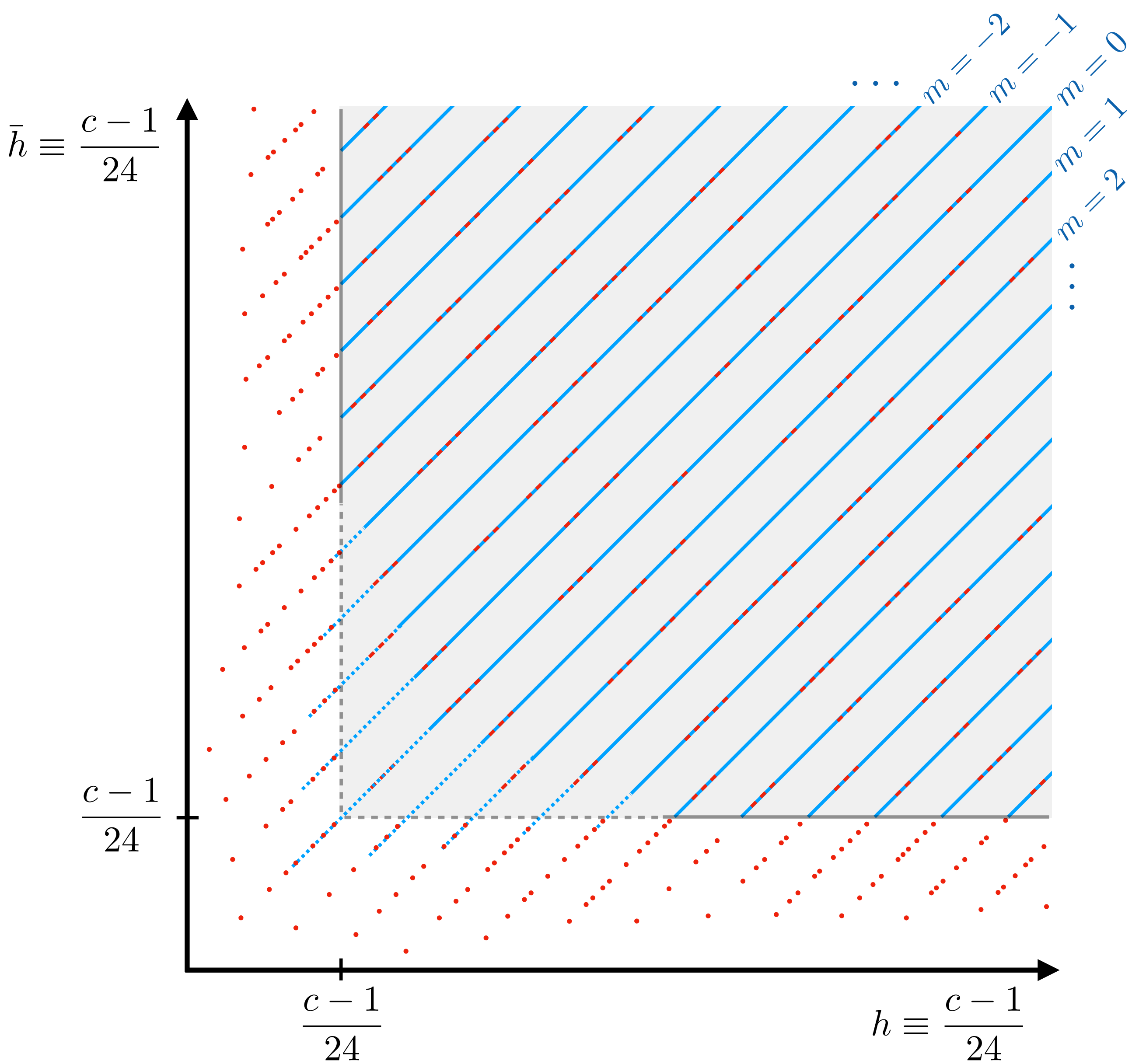


- Cardy formula:

low temperature dominated by ground state $\xrightarrow{\gamma_S}$ exponentially dense high energy spectrum

- Focus on **chaotic part of spectrum** that's unconstrained by symmetries



1. Only consider primary states

$$\rho(E) \;\to\; \rho_P(E)$$

2. Subtract "background" due to sporadic states below extremality

$$\rho_P(E) = \hat{\rho}_{\text{sparse}}(E) + \tilde{\rho}_P(E)$$

$\sim$ light states $+$ Cardy (self-averaging)

fluctuations around dense average (oscillatory)

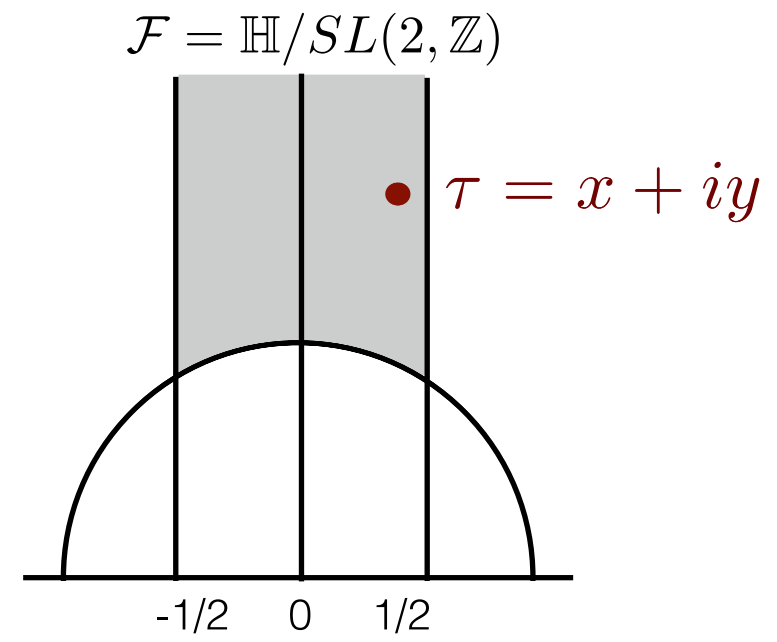
- $\tilde{\rho}_P$ can be defined in modular invariant way [Benjamin/Collier/Fitzpatrick/Maloney/Perlmutter '21]
- RMT correlations in $\tilde{\rho}_P^m$ are now a reasonable assumption!

- Want to discuss RMT statistics without giving up modular invariance

- E.g., $\langle \widetilde{Z}_P^m(y_1 = \beta + iT)\, \widetilde{Z}_P^m(y_2 = \beta - iT) \rangle \sim \dfrac{T}{2\pi\beta}$

  is not a modular invariant statement

  (and modular transformations mix spin)



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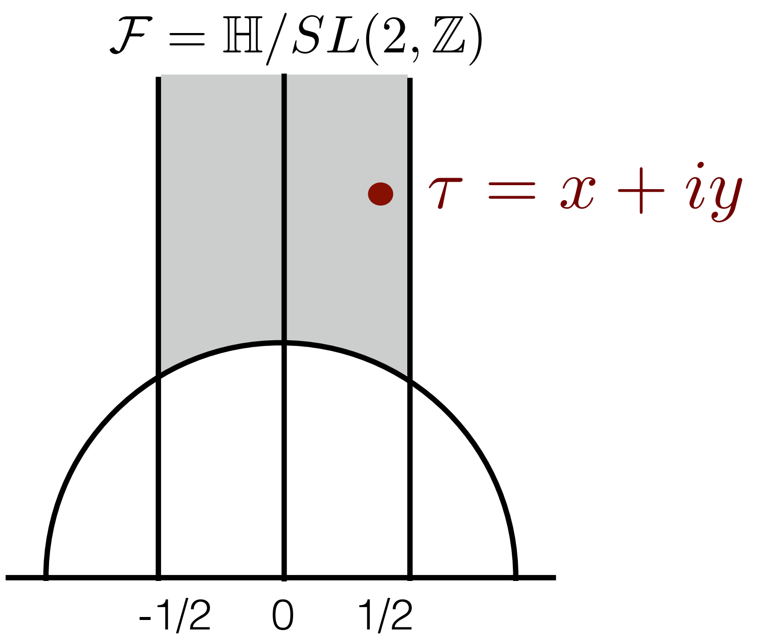

- Work in variables where modular invariance is manifest!

$$\widetilde{Z}_{\mathrm{P}}(\tau) = \text{ const. } + \int_{s \equiv \frac{1}{2} + i\mathbb{R}} ds\, \left(\widetilde{Z}_{\mathrm{P}},\, E_s\right) E_s(\tau) + \sum_{n \geq 1} \left(\widetilde{Z}_{\mathrm{P}},\, \overline{\nu}_n\right) \overline{\nu}_n(\tau)$$

[Rankin '39] [Selberg '40] ...

[Benjamin/Collier/Fitzpatrick/Maloney/Perlmutter '21]

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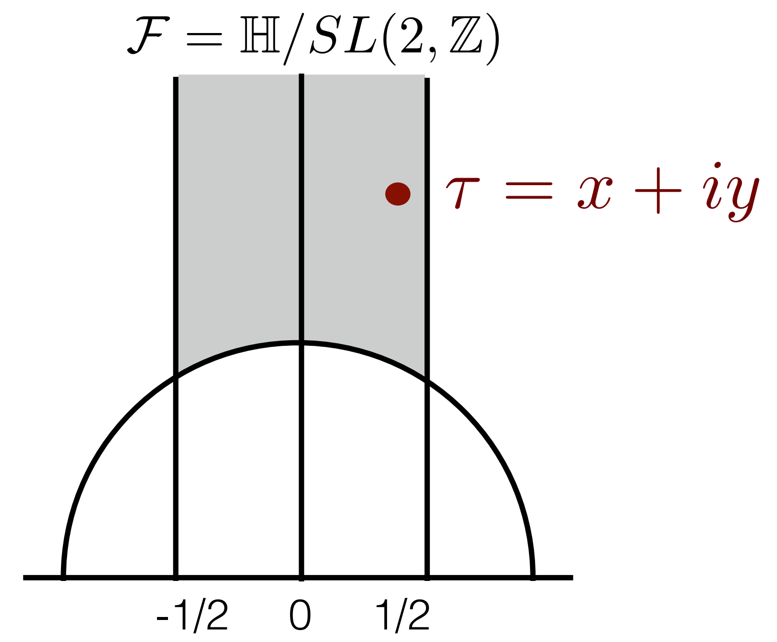

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[Rankin '39] [Selberg '40] ...

