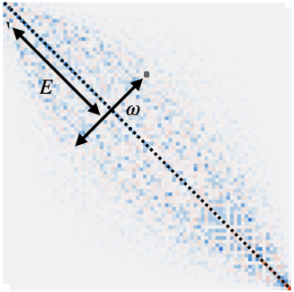
Eigenstate thermalization and free probability



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Quantum chaos @ Bernoulli

Outline

- "Toy ETH": a RMT perspective
- ETH in many-body systems
- Connections with free probability

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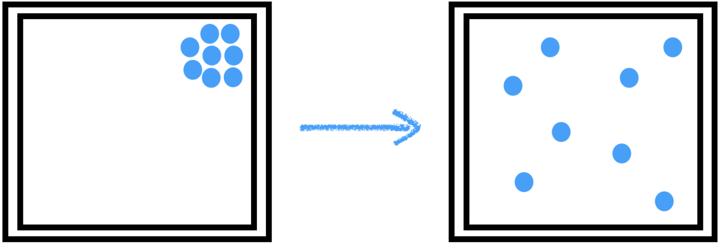
Context

ETH explains equilibration of *isolated quantum many-body* systems



In our work: need to characterise better ETH ansatz, all about dynamics *at equilibrium*

Dynamics and ETH

Heisenberg picture (evolution of the operators):

$$A(t) = e^{iHt} A e^{-iHt} = \sum_{ij} e^{i(E_i - E_j)t} A_{ij} |E_i\rangle\langle E_j|$$

$$A_{ij} = \langle E_i | A | E_j \rangle$$

Look at matrix elements of observables in the basis of the energy

Characterise them “statistically"

Toy ETH

(Diagonal) matrix Λ (observable) of size $\mathscr{D}$

Matrix elements in a random basis $A = O\Lambda O^T$

$$\overline{A_{ii}} = \frac{1}{\mathscr{D}}\sum_i \lambda_i = m_1^A = \kappa_1^A$$

$$\overline{A_{ij}} = 0 \qquad i \neq j$$

$$\overline{A_{ij}^2} = \frac{1}{\mathscr{D}}\left[\frac{1}{\mathcal{N}}\sum_i \lambda_i^2 - \left(\frac{1}{\mathcal{N}}\sum_i \lambda_i\right)^2\right] = \frac{1}{\mathscr{D}}\left[m_2^A - (m_1^A)^2\right] = \frac{1}{\mathscr{D}}\kappa_2^A$$

Toy ETH: ansatz

$$A_{ij} = \kappa_1^A \delta_{ij} + \sqrt{\frac{\kappa_2^A}{\mathscr{D}}} R_{ij}$$

$$\overline{R_{ij}} = 0 \qquad \overline{R_{ij}^2} = 1$$

No info about correlations

$$\overline{A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_p i_1}} \simeq \frac{1}{\mathscr{D}^{(p-1)}} \kappa_p^A \qquad i_1 \neq i_2 \neq \dots \neq i_p$$

Free cumulants for one matrix

Stieltjes transform: generator of the moments

$$S(z) = \left\langle \frac{1}{z - A} \right\rangle = \frac{1}{z} \sum_{n=0}^{\infty} \frac{m_n^A}{z^n}$$

$\mathscr{R}$-transform: generator of free cumulants

$$\mathscr{R}(z) = S^{-1}(z) - \frac{1}{z} = \sum_{n=0}^{\infty} \kappa_{n+1}^A z^n$$

Free cumulants for one matrix

Free cumulants in terms of moments $m_n^A = \frac{1}{\mathscr{D}} \mathrm{Tr} A^n$:

$$\kappa_1^A = m_1^A$$

$$\kappa_2^A = m_2^A - (m_1^A)^2$$

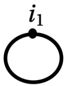
$$\kappa_3^A = m_3^A - 3m_2^A m_1^A + 2(m_1^A)^3$$

$$\kappa_4^A = m_4^A - 4m_3^A m_1^A - 2(m_2^A)^2 + 10m_2^A (m_1^A)^2 - 5(m_1^A)^4$$

