# Thermalization and Chaos in QFT

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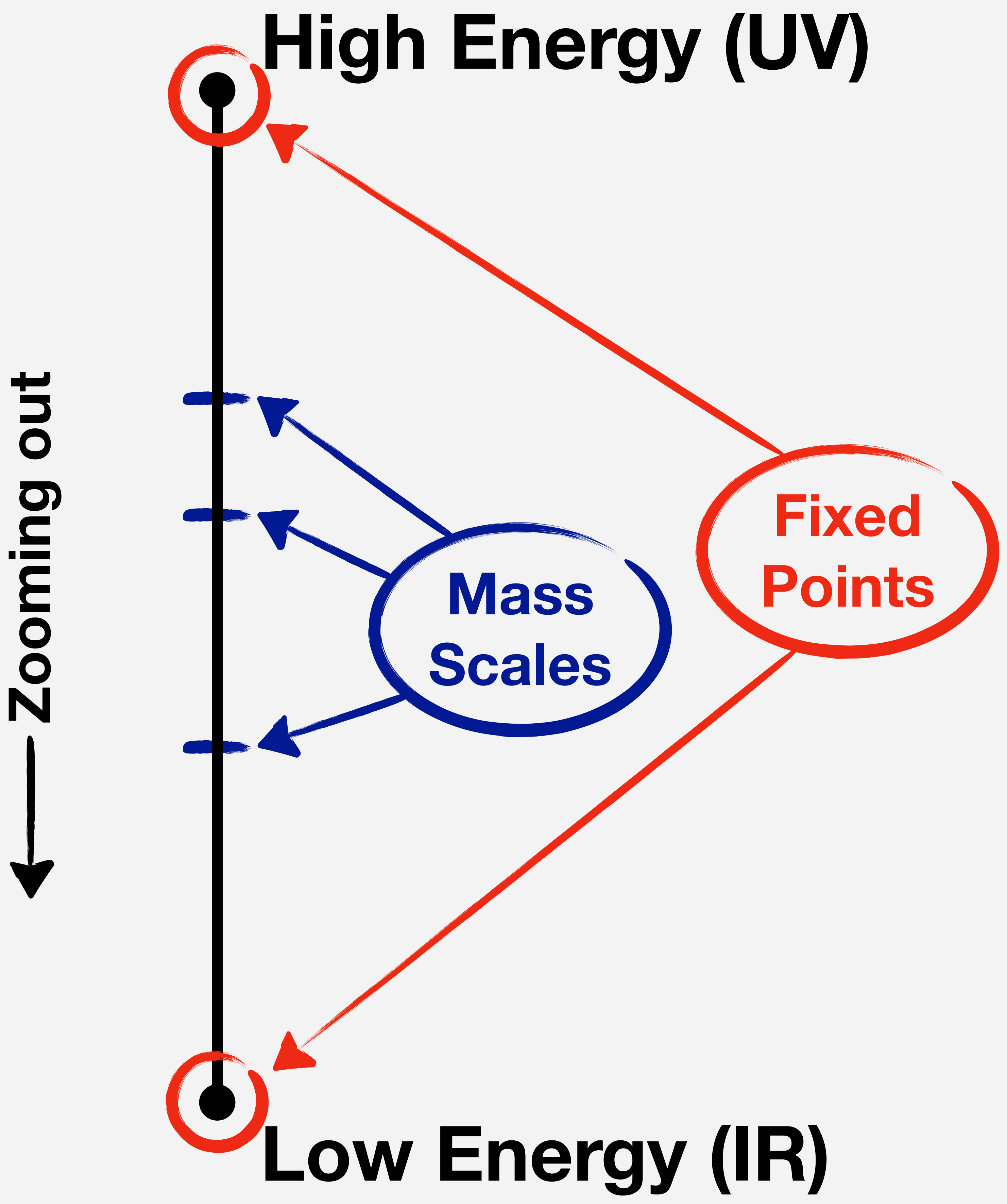
Based on work done in collaboration with:
Luca Delacrétaz, Liam Fitzpatrick, Ami Katz
arXiv: 2105.02229, 2207.11261