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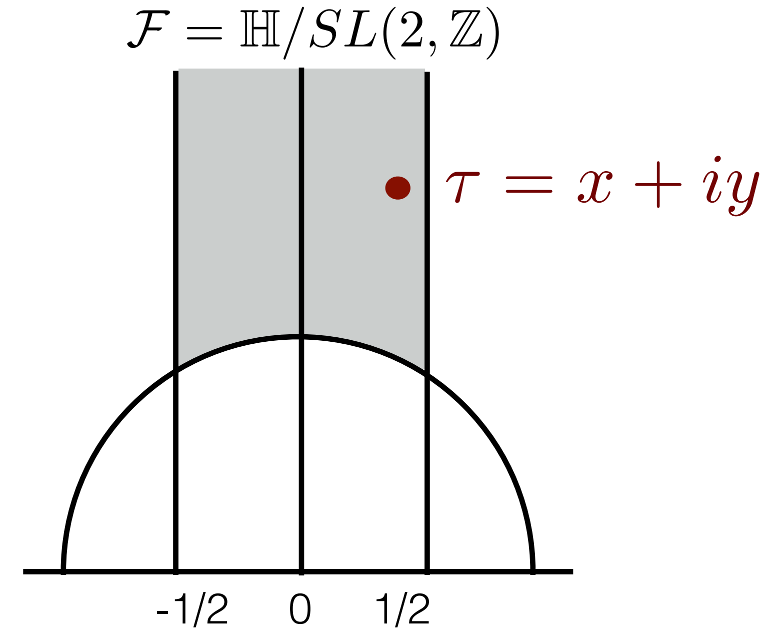
  is not a modular invariant statement
  (and modular transformations mix spin)



- **Work in variables where modular invariance is manifest!**

$$\widetilde{Z}_{\mathrm{P}}(\tau) = \text{ const.} + \int_{s\equiv\frac{1}{2}+i\mathbb{R}} ds\, \underbrace{\left(\widetilde{Z}_{\mathrm{P}},\, E_s\right)}_{z_{s\equiv\frac{1}{2}+i\alpha}} E_s(\tau) + \sum_{n\geq 1} \underbrace{\left(\widetilde{Z}_{\mathrm{P}},\, \overline{\nu}_n\right)}_{z_n} \overline{\nu}_n(\tau)$$

encode all information about **spectrum**.

—> encode **spectral statistics** in $\langle z_{s_1} z_{s_2}\rangle$, $\langle z_{n_1} z_{n_2}\rangle$ etc.

[Rankin '39] [Selberg '40] ...

[Benjamin/Collier/Fitzpatrick/Maloney/Perlmutter '21]

$$\widetilde{Z}_{\mathrm{P}}(\tau) = \text{const.} + \int_{s\equiv\frac{1}{2}+i\mathbb{R}} ds\, \left(\widetilde{Z}_{\mathrm{P}},\, E_s\right) E_s(\tau) + \sum_{n\geq 1} \left(\widetilde{Z}_{\mathrm{P}},\, \overline{\nu}_n\right) \overline{\nu}_n(\tau)$$

1. Real-analytic Eisenstein series ("scattering states"):

$$\Delta_{\mathcal{F}} E_s = s(1-s) E_s \qquad s \in \frac{1}{2} + i\mathbb{R}$$

2. Maass cusp forms ("bound states"):

$$\Delta_{\mathcal{F}} \nu_n = \left(\frac{1}{4} + R_n^2\right) \nu_n \qquad R_n = 13.780,\ 17.739,\ 19.424, \ldots \quad (n = 1, 2, 3, \ldots)$$

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1. Real-analytic Eisenstein series ("scattering states"):

$$\Delta_{\mathcal{F}} E_s = s(1-s) E_s \qquad s \in \frac{1}{2} + i\mathbb{R}$$

$$E_{s=\frac{1}{2}+i\alpha}(\tau = x+iy) = \sum_{m\geq 0} \cos(2\pi m x)\, \tilde{a}_m^{(\alpha)}\, \sqrt{y}\, K_{i\alpha}(2\pi m y)$$

Fourier coefficients: $\tilde{a}_m^{(\alpha)} = \frac{4}{\Gamma(-i\alpha)\zeta(-2i\alpha)} \sum_{d|m} \left(\frac{d^2}{m\pi}\right)^{i\alpha}$

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Fourier coefficients: not known analytically (*arithmetic chaos)*

interesting math conjectures (e.g., $-2 \leq a^{(n)}_{m\ \text{prime}} \leq 2$)

Spin *m=0* ‘ramp’

- ‘Ramp’ in the spectrum of scalar operators:

$$\frac{1}{(4\pi i)^2}\iint_{\frac{1}{2}+i\mathbb{R}} ds_1 ds_2 \, \langle z_{s_1} z_{s_2} \rangle_{\mathrm{ramp}} \, E^0_{s_1}(x_1+iy_1) E^0_{s_2}(x_2+iy_2) \sim \frac{1}{\pi}\frac{y_1 y_2}{y_1+y_2} + (\mathrm{subleading})$$

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- Ignore 'subleading' and invert the integral transform: [**FH**/Marteau/Reeves/Rozali]

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- For leading term (linear ramp), only need asymptotics: [Di Ubaldo/Perlmutter]

$$\langle z_{s_1} z_{s_2}\rangle_{\mathrm{ramp}} \sim e^{-\pi\alpha_1} \times 4\pi\delta(\alpha_1+\alpha_2)$$

different functions with these asymptotics give different subleading corrections

Spin *m=0* ‘ramp’

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Spin *m=0* 'ramp'

- 'Ramp' in the spectrum of scalar operators:

<u>time domain</u> $\longleftrightarrow$ <u>modular invariant basis</u>

$$\langle \widetilde{Z}_P^m(\beta+iT)\widetilde{Z}_P^m(\beta-iT)\rangle \sim \frac{T^2}{2\pi\beta} + (\text{subleading})$$

required for mod.inv.

$$\langle z_{\frac{1}{2}+i\alpha_1} z_{\frac{1}{2}+i\alpha_2}\rangle \sim \left(e^{-\pi\alpha_1} + \text{subleading}\right) \times 4\pi\delta(\alpha_1+\alpha_2)$$

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How to find possible *modular completions?*

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Any choice is an allowed modular completion.

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Important example: holography

$$(\text{subleading})_{\text{AdS}_3\text{ wormhole}}$$

$$= \sum_c S(m_1,m_2,c)(\cdots)$$

[Cotler/Jensen '20]

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[Collier] [Di Ubaldo/Perlmutter '23]
[FH/Reeves/Rozali '23]

- What about RMT universality in spin *m>0* spectrum?
- Can be encoded in Maass cusp forms
- Shortcut: spectral decomposition follows from a trace fomula
  [Bruggeman '78][Kuznetsov '81][FH/Reeves/Rozali '23]

$$\delta_{m_1 m_2}\,\mathcal{I}[g](y_1,y_2)\;+\;\mathcal{G}_{m_1 m_2}[g](y_1,y_2)$$
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[Bruggeman '78][Kuznetsov '81][FH/Reeves/Rozali '23]

$$\mathcal{I}[g](y_1, y_2) = \frac{y_1 y_2}{\pi^2} \int_{\mathbb{R}} d\alpha\, \alpha\, \tanh(\pi\alpha)\, g(\alpha)\, K_{i\alpha}(2\pi m_1 y_1) K_{i\alpha}(2\pi m_2 y_2)$$

$$\mathcal{G}_{m_1 m_2}[g](y_1, y_2) = 8i\sqrt{y_1 y_2} \sum_{c\geq 1} \frac{S(|m_1|, |m_2|; c)}{c} \int_{\mathbb{R}} \frac{d\alpha}{4\pi} \frac{\alpha\, g(\alpha)}{\cosh(\pi\alpha)} J_{2i\alpha}\left(\frac{4\pi\sqrt{|m_1 m_2|}}{c}\right) K_{i\alpha}(2\pi m_1 y_1) K_{i\alpha}(2\pi m_2 y$$
$$+ \frac{16}{\pi}\sqrt{y_1 y_2} \sum_{c\geq 1} \frac{S(|m_1|, -|m_2|; c)}{c} \int_{\mathbb{R}} \frac{d\alpha}{4\pi}\, \alpha\, g(\alpha) \sinh(\pi\alpha)\, K_{2i\alpha}\left(\frac{4\pi\sqrt{|m_1 m_2|}}{c}\right) K_{i\alpha}(2\pi m_1 y_1)$$

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[Bruggeman '78][Kuznetsov '81][**FH**/Reeves/Rozali '23]

spin-diagonal term

"modular completion" (subleading for $T \gg \beta$)