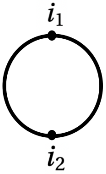
Differ from standard cumulants from order $n = 4$ on

(see later about free probability)

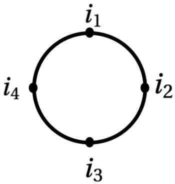
Toy ETH: diagrams



$$\overline{A_{i_1 i_1}} = \kappa_1^A$$



$$\overline{|A_{i_1 i_2}|^2} = \frac{1}{\mathcal{D}} \kappa_2^A \quad i_1 \neq i_2$$

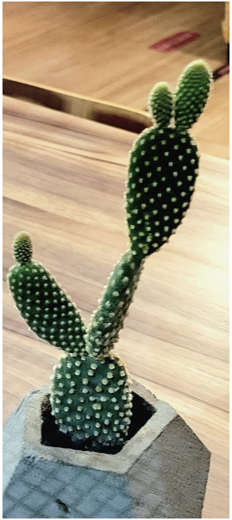


$$\overline{A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4} A_{i_4 i_1}} = \frac{1}{\mathcal{D}^{p-1}} \kappa_p^A$$

$$i_1 \neq i_2 \neq i_3 \neq i_4 \quad p = 4$$

Toy ETH: ansatz

Possible to characterise all diagrams relevant for moments in the large system size



Toy ETH: moments

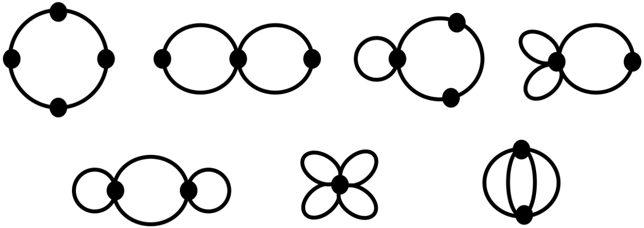
$$\frac{1}{\mathscr{D}}\mathrm{Tr}A^4 = \frac{1}{\mathscr{D}}\sum_{i_1\neq i_2\neq i_3\neq i_4} A_{i_1 i_2}A_{i_2 i_3}A_{i_3 i_4}A_{i_4 i_1} + \frac{2}{\mathscr{D}}\sum_{i_1\neq i_2\neq i_3} A_{i_1 i_2}A_{i_2 i_1}A_{i_1 i_3}A_{i_3 i_1}$$

$$+\frac{4}{\mathscr{D}}\sum_{i_1\neq i_2\neq i_3} A_{i_1 i_1}A_{i_1 i_2}A_{i_2 i_3}A_{i_3 i_1} + \frac{4}{\mathscr{D}}\sum_{i_1\neq i_2} A_{i_1 i_1}A_{i_1 i_2}A_{i_2 i_1}A_{i_1 i_1}$$

$$+\frac{2}{\mathscr{D}}\sum_{i_1\neq i_2} A_{i_1 i_1}A_{i_1 i_2}A_{i_2 i_2}A_{i_2 i_1} + \frac{1}{\mathscr{D}}\sum_{i_1} A^4_{i_1 i_1} + \frac{1}{\mathscr{D}}\sum_{i_1\neq i_2} A_{i_1 i_2}A_{i_2 i_1}A_{i_1 i_2}A_{i_2 i_1}$$