# Big Picture

Quantum Mechanics + Special Relativity = **Quantum Field Theory**



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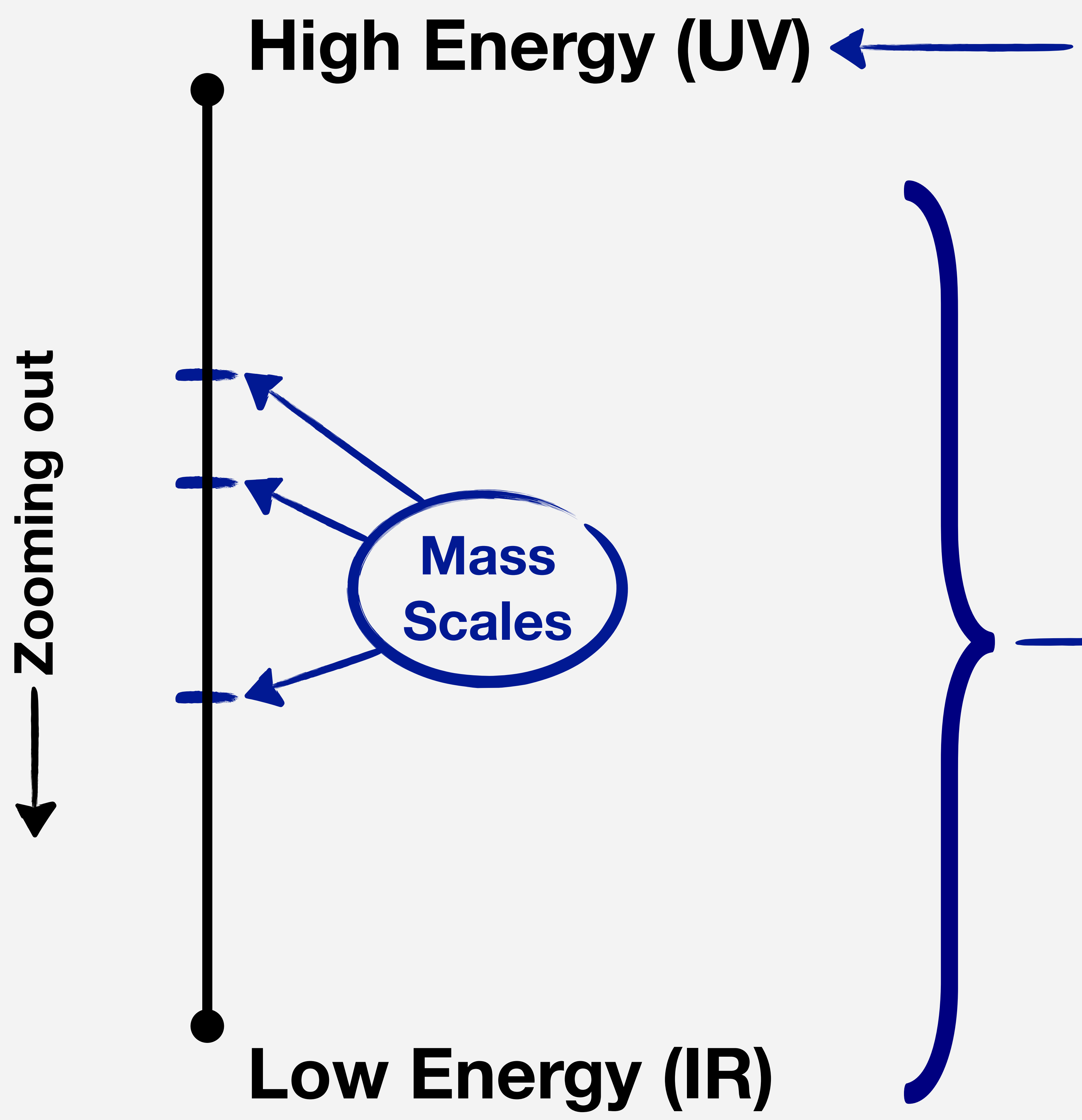