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Eisenstein series

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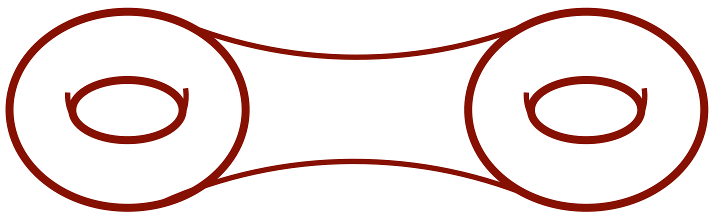
[Cotler/Jensen '20]

wormhole amplitude
in AdS3 gravity



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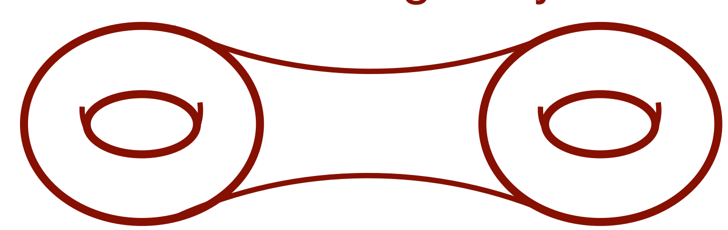

Gravity amplitude = simplest (minimal) completion of the 'bare ramp' into a SFF consistent with modular invariance

"MaxRMT" principle

[Di Ubaldo/Perlmutter '23]

… [FH/Reeves/Rozali '23]

Spinning ramps and arithmetic chaos

- Spinning ramps encoded in "random sum of cusp forms"! How??

$$\frac{1}{\pi}\frac{y_1 y_2}{y_1+y_2}\, e^{-2\pi m(y_1+y_2)} + \ldots \;=\; \sum_{n\geq 1} \frac{1}{2\,\cosh(\pi R_n)}\, \frac{\nu_n^m(y_1)\nu_n^m(y_2)}{\|\nu_n\|^2}$$

spin *m* 'ramp'
(quantum chaos)

'arithmetic chaos'
[Sarnak '93] [Hejhal/Arno '93] [Steil '94]

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spin *m* ‘ramp’
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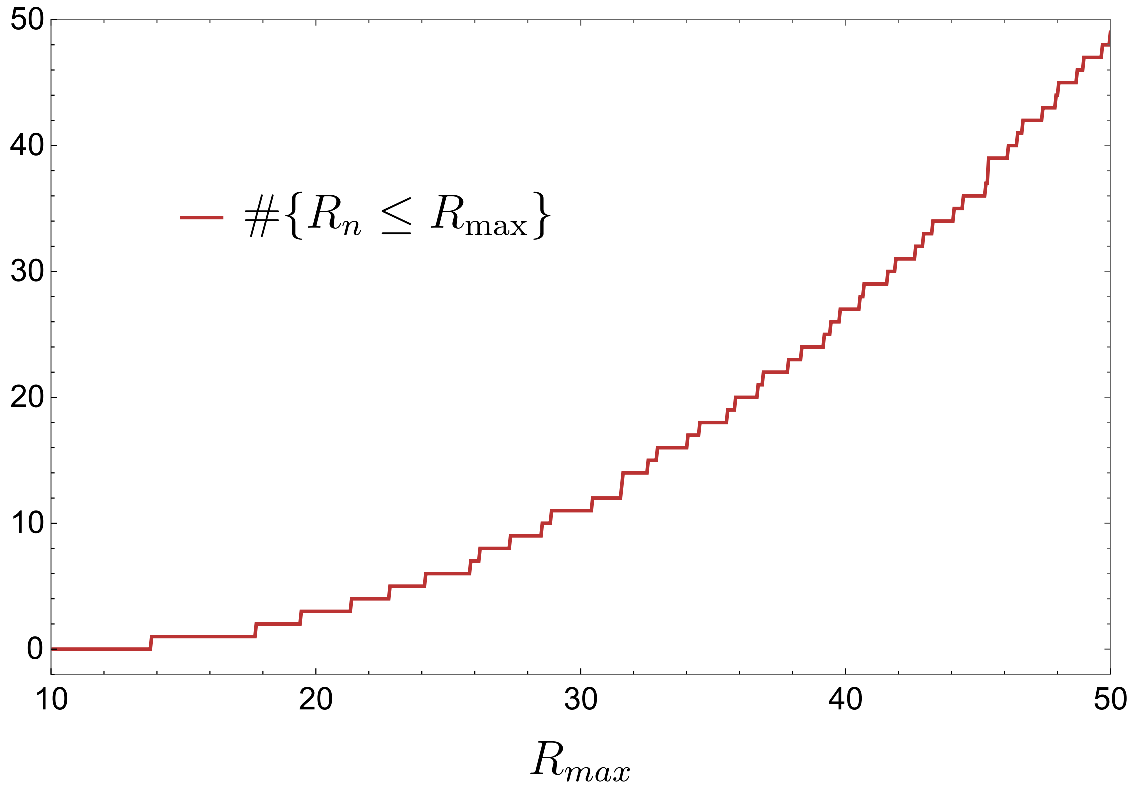
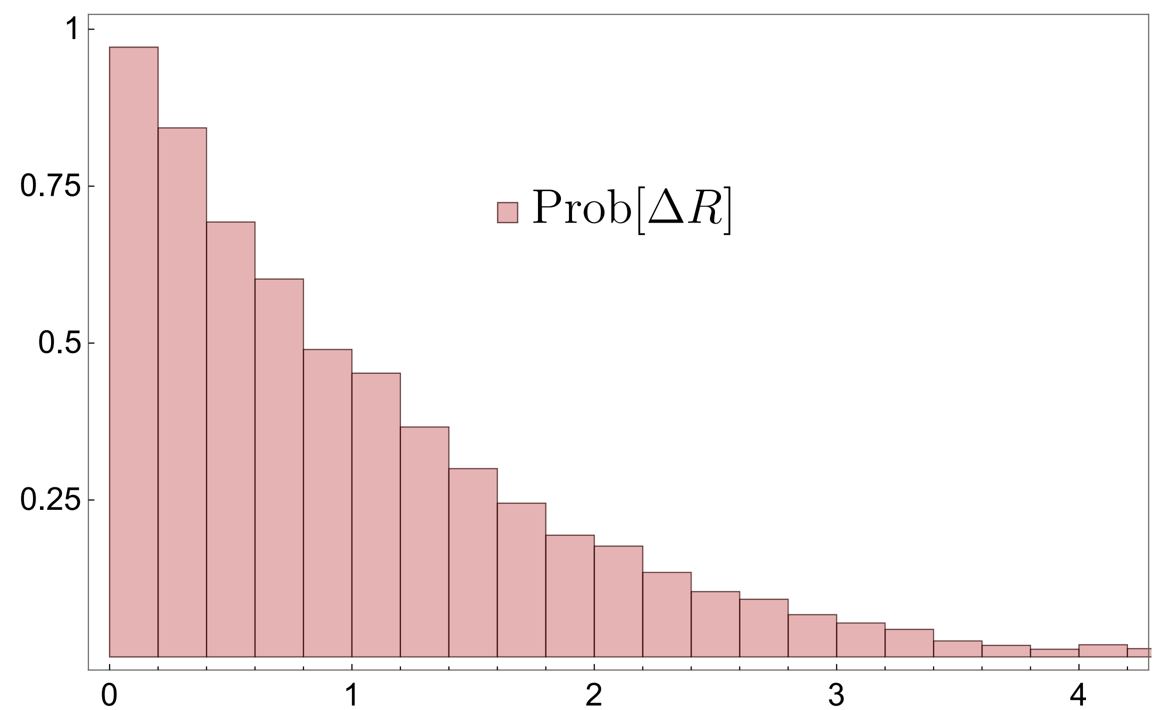
‘arithmetic chaos’
[Sarnak ’93] [Hejhal/Arno ’93] [Steil ’94]

- Eigenvalues R_n: sporadic numbers, Poisson distributed

$$R_n = 13.7798.., \quad 17.7386.., \quad 19.4235.., \quad 21.3158.., \quad 22.7859.., \quad 24.1124.., \quad 25.8262.., \quad \ldots$$





[Then ’04] [2309.00611]