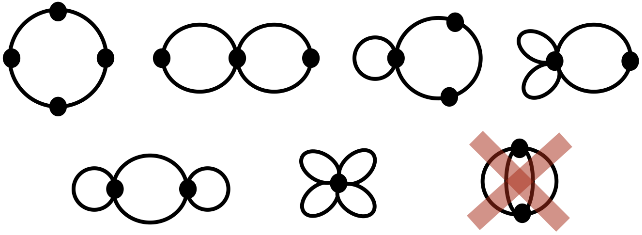
Toy ETH: moments and diagrams

$$\frac{1}{\mathcal{D}}\mathrm{Tr}A^4 =$$



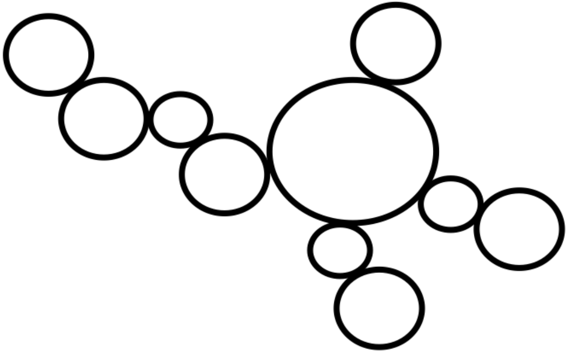
Toy ETH: moments and diagrams

$$\frac{1}{\mathcal{D}}\mathrm{Tr}A^4 =$$



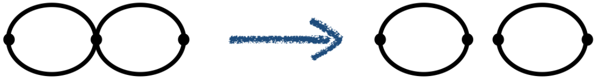
Only cacti dominate in the large size limit

A cactus



(of high order ...)

Cactus diagrams and free cumulants



$$E(Cactus) \sim \prod_{i=1}^{\#loops} \kappa_{n_i}$$

Free cumulants natural building block

Maillard et al, J. Stat Mech. (2019)
C. Male et al., Traffic

A simple proof

… based on small rank HCIZ integral:

$$\mathscr{I}_\beta(A,B) = \int_{G(N)} D\Omega \, e^{\frac{\beta N}{2} \mathrm{Tr} B\Omega A\Omega^\dagger}$$

over (flat) Haar measure of the compact group
$\Omega \in G(N) = O(N), U(N)$ or $Sp(N)$.

In the small rank limit this integral related to the $\mathscr{R}$-transform

Why (free) cumulants?

In classical probability

- Cumulants distinguish Gaussian distribution ($c_n = 0 \ \forall n > 2$) from the rest
- For *independent* variables cumulants are additive:

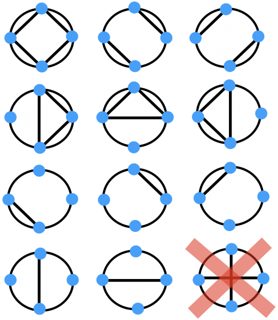
$$Z = X + Y \qquad c_Z = c_X + c_Y$$

Analogously free cumulants for matrices
But in classical probability much easier to write the generating function for the multivariate case and define mixed cumulants.

Non-crossing partitions

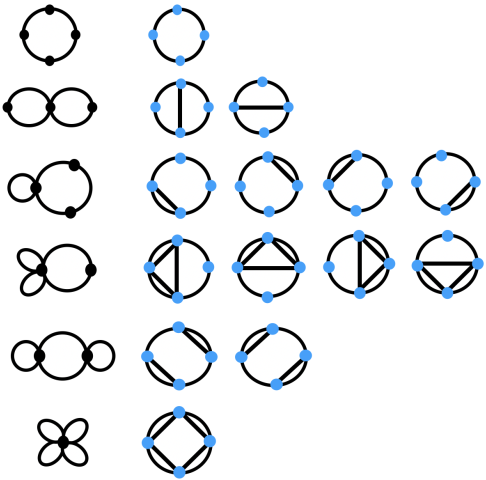
In free probability NCP characterise moment-cumulant formula:

$$m(A_1, \ldots, A_p) = \sum_{\pi \in NCP} \prod_{\substack{V \in \pi \\ V=(i_1,\ldots,i_l)}} \kappa_l(A_{i_1}, \ldots, A_{i_l})$$