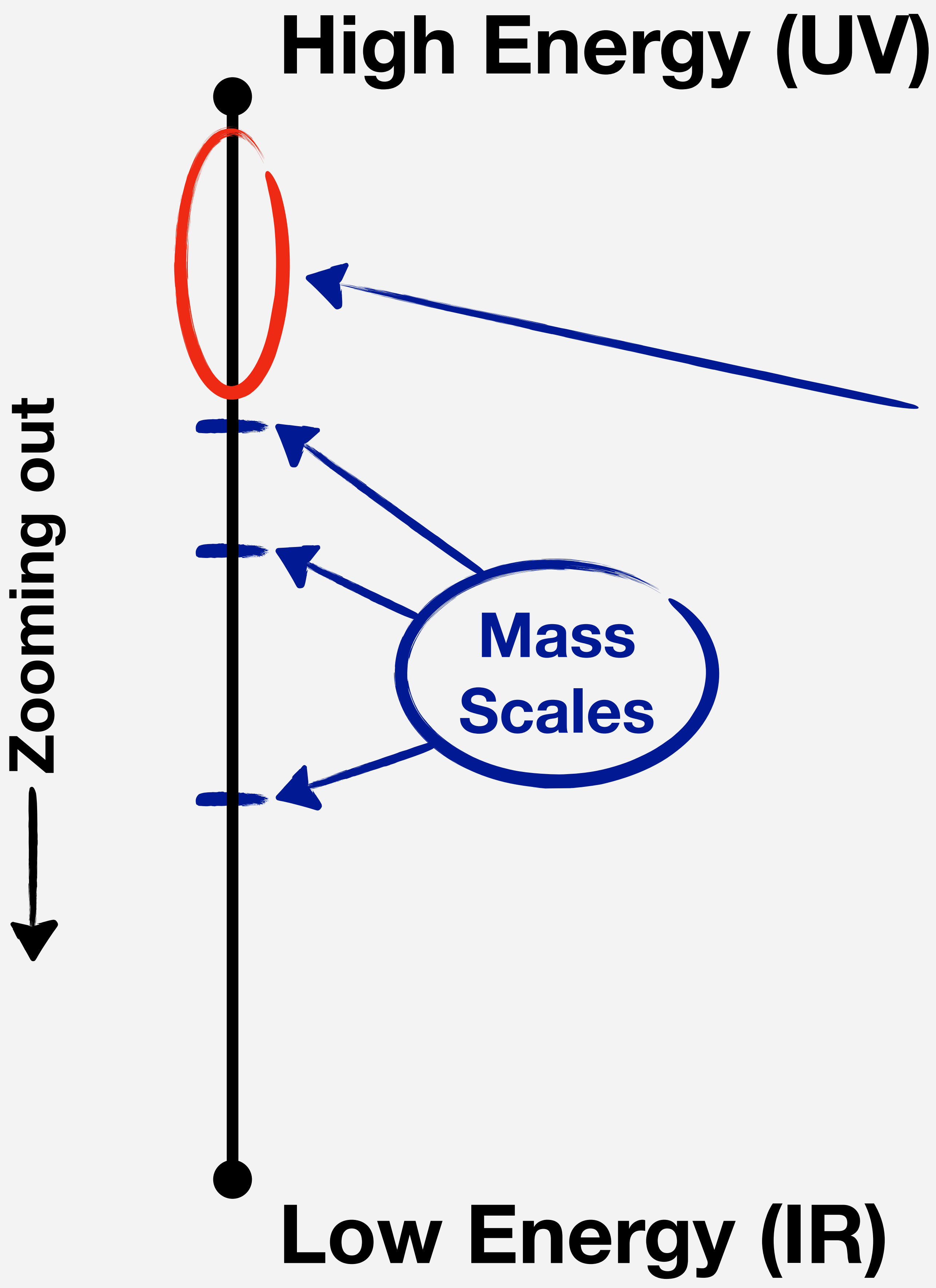

**How does short-distance data…**

$\Delta_i, \, C_{ijk}$

**…determine long-distance dynamics?**

$m_i^2, \, \langle \mathcal{O}(t)\mathcal{O}(0) \rangle, \, \mathcal{M}(s,t), \, \ldots$