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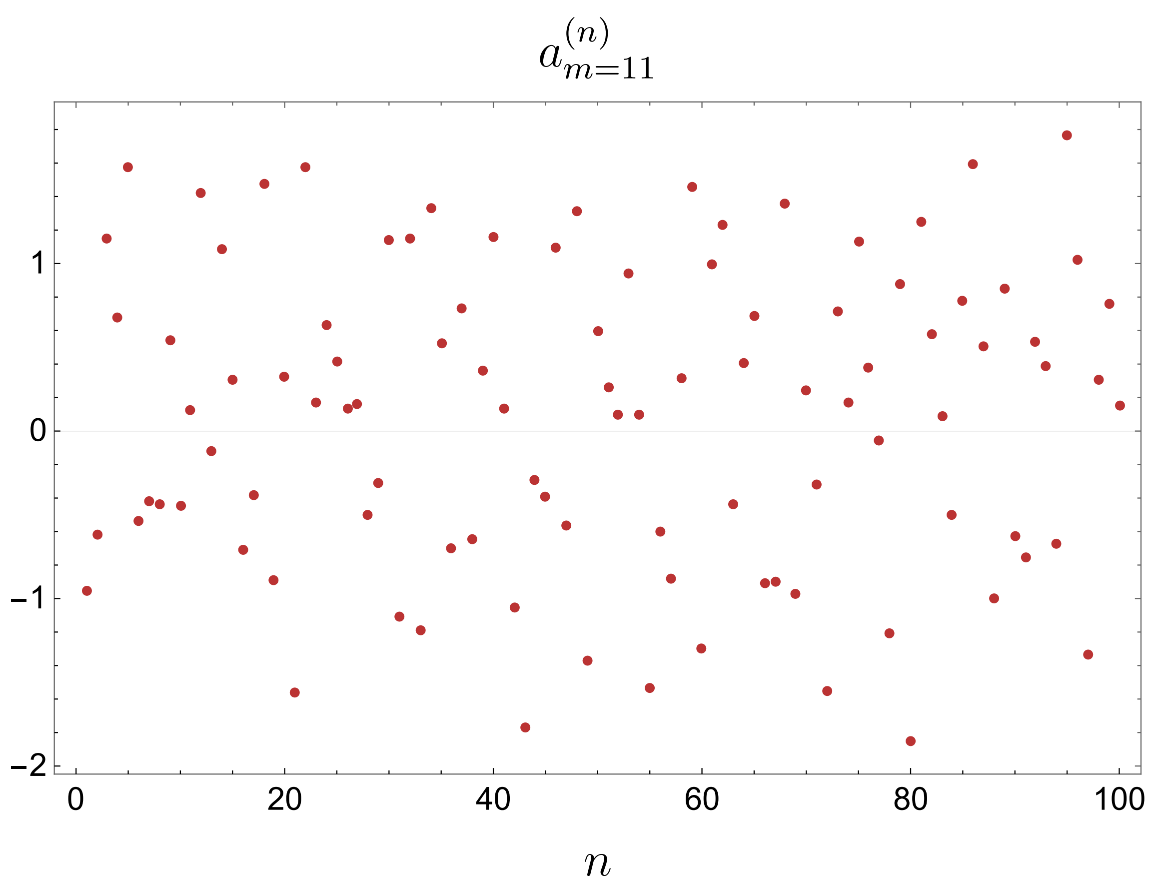
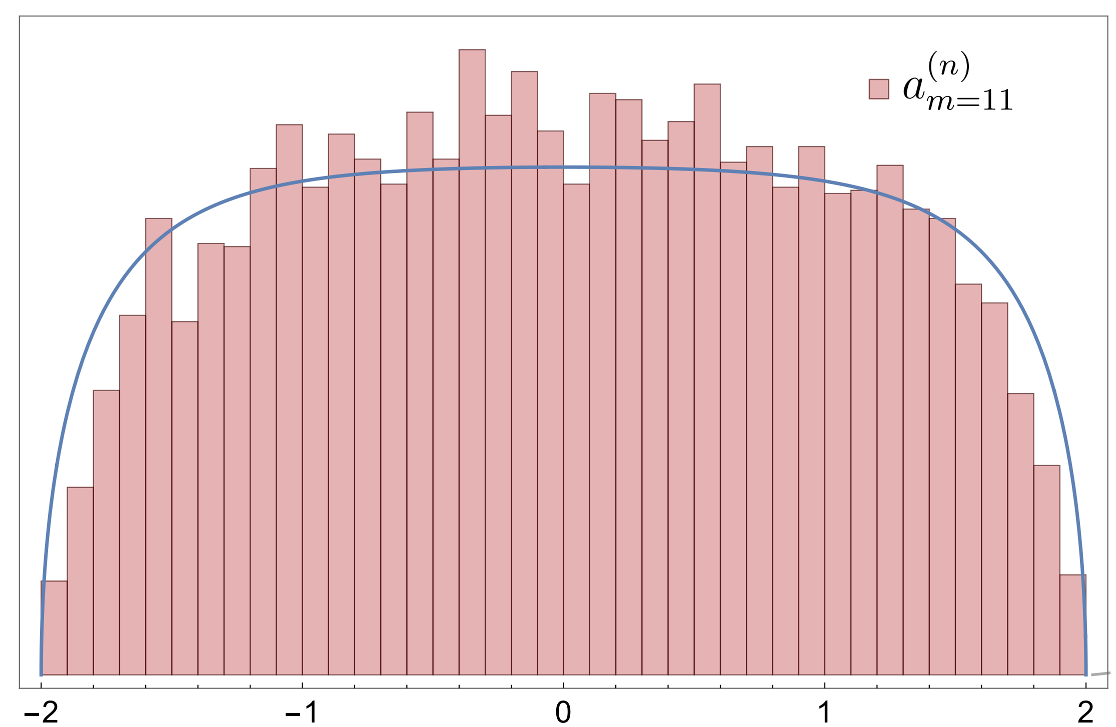
[Sarnak '93] [Hejhal/Arno '93] [Steil '94]

- Fourier coefficients $a_m^{(n)}$: also erratic, Poisson

$$a_{m=11}^{(n)} = -0.954.., \quad -0.621.., \quad +1.154.., \quad +0.678.., \quad +1.575.., \quad -0.533.., \quad \ldots$$





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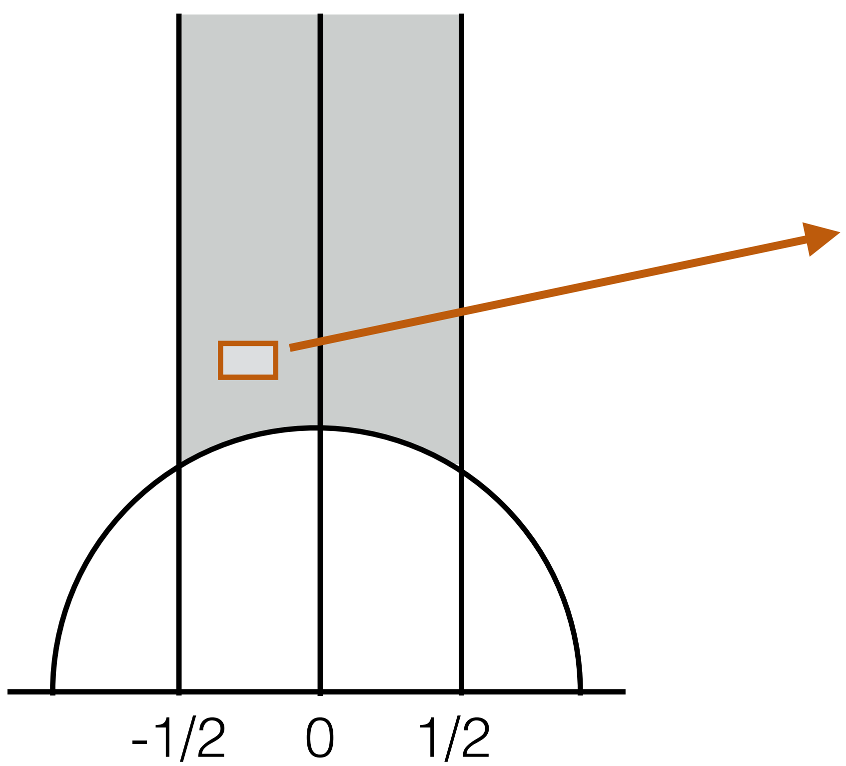
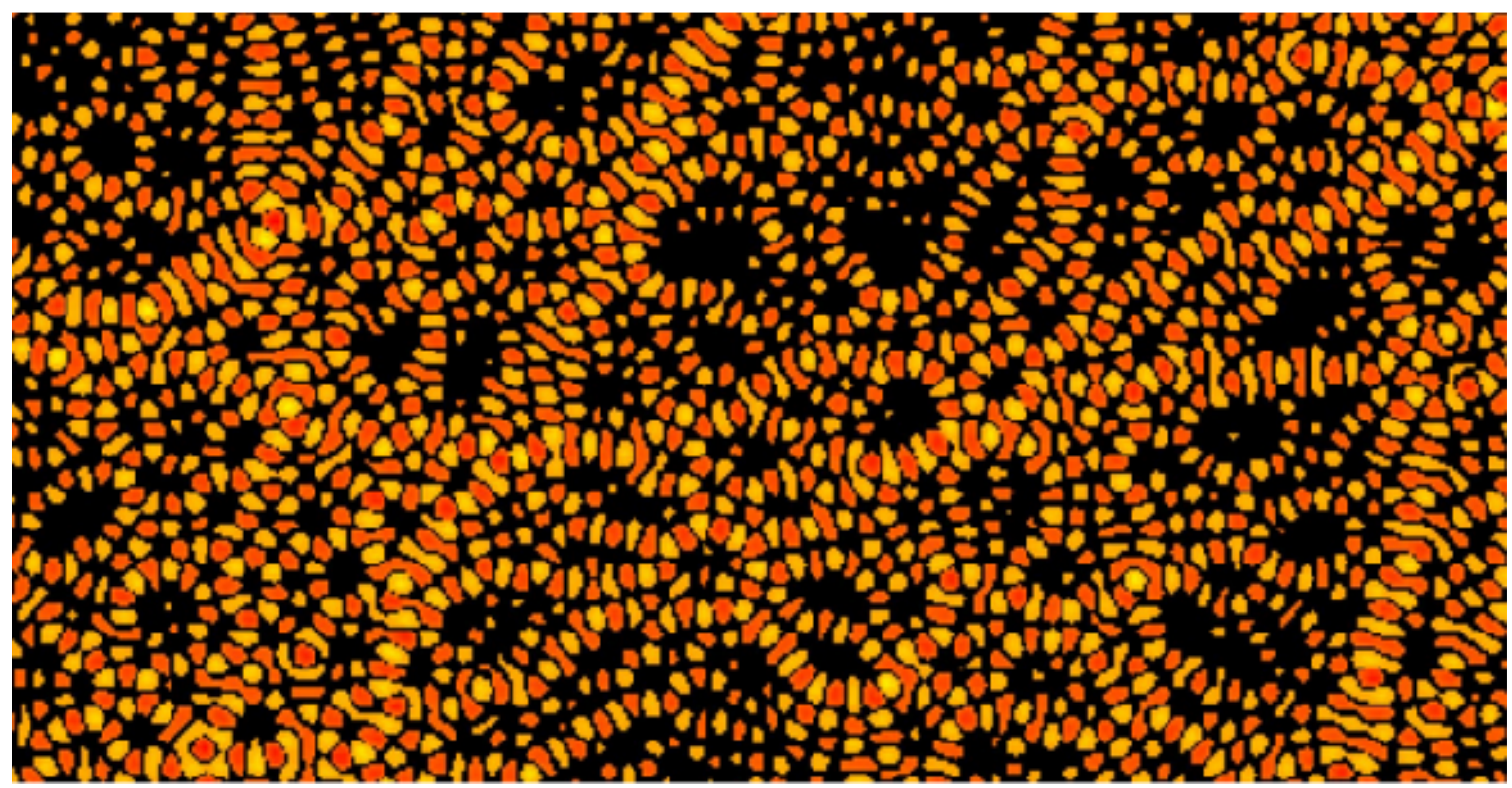
e.g.: cusp form with $R_n = 40000.000164..$