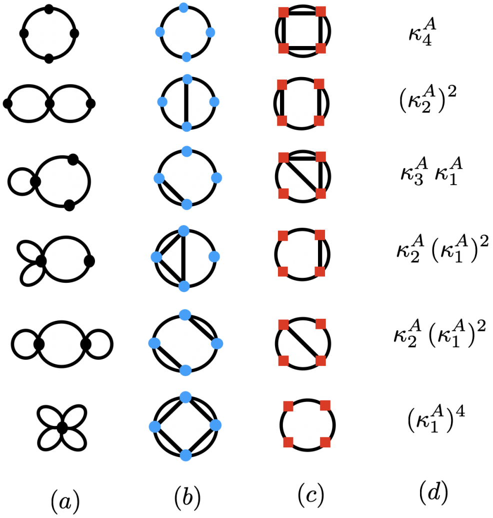


Speicher (1994)

Cacti and non-crossing partitions



… and the dual lattice of NCP



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- "Toy ETH": a RMT perspective
- ETH in many-body systems
- Connections with free probability

The many-body problem

E.g. H spin chain, physical observable:

$$A = \sigma_i^z$$

H and A matrices of size $\mathcal{N} = 2^N$

$$\rho(E = Ne) = \sum_{\alpha=1}^{\mathcal{N}} \delta(E - E_i) \propto e^{S(E)} \simeq e^{Ns(e)}$$

Level spacing exponentially small!

Equilibrium correlation functions (Moments)

$$\mathrm{Tr}\left[\rho_\beta A(t) A(0)\right] \underbrace{=}_{\text{Time translation invariance}} \mathrm{Tr}\left[\rho_\beta A(t+\tau) A(\tau)\right]$$

Density matrix (in general no cyclic invariance)

$$\rho_\beta = \frac{e^{-\beta H}}{Z} \qquad Z = \sum_{\alpha=1}^{\mathcal{N}} e^{-\beta E_\alpha}$$

- $\rho_\beta \geq 0$
- $\mathrm{Tr}\rho_\beta = 1$
- Peaked function at some characteristic energy e_β fixed by β

Randomness

Perturbed Hamiltonian:

$$H + \lambda_N V$$

Non perturbative corrections by nearby states

$$|E_n\rangle = |E_n^{(0)}\rangle + \lambda_N \sum_{m \neq n} \frac{\langle E_m^{(0)}|V|E_n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}} |E_m^{(0)}\rangle + \ldots$$

But no change in physics

Randomness

$$U = \begin{pmatrix} U^1 & 0 & 0 & \dots & 0 \\ 0 & U^2 & 0 & \dots & 0 \\ 0 & 0 & U^3 & \dots & 0 \\ & \vdots & & & \\ 0 & 0 & 0 & \dots & U^n \end{pmatrix}$$

Toy-ETH too simple. No full rotational invariance of the basis that we choose: $\rightarrow$ information on energy indices. But hope diagrammatic survives.

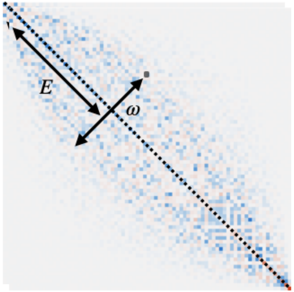
Eigenstate thermalization

Single eigenstates provide equilibrium statistical averages
$\langle E_i | A | E_i \rangle$ varies smoothly with the energy E_i

For dynamics necessary off-diagonal matrix elements

J. Deutsch (1991), M. Srednicki (1994)
Review: D'Alessio, Kafri, Polkovnikov, Rigol (2016)

Eigenstate thermalization ansatz



$$A_{ij} = \mathscr{A}(e)\delta_{ij} + e^{-Ns(e)/2} f_e(\omega) R_{ij}$$

$$E = (E_i + E_j)/2 \quad e = E/N \quad \omega = E_i - E_j$$

R_{ij} (pseudo)-random numbers

$$\overline{R_{ij}} = 0 \quad \overline{R^2_{ij}} = 1$$

M. Srednicki (1999)

One and two-time correlation functions

$$\langle A \rangle_\beta \xrightarrow{N\to\infty} \mathscr{A}(e_\beta)$$

$$\langle A(t)A(0)\rangle_\beta - \langle A\rangle_\beta^2 \xrightarrow{N\to\infty} \int \mathrm{d}\omega \; e^{-\beta\omega/2} e^{i\omega t} |f_{e_\beta}(\omega)|^2$$