# Goal for Today




**What are the dynamics of high-energy states in QFT?**

$E_\alpha,\ \langle\psi(t)\psi(0)\rangle,\ \langle E_\alpha|\mathcal{O}|E_\alpha\rangle,\ \ldots$

# Punchline

- QFTs are generically **chaotic**
- ~~Naive intuition: UV states are described by CFT (relevant deformation)~~
- **Macroscopic** features match **CFT**, but **microscopic** features are **sensitive** to deformation
- High-energy dynamics governed by **Eigenstate Thermalization Hypothesis (ETH)** and **Random Matrix Theory (RMT)**
- States near **multi-particle thresholds** violate ETH, with **semiclassical description**

# Menu

- Model: 1+1d $\phi^4$ theory
- Method: Conformal truncation
- Understanding high-energy states
- Thermalization and chaos in QFT
- “Scar” states at weak coupling

# Menu

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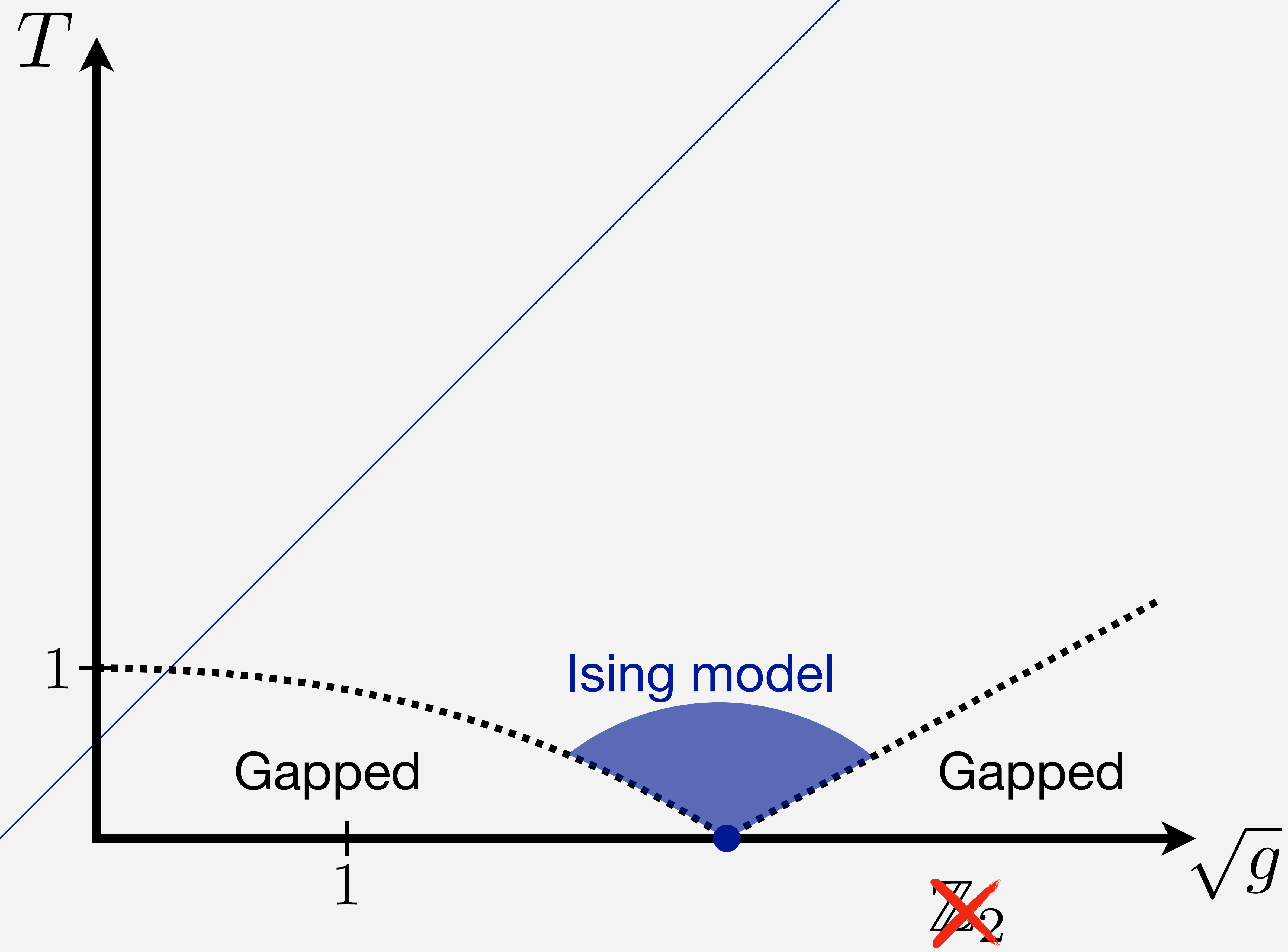
} ⟵ Lots of plots