[H. Then, Math. Comp. 74 (2004)]

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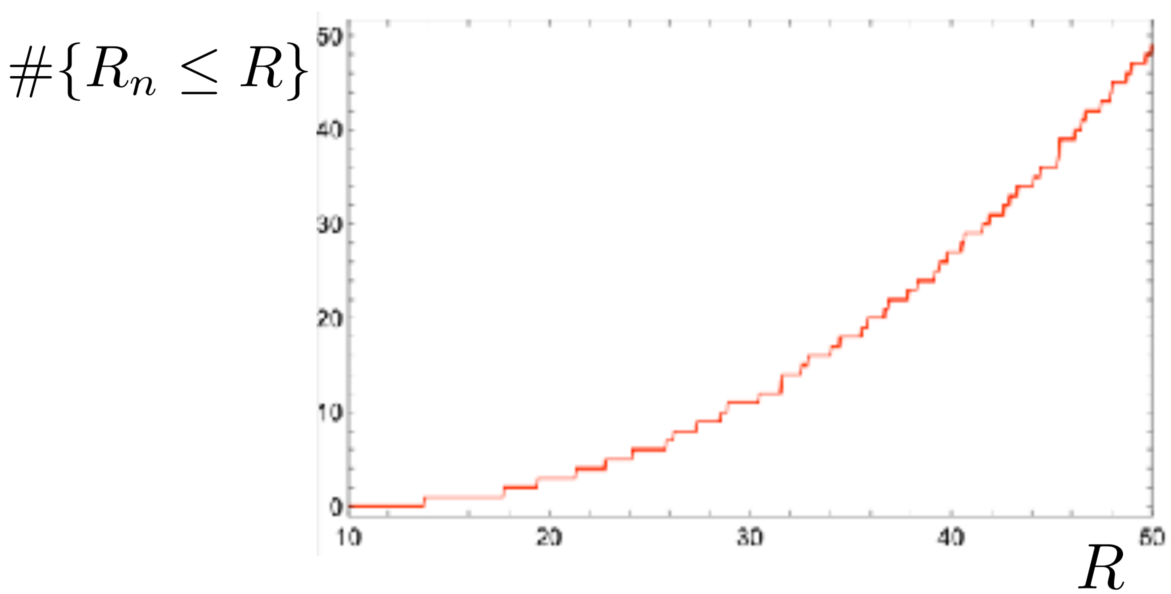
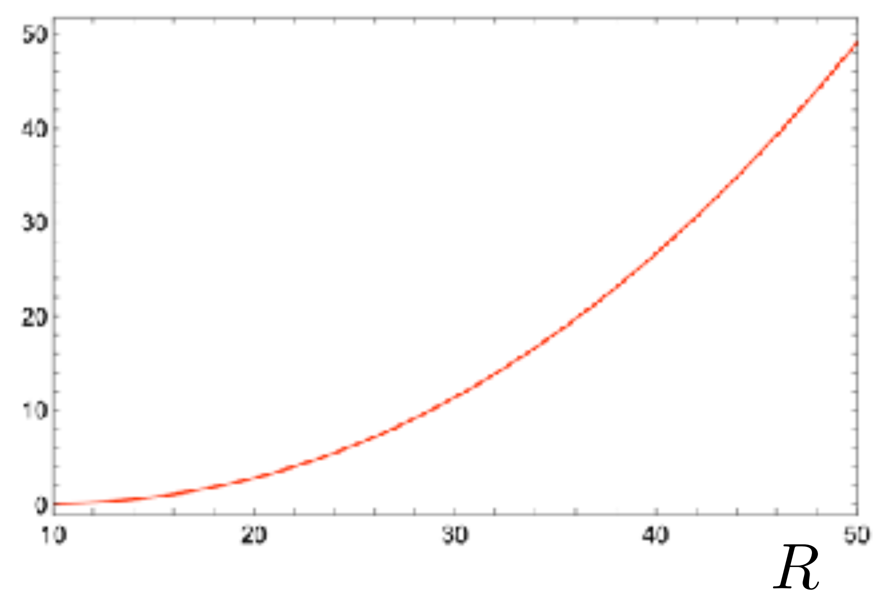
- Can understand this in surprising detail [**FH**/Reeves/Rozali '23]
  1. Numerically: sum over O(10^4) cusp forms
  2. Analytically: trace formula
  3. Analytically: approximate ($\bigstar$) using **statistical properties** of R_n & $a_m^{(n)}$



$\approx$



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Wormhole amplitude again implements RMT universality in the most extreme way:

$\langle z_n z_n \rangle_{\text{wormhole}}$ encodes <u>complete</u> statistical information (all moments) of <u>all</u> cusp form data.

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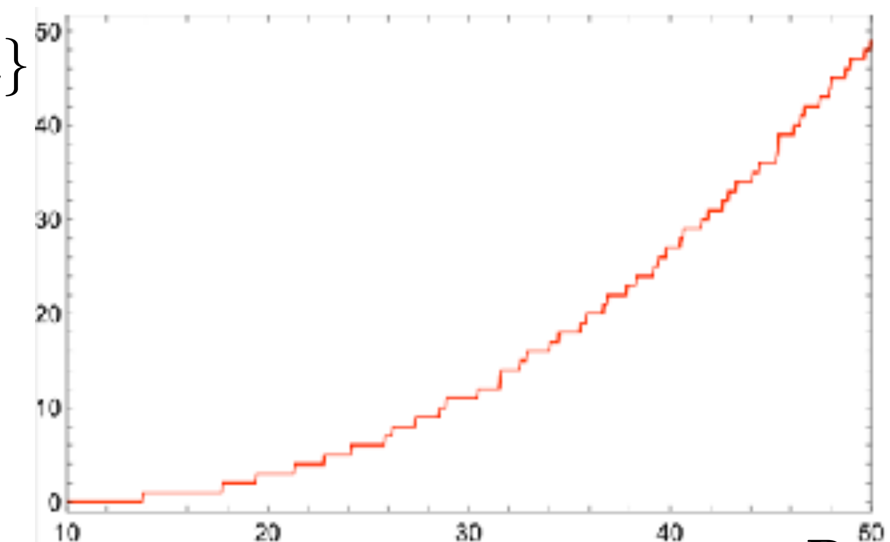
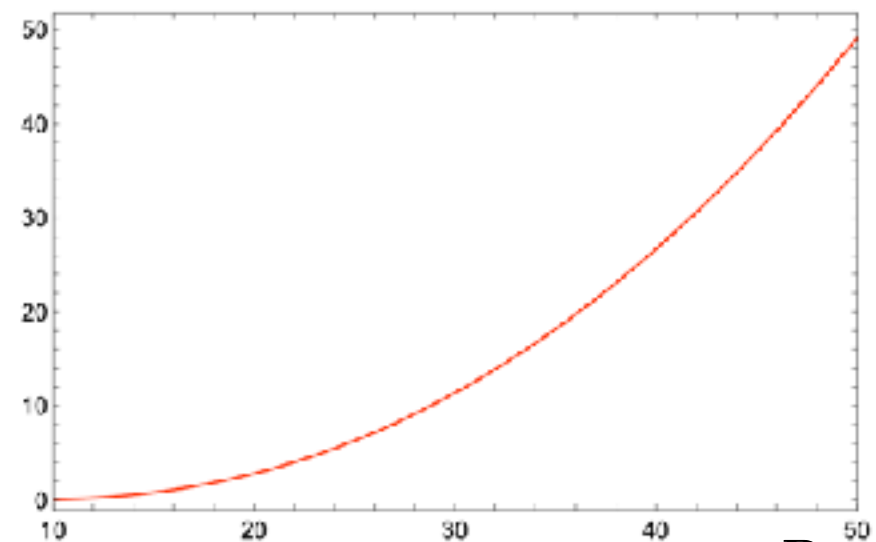
$\#\{R_n \leq R\}$



$\approx$



$$\sum_{n\geq 1}\frac{1}{2\cosh(\pi R_n)}\,\frac{\nu_n^m(y_1)\nu_n^m(y_2)}{\|\nu_n\|^2} \;\longrightarrow\; \int_0^\infty dR\,\bar{\mu}(R)\,(\cdots)K_{iR}(2\pi m y_1)K_{iR}(2\pi m y_2)$$

Topological expansion

[Boruch/Di Ubaldo/**FH**/Perlmutter/Rozali (to appear)]