Self-averaging assumption w.r.t. "fictitious ensemble"

One and two-point function independent of correlations between different matrix elements

Multi-point correlation functions

$$C_4^{\beta}(t_1, t_2, t_3) = \mathrm{Tr}\left[\rho_\beta A(t_1) A(t_2) A(t_3) A(0)\right]$$

$C_4^{\beta}(t, 0, t)$ Out-of-Time-Order Correlator

"quantum Lyapunov exponent"

Larkin and Ovchinikov (1969)
Kitaev (2015)
Maldacena, Shenker and Stanford (2016)

Multi-point correlation functions

In the energy eigenbasis

$$C_p^\beta(t_1,\ldots,t_{p-1}) = \sum_{i_1,\ldots,i_p}\left[\frac{e^{-\beta E_{i_1}}}{Z} A_{i_1 i_2}(t_1) A_{i_2 i_3}(t_2)\ldots A_{i_p i_1}(0)\right]$$

For any $p > 2$ products of different matrix elements!

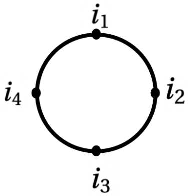
Argument for correlations

$|f_e(\omega)|^2$ Fourier transform of $C_2^\beta(t)$

A_{ij} independent variables $\rightarrow$ all multi-point functions determined solely by $f_e(\omega)$, i.e. by $C_2^\beta(t)$

Unreasonable in general

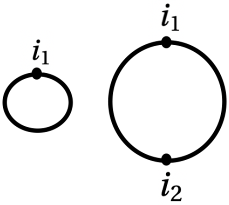
Beyond independent matrix elements



Multipoint functions

$$\overline{A_{i_1 i_2} A_{i_2 i_3} A_{i_3 i_4} A_{i_4 i_1}} \quad \text{for} \quad i_1 \neq i_2 \ldots \neq i_n$$

$$\propto f_e^{(4)}(\omega_1, \omega_2, \omega_3) \to C_4^{\beta}(t_1, t_2, t_3)$$



Same spirit as usual ETH (and toy ETH)

$$f_e^{(1)} = \mathscr{A}(e) \qquad f_e^{(2)}(\omega) = |f_e(\omega)|^2$$

Generalized ETH

$$\overline{A_{i_1 i_2} A_{i_2 i_3} \dots A_{i_n i_1}} \simeq e^{-(n-1)Ns(e)} f_e^{(n)}(\omega_1, \dots, \omega_{n-1})$$

for $i_1 \neq i_2 \dots \neq i_n$ $\quad e = \frac{1}{n}\sum_{k=1}^{n} e_{i_k}$ $\quad \omega_k = E_{i_k} - E_{i_{k+1}}$

+ same diagrams toy ETH (factorising cacti)

$$\overline{A_{i_1 i_2} A_{i_2 i_1} A_{i_1 i_3} A_{i_3 i_1}} \simeq \overline{A_{i_1 i_2} A_{i_2 i_1}}\ \overline{A_{i_1 i_3} A_{i_3 i_1}}$$

weakly correlated variables

Foini and Kurchan (2019)

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Moment-cumulant formula

$$C(a_1, \ldots, a_n) = \sum_{\pi \in NCP(n)} \kappa_\pi(a_1, \ldots, a_n)$$

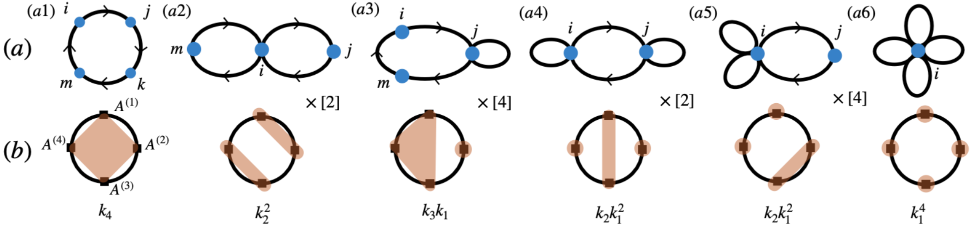
$$\kappa_\pi(a_1, \ldots, a_n) = \prod_{\substack{V \in \pi \\ V=(i_1,\ldots,i_l)}} \kappa_l(a_{i_1}, \ldots, a_{i_l})$$