# Model: $1+1$d $\phi^4$ theory

$$\mathcal{L} = (\partial\phi)^2 - m^2\phi^2 - g\phi^4$$

- $\mathbb{Z}_2$ symmetry: $\phi \to -\phi$
- Critical point: **Ising model**

**Goal: study high-energy states at both weak and strong coupling**



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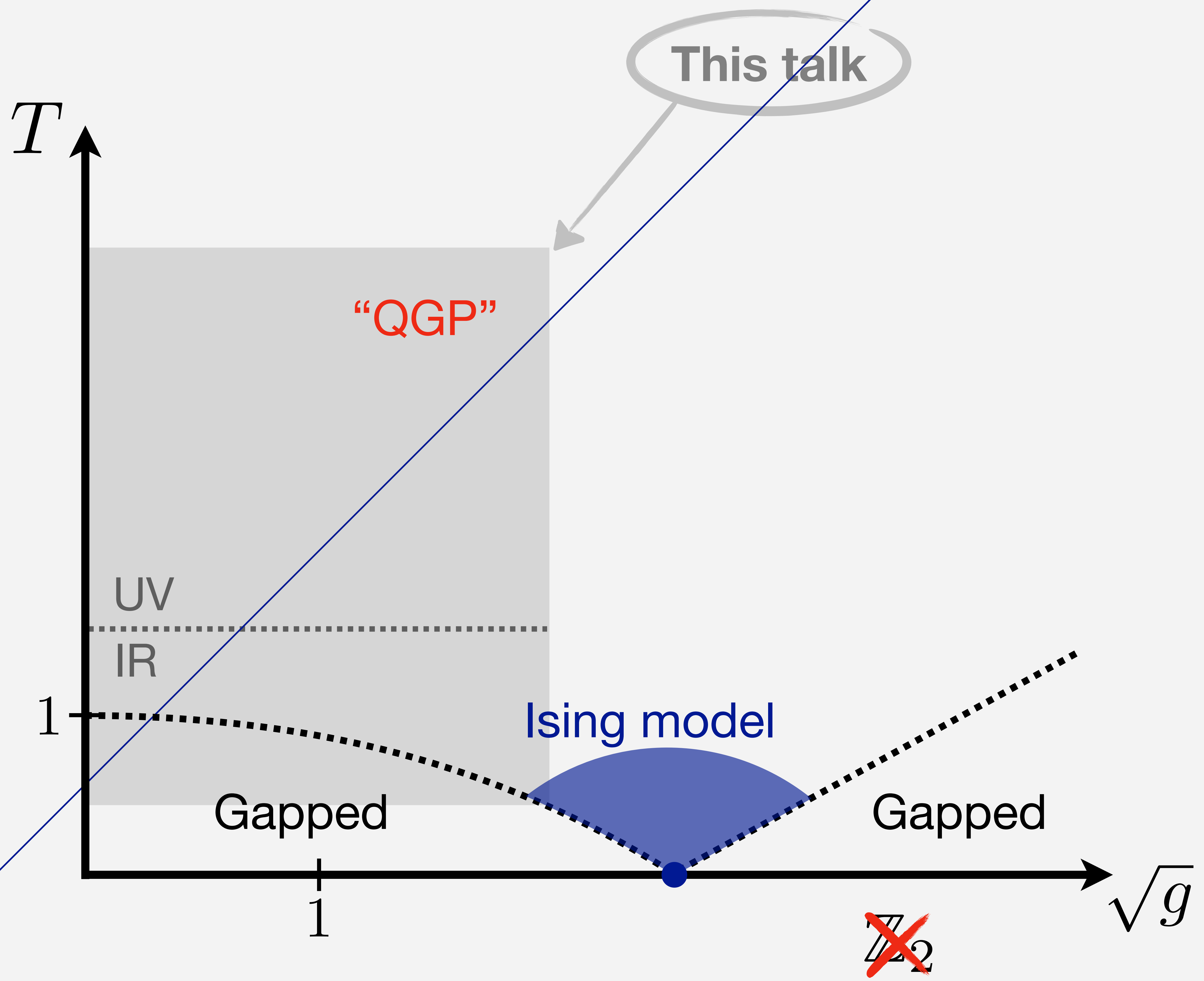
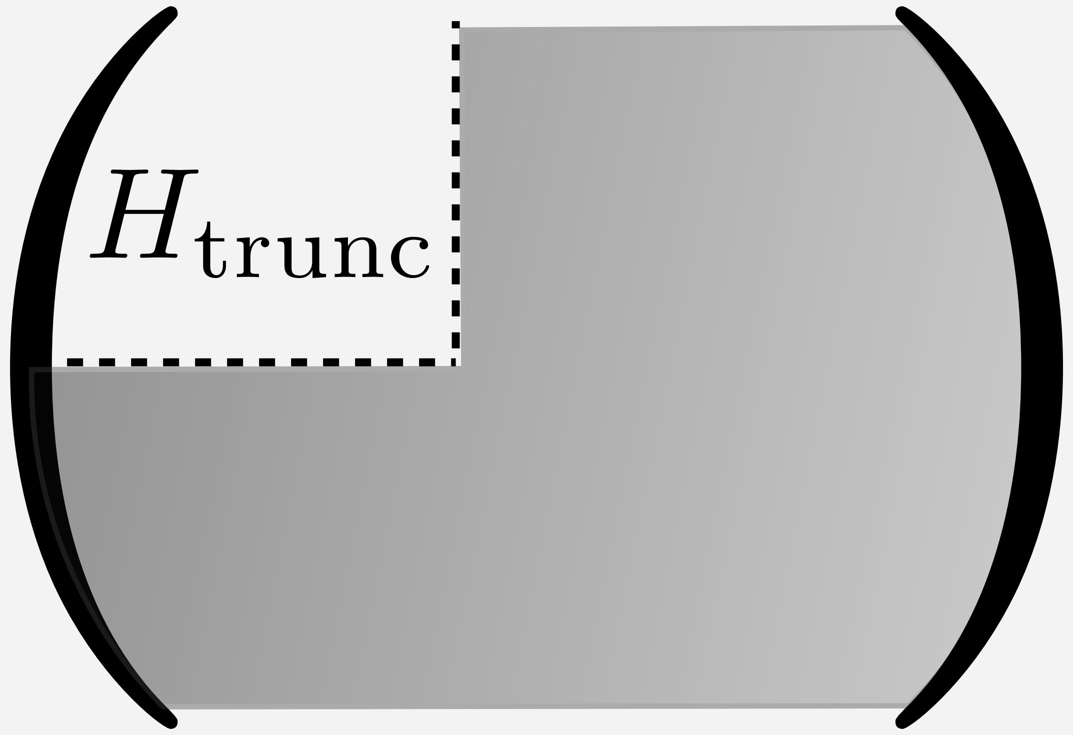


# Hamiltonian Truncation

- Old idea (**Rayleigh-Ritz**) for approximating energy eigenstates
- Basic steps:

1) **Discretize** Hilbert space
2) **Truncate** to finite-dimensional subspace
3) **Diagonalize** truncated Hamiltonian

$$H = H_0 + V =$$




**Solvable theory**
**(e.g.** $\mathscr{L} = (\partial\phi)^2$**)**

**NOT small perturbation!**
**(e.g.** $m^2\phi^2 + g\phi^4$**)**

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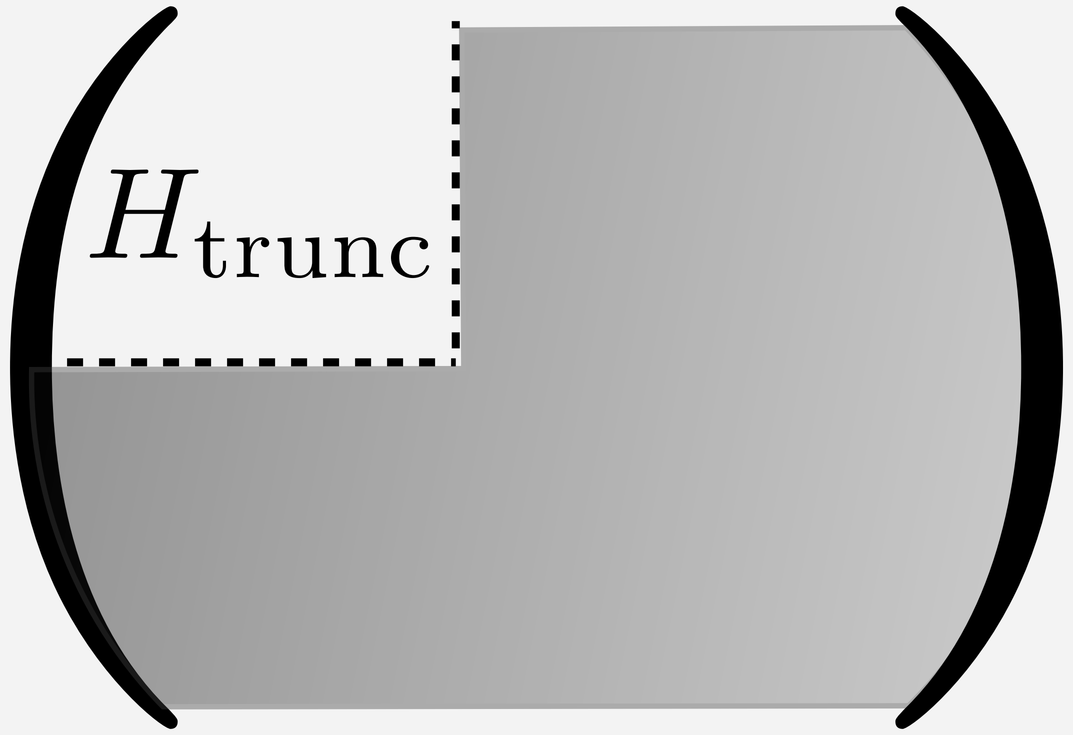



**Solvable theory** (e.g. $\mathscr{L} = (\partial\phi)^2$) ← $H_0$

**NOT small perturbation!** (e.g. $m^2\phi^2 + g\phi^4$) ← $V$

**This talk**

DLCQ Pauli et al ’85, TCSA Yurov et al ’90, Massive Fock Space Rychkov et al ’14, Conformal Truncation Katz et al ’16, HTET Cohen et al ’21, RCMPS Tilloy ’21, …

# Conformal Truncation

$$\mathcal{L}_0 = (\partial\phi)^2$$

1) **Discretize:** Hilbert space of UV CFT = Fock space of massless scalar

$$|p_1, \ldots, p_n\rangle \qquad (n = 0, 1, 2, \ldots)$$

Span Hilbert space with basis of **polynomials**

$$|\Psi_{\mathbf{k}}(p)\rangle \sim \int dp_1 \cdots dp_n \, \delta(p - \textstyle\sum_i p_i) \, p_1^{k_1} \cdots p_n^{k_n} |p_1, \ldots, p_n\rangle$$