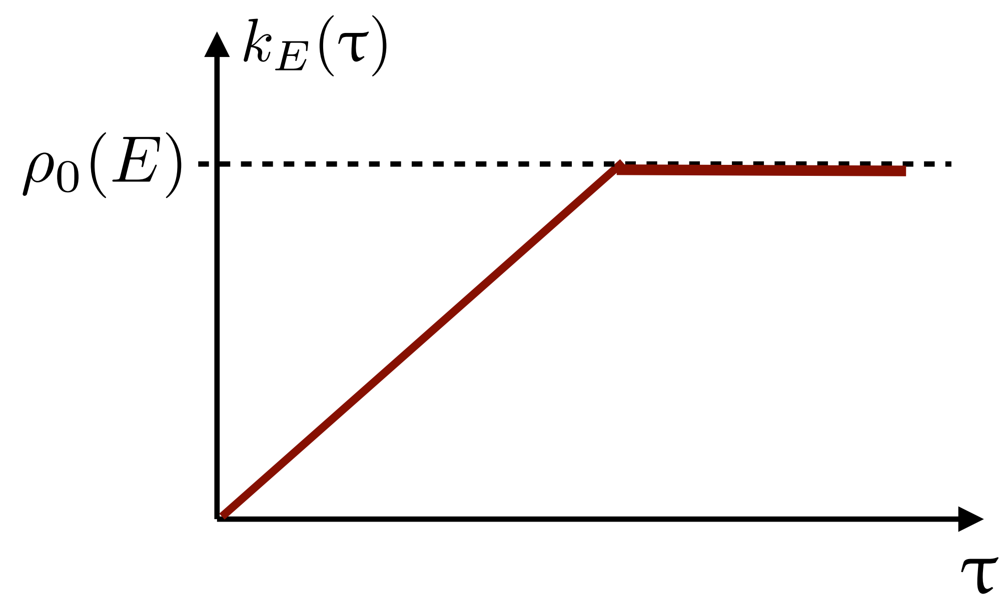
- So far we discussed the "linear ramp": $\mathrm{SFF}_\beta(T) \propto \dfrac{T}{\beta} + [\text{modular completion}]$

- RMT universality says more: [Mehta, Gaudin, Dyson, ...]

$$\mathrm{SFF}_\beta(T) = \int_0^\infty \frac{dE}{2\pi}\, e^{-2\beta E}\, k_E(T)$$

e.g., GUE with $T \to \infty$ $(\tau = Te^{-S_0}$ fixed$)$:

$$k_E(\tau) = \begin{cases} \tau & \text{for} \quad \tau \leq \rho_0(E) \\ \rho_0(E) & \text{for} \quad \tau \geq \rho_0(E) \end{cases}$$



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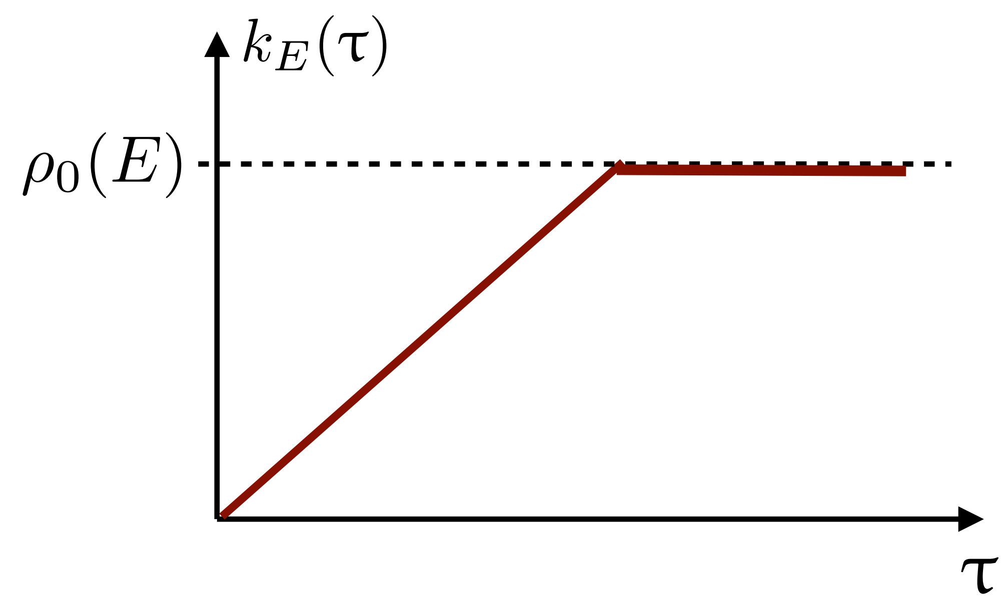
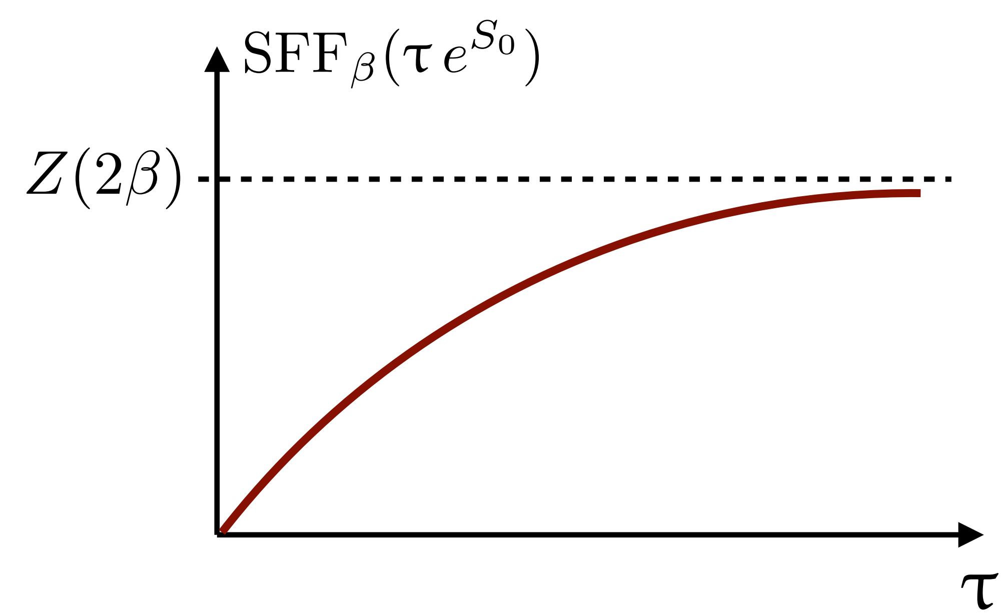


$$\mathrm{SFF}_\beta(T) = \frac{T}{4\pi\beta} + \sum_{g\geq 1} c_g(\rho_0;\beta)\, \tau^{2g+1}$$

"genus expansion"

(non-perturbative e^{-S_0} corrections to leading ramp)



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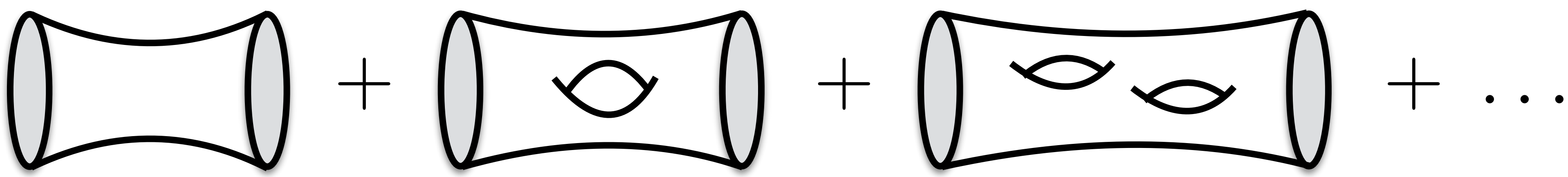
- Understood in (1+1)-diml. gravity dual to RMT

[Saad/Shenker/Stanford '19] ... [Saad/Stanford/Yang/Yao '22] [Blommaert/Kruthoff/Yao '22] ...



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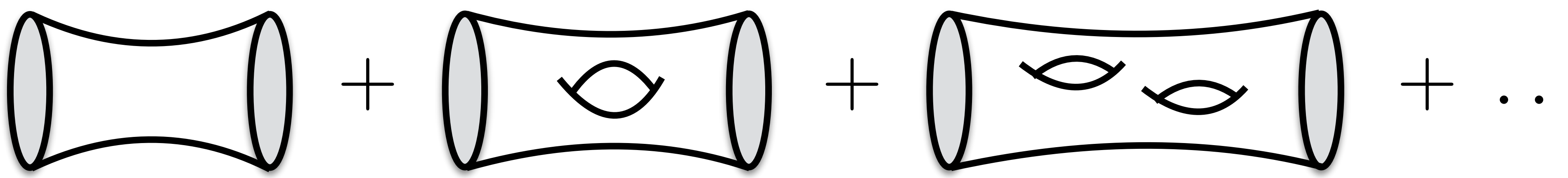

- In modular invariant CFT, expect:

$$\mathrm{SFF}_\beta(T) = \underbrace{\left(\frac{T}{4\pi\beta} + [\text{modular completion}]\right)}_{\text{diagonal spectral correlations}} + \sum_{g\geq 1} \underbrace{\left(c_g(\rho_0;\beta) + [\text{modular completion}]\right)\tau^{2g+1}}_{\text{off-diagonal?}}$$

$$\langle z_{\frac{1}{2}+i\alpha_1} z_{\frac{1}{2}+i\alpha_2}\rangle \sim \left(e^{-\pi\alpha_1} + \ldots\right) \times 4\pi\delta(\alpha_1+\alpha_2)$$