$C(a_1, \ldots, a_n) \rightarrow$ moments

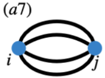
$\kappa_l(a_{i_1}, \ldots, a_{i_l}) \rightarrow$ free cumulants

$NCP(n)$ non-crossing partitions of n

Cactus diagrams and non-crossing partitions



- Each edge carries an index of time (as different matrices)
- ! No cyclic invariance
- Use scaling of $f^{(n)}$ on e and ω's

Neglect 

An example

From FP it is immediate to find the following decomposition:

$$\begin{aligned}\langle A(t_1)A(t_2)A(t_3)A(0)\rangle_\beta &= \kappa_4^\beta(t_1,t_2,t_3)\\ &+\kappa_2^\beta(t_1-t_2)\,\kappa_2^\beta(t_3)+\kappa_2^\beta(t_2-t_3)\,\kappa_2^\beta(t_1)\end{aligned}$$

(assuming $\langle A\rangle_\beta = 0$)

Free cumulant

Random matrix elements of one operator at different times

One matrix vs Infinitely many matrices!

$$\kappa_n^\beta(t_1,\ldots,t_{n-1},0) = \frac{1}{Z}\sum_{i_1\neq i_2\neq\ldots\neq i_n} e^{-\beta E_{i_1}} A_{i_1 i_2}(t_1)\ldots A_{i_n i_1}(0)$$

$$= \int \mathrm{d}\omega_1\ldots\mathrm{d}\omega_{n-1} e^{i\vec{\omega}\cdot\vec{t}} e^{-\beta\vec{\omega}\cdot\vec{l}_n} f_{\epsilon_\beta}^{(n)}(\omega_1,\ldots,\omega_{n-1})$$

$$\vec{l}_n = \left(\tfrac{n-1}{n},\ldots,\tfrac{1}{n}\right)$$

Pappalardi, Foini and Kurchan (2022)

Free cumulant

One matrix vs Infinitely many matrices!

$$\kappa_n^\beta(t_1, \ldots, t_{n-1}, 0) =$$

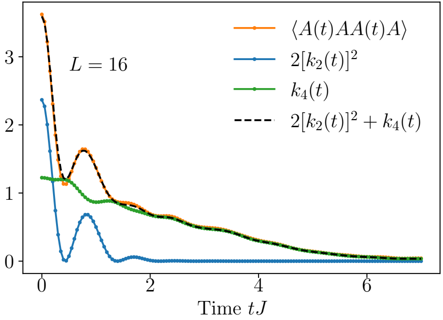
$$\frac{1}{Z} \sum_{i_1 \neq i_2 \neq \ldots \neq i_n} e^{-\beta E_{i_1}} e^{i(E_{i_1} - E_{i_2}) t_1 + \cdots + i(E_{i_{n-1}} - E_{i_n}) t_{n-1}} A_{i_1 i_2} \ldots A_{i_n i_1}$$

At large times free cumulants vanish, matrices mutually free

Pappalardi, Foini and Kurchan (2022)

Numerical verification

$$H = \sum_{l=1}^{N} \sigma_l^x \sigma_{l+1}^x + h_z \sum_{l=1}^{N} \sigma_l^z + h_x \sum_{l=1}^{N} \sigma_l^x$$



Pappalardi, Fritzsch, Prosen (2023)

In summary ...

- Propose a (simple) ansatz able to account for correlations between matrix elements. Relevant for multi-point functions
- Recognise importance of free probability in connection with quantum statistical mechanics

The team

Jorge Kurchan

Silvia Pappalardi

The team

Jorge Kurchan

Silvia Pappalardi

Thank you!