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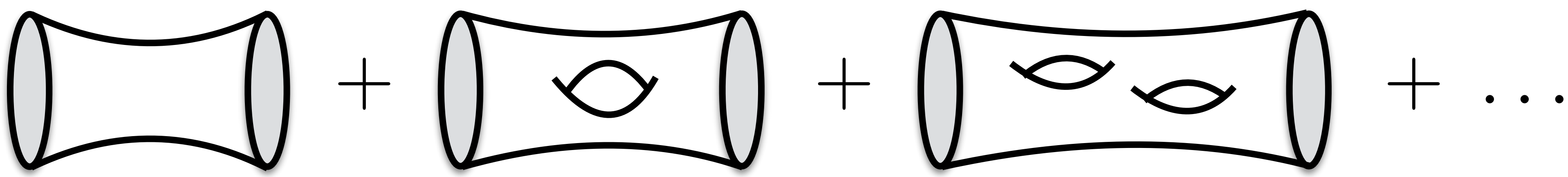

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- Two steps:
  1. Define topological expansion in 2d (for general $\rho_0(E)$)
  2. Apply to $\rho_0(E)$ describing irrational CFT

Toy model: GUE Airy RMT2

- Consider $\rho(E) = \sqrt{E}\, e^{S_0}$
- In this case, the genus expansion is well-known:

$$\langle \widetilde{Z}_P^0(y_1)\widetilde{Z}_P^0(y_2)\rangle = \frac{1}{2\pi}\frac{y_1 y_2}{y_1+y_2} + \sum_{g=1}^{\infty}\frac{(-1)^g}{2\pi g!(2g+1)e^{2gS_0}}(y_1+y_2)^{g-1}(y_1 y_2)^{g+1}$$

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- Claim: this can be uplifted to a modular invariant expression:

$$\langle \widetilde{Z}_P(\tau_1)\widetilde{Z}_P(\tau_2)\rangle = \frac{1}{(4\pi i)^2}\iint_{\frac{1}{2}+i\mathbb{R}} ds_1 ds_2\, \langle z_{s_1} z_{s_2}\rangle_{\tau\text{-GUE Airy}}\, E_{s_1}(\tau_1)E_{s_2}(\tau_2) + [\text{cusp forms}]$$

$$\langle z_{\frac{1}{2}+i\alpha_1} z_{\frac{1}{2}+i\alpha_2}\rangle_{\tau\text{-GUE Airy}} = \frac{e^{-2i\alpha_+}}{\alpha_+(2\alpha_+-i)} \times \Gamma(\cdots)\,\Gamma(\cdots)$$

$$\alpha_+ \equiv \frac{1}{3}(\alpha_1+\alpha_2),\ \alpha_- \equiv \frac{1}{3}(\alpha_1-\alpha_2)$$

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  - Check (and derivation): $E_s(\tau) \mapsto y^s$ reproduces $(\bigstar)$ …

    … and everything else is subleading

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- Fully resummed genus expansion encoded in analytic structure
- To develop genus expansion, evaluate $\alpha_\pm$ integrals by contour deformation

poles: $\alpha_+ = 0 \longrightarrow$ ramp (g=0)

$\alpha_+ = -ig \longrightarrow$ genus g contribution $\sim e^{-2gS_0}$

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<u>Outlook:</u>

- We have a systematic way to produce $\langle z_{s_1} z_{s_2}\rangle$ given a spectral curve $\rho(E)$
- Defines the modular invariant uplift of universal RMT
- Minimal modular invariant uplift of g=0 RMT => AdS$_3$ wormhole.
  What do we learn about gravity from minimal uplift of g>0 terms?

Summary

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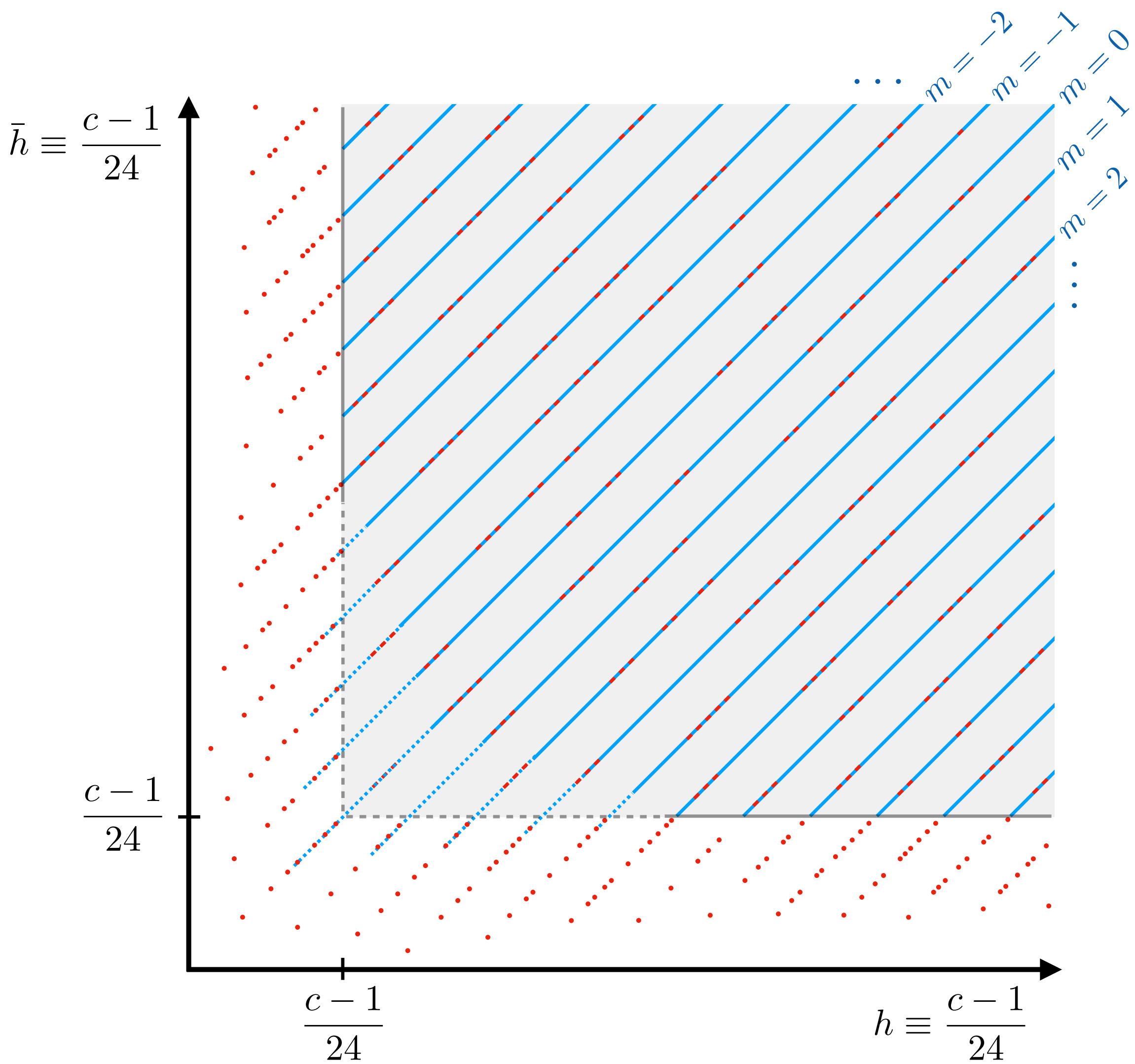
- Explore assumption of RMT universality in 2d CFT
- Tool: mod. inv. $SL(2,\mathbb{Z})$ spectral decomposition
- Kuznetsov trace formula determines subleading corrections, required by modular invariance
  - Linear ramp: diagonal correlations on $L^2(\mathcal{F}) \times L^2(\mathcal{F})$
  - **Minimal modular completion**: $\mathbb{T}^2 \times I$ wormhole
- Spinning ramps encode statistical information about arithmetic chaos of Maass cusp forms. For AdS_3 wormhole this information is **maximal**.
- Beyond the diagonal: plateau, GOE ramp, genus expansion, …

  —> general definition of **modular invariant "RMT_2"**

More details

Definition of $\widetilde{Z}_P(\tau)$



1. Primary partition function:

$$Z_{\rm P}(\tau \equiv x+iy) = y^{1/2}|\eta(\tau)|^2\, Z(\tau)$$

2. Consider states above "black hole threshold": $\min(h,\bar{h}) > \dfrac{c-1}{24}$

   - Remove states below threshold and their 'modular completion' [Benjamin/Collier/Fitzpatrick/Maloney/Perlmutter '21]

$$\widetilde{Z}_{\rm P} = Z_{\rm P} - \hat{Z}_{\rm light}$$

e.g.: $\hat{Z}_{\rm light} = \displaystyle\sum_{\gamma\in SL(2,\mathbb{Z})/\Gamma_\infty} Z_{\rm light}(\gamma\tau)\,, \quad Z_{\rm light} = \sum_{\substack{h,\bar{h}:\\ \min(h,\bar{h})\leq \frac{c-1}{24}}} q^{h-\frac{c-1}{24}}\,\bar{q}^{\bar{h}-\frac{c-1}{24}}$

3. Fix spin $m = h-\bar{h}$: $\boxed{\widetilde{Z}^m_{\rm P}(y)} \equiv \displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} dx\, \widetilde{Z}_{\rm P}(x+iy)\, e^{2\pi i m x}$

Hecke algebra

$$T_m f(\tau) = \frac{1}{\sqrt{m}} \sum_{\substack{a,b,d: \\ ad=m \\ 0\leq b\leq d-1}} f\left(\frac{a\tau+b}{d}\right) \qquad [T_m,\, \Delta_{\mathcal{F}}] = 0$$

$$\Rightarrow \qquad T_m \nu_n(\tau) = a_m^{(n)}\, \nu(\tau) \qquad T_m E_{s=\frac{1}{2}+i\alpha}(\tau) = \frac{1}{2}\, a_m^{(\alpha)}\, \nu(\tau)$$

- Many constraints between “random” Fourier coefficients

$$a_m^{(n)} a_{m'}^{(n)} = \sum_{\substack{\ell|(m,m') \\ \ell>0}} a_{\frac{mm'}{\ell^2}}^{(n)}$$

  - $a_p^{(n)}$ for prime spins *p* determine all other $a_m^{(n)}$

e.g.: $a_p^{(n,\pm)} a_{p'}^{(n,\pm)} = a_{pp'}^{(n,\pm)} \quad (p \neq p' \text{ prime})$

Approximate sum over cusp forms

- Spinning ramp is encoded in “random sum of cusp forms”! How??

$$\frac{1}{\pi}\frac{y_1 y_2}{y_1+y_2}\, e^{-2\pi m(y_1+y_2)} + \ldots \;=\; \sum_{n\geq 1} \frac{1}{2\,\cosh(\pi R_n)}\, \frac{\nu_n^m(y_1)\nu_n^m(y_2)}{\|\nu_n\|^2}$$

‘arithmetic chaos’

‘ramp’
(quantum chaos)

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$$= \sum_{n\geq 1} \frac{4}{L^{(n)}_{\nu\times\nu}(1)}\, \nu_n^m(y_1)\nu_n^m(y_2)$$

$$L^{(n)}_{\nu\times\nu}(s) = \frac{\zeta(2s)}{\zeta(s)} \sum_{m\geq 1} \frac{\left(a_m^{(n)}\right)^2}{m^s}$$

$$= \prod_{p\ \mathrm{prime}} \frac{1}{1-\left(a_p^{(n)}\right)^2\left(p^{-s}-p^{-2s}\right)+\left(p^{-s}-p^{-2s}-p^{-3s}\right)}$$

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$$= \sum_{n\geq 1} 4 \prod_{p \text{ prime}} \left[1-\left(a_p^{(n)}\right)^2\left(p^{-1}-p^{-2}\right)+\left(p^{-1}-p^{-2}-p^{-3}\right)\right] \times \left(a_m^{(n)}\right)^2 \sqrt{y_1 y_2}\, K_{iR_n}(2\pi m y_1) K_{iR_n}(2\pi m y_2)$$

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- Large $y_{1,2}$: larger R_n dominate

$$\sum_{n\geq 1} f(R_n) \approx \int_{R_{\min}}^{\infty} dR\, \bar{\mu}(R)\, f(R)$$

- R_n become very dense

$$\sum_{n\geq 1} f\left(R_n, (a_m^{(n)})^2\right) \approx \int_{R_{\min}}^{\infty} dR\, \bar{\mu}(R)\, f\left(R, \overline{(a_m^{(n)})^2}\right)$$

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- **Coarse-grain** over sum of cusp forms! In particular:

$$\sum_{n=n_{\min}}^{n_{\max}}\left\{\prod_{p\text{ prime}}\left[1-(a_p^{(n)})^2(p^{-1}-p^{-2})+(p^{-1}-p^{-2}-p^{-3})\right]\times(a_m^{(n)})^2\right\}\approx\frac{6}{\pi^2}$$

- Asymptotically exact! Contains **full information about statistical distribution** of Fourier coefficients $a_m^{(n)}$ for all spins [2309.00611]

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  - L-function is spin-independent but contains info about all spins “democratically” —> $\langle z_{n_1} z_{n_2} \rangle$ gives all independent spinning ramps

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  - Asymptotically exact! Contains **full information about statistical distribution** of Fourier coefficients $a_m^{(n)}$ for all spins [2309.00611]
  - For large *n*: $\frac{6}{\pi^2}$ is required in order to get the universal ramp
  - Corrections to $\frac{6}{\pi^2}$: theory-dependent subleading corrections

- Spinning ramp is encoded in “random sum of cusp forms”! How??

$$\frac{1}{\pi}\frac{y_1y_2}{y_1+y_2}e^{-2\pi m(y_1+y_2)}+\ldots=\sum_{n\geq 1}4\prod_{p\text{ prime}}\left[1-\left(a_p^{(n)}\right)^2\left(p^{-1}-p^{-2}\right)+\left(p^{-1}-p^{-2}-p^{-3}\right)\right]\times\left(a_m^{(n)}\right)^2\sqrt{y_1y_2}\,K_{iR_n}(2\pi m y_1)K_{iR_n}(2\pi m y_2)$$

$$\approx\int dR\,\bar{\mu}(R)\,4\times\frac{6}{\pi^2}\times\sqrt{y_1y_2}K_{iR}(2\pi m y_1)K_{iR}(2\pi m y_2)$$

- **Coarse-grain** over sum of cusp forms! In particular:

$$\sum_{n=n_{\min}}^{n_{\max}}\left\{\prod_{p\text{ prime}}\left[1-(a_p^{(n)})^2(p^{-1}-p^{-2})+(p^{-1}-p^{-2}-p^{-3})\right]\times(a_m^{(n)})^2\right\}\approx\frac{6}{\pi^2}$$

  - Asymptotically exact! Contains **full information about statistical distribution** of Fourier coefficients $a_m^{(n)}$ for all spins [2309.00611]

  - L-function is spin-independent but contains info about all spins “democratically” —> $\langle z_{n_1} z_{n_2}\rangle$ gives all independent spinning ramps

Average over cusp form data

$$\sum_{n\geq 1}\left\{\prod_{p\ \mathrm{prime}}\left[1-(a_p^{(n)})^2(p^{-1}-p^{-2})+(p^{-1}-p^{-2}-p^{-3})\right]\times(a_m^{(n)})^2\right\}=\frac{6}{\pi^2}$$

Proof:

- Prime decomposition:

$$m=p_1^{k_1}\cdots p_r^{k_r}\qquad\Rightarrow\qquad\left(a_m^{(n)}\right)^2=\left(a_{p_1^{k_1}}^{(n)}\right)^2\cdots\left(a_{p_r^{k_r}}^{(n)}\right)^2$$

- Hecke algebra:

$$a_m^{(n)}a_{m'}^{(n)}=\sum_{\substack{\ell|(m,m')\\ \ell>0}}a^{(n)}_{\frac{mm'}{\ell^2}}$$

e.g.:

$$a^{(n)}_{p_1^{k_1}\cdots p_r^{k_r}}=a^{(n)}_{p_1^{k_1}}\cdots a^{(n)}_{p_r^{k_r}}$$

$$a^{(n)}_{p^k}=a^{(n)}_{p^{k-1}}\,a^{(n)}_p-(1-\delta_{k,1})\,a^{(n)}_{p^{k-2}}$$

- Distributions of prime spin coefficients:

$$\boldsymbol{\mu}_p(a)=\begin{cases}\dfrac{(p+1)\sqrt{4-a^2}}{2\pi\left(\left(p^{1/2}+p^{-1/2}\right)^2-a^2\right)} & \text{if } |a|<2\\ 0 & \text{otherwise}\end{cases}$$

E.g. if m prime: only need $\overline{\left(a_p^{(n)}\right)^2}=\dfrac{p+1}{p}$, $\overline{\left(a_p^{(n)}\right)^4}=\dfrac{2p^2+3p+1}{p^3}$

For generic m : need all moments of $\boldsymbol{\mu}_p(a)$

A generalization

$$\delta_{m_1 m_2} \frac{\sqrt{y_1 y_2}}{\pi^2} \int_{\mathbb{R}} d\alpha\, \alpha \tanh(\pi\alpha)\, g(\alpha)\, K_{i\alpha}(2\pi m_1 y_1) K_{i\alpha}(2\pi m_2 y_2) + \mathcal{G}_{m_1 m_2}[g]$$

$$= \int_{\mathbb{R}} \frac{d\alpha}{4\pi} \frac{g(\alpha)}{2\cosh(\pi\alpha)} E^{m_1}_{\frac{1}{2}+i\alpha}(y_1) E^{m_2}_{\frac{1}{2}+i\alpha}(y_2) + \sum_{n\geq 1} \frac{g(R_n)}{2\cosh(\pi R_n)} \frac{\nu_n^{m_1}(y_1)}{\|\nu_n\|} \frac{\nu_n^{m_2}(y_2)}{\|\nu_n\|}$$

- Pure gravity wormhole: $g(\alpha) = 1 \quad \Rightarrow \quad \mathrm{LHS} = [\mathrm{ramp}] + \mathcal{G}_{m_1 m_2}$
- Narain theories: D free lattice bosons with $U(1)^D \times U(1)^D$ symmetry

[Maloney/Witten '20][Afkhami-Jeddi/Cohn/Hartman/Tajdini '20]

$$Z_{\mathrm{P(D)}} = y^{D/2} |\eta(x+iy)|^{2D} Z_{\mathrm{Narain}}$$

$$(y_1 y_2)^{-D/2} \langle Z^{m_1}_{\mathrm{P(D)}}(y_1) Z^{m_2}_{\mathrm{P(D)}}(y_2) \rangle \Big|_{y_{1,2}\to\beta\pm iT} \propto \beta^{-D/2} e^{-4\pi|m_1|\beta} \delta_{m_1 m_2}$$

plateau! (no ramp)

spectral decomposition of SFF follows from $g(\alpha) \propto \dfrac{\left|\Gamma\left(\frac{D-1}{2} + i\alpha\right)\right|^2}{\left|\Gamma\left(\frac{1}{2} + i\alpha\right)\right|^2}$