Equivalent to basis of local **CFT operators**

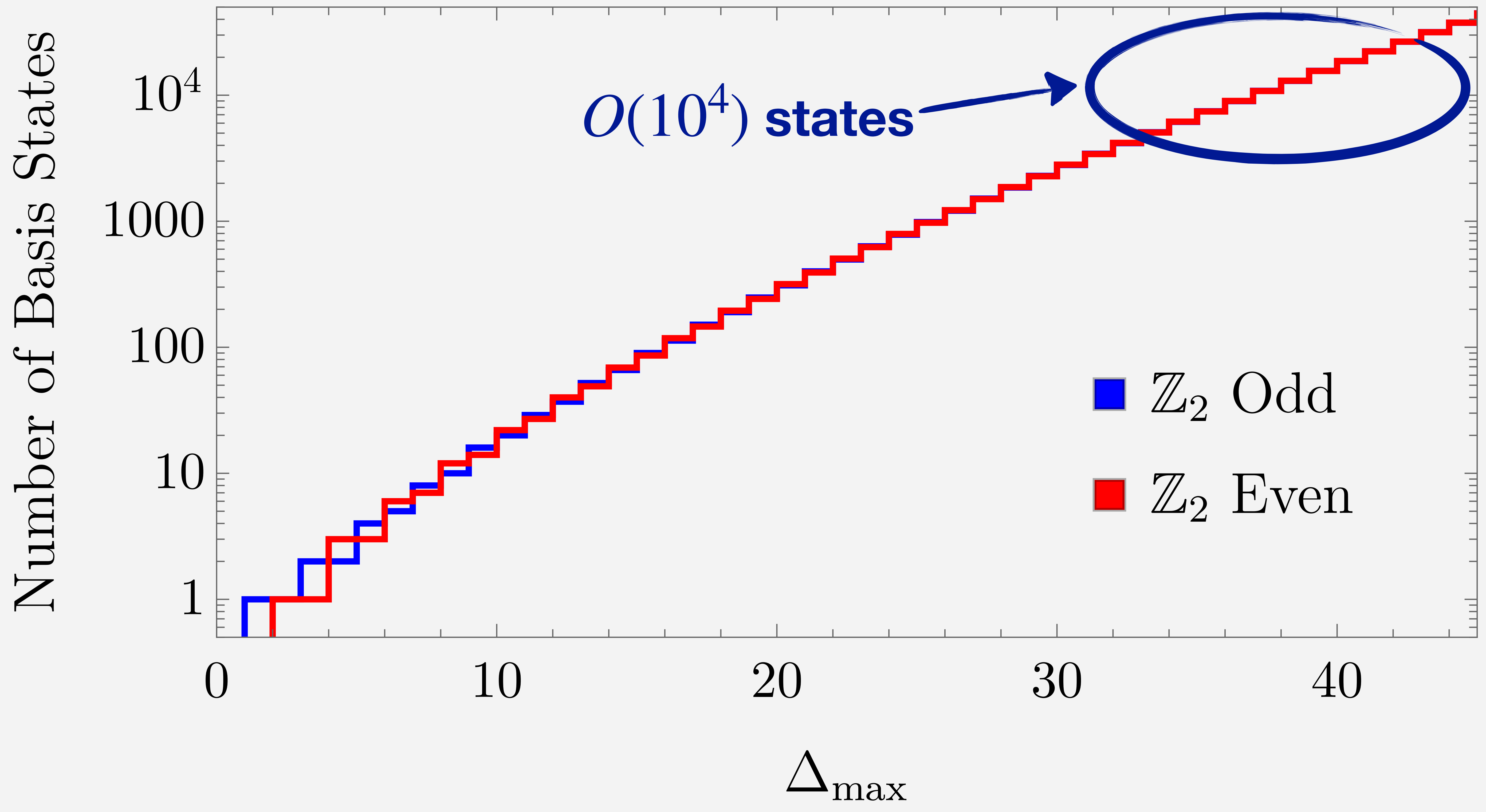
$$|\Psi_{\mathbf{k}}(p)\rangle \Leftrightarrow \mathcal{O} \sim \partial^{k_1}\phi \cdots \partial^{k_n}\phi$$

**Use UV conformal symmetry to organize Hilbert space**

# Conformal Truncation

$\mathcal{L}_0 = (\partial\phi)^2$

2) **Truncate:** restrict to polynomials with degree $\Delta \leq \Delta_{\max}$ (low-dimension operators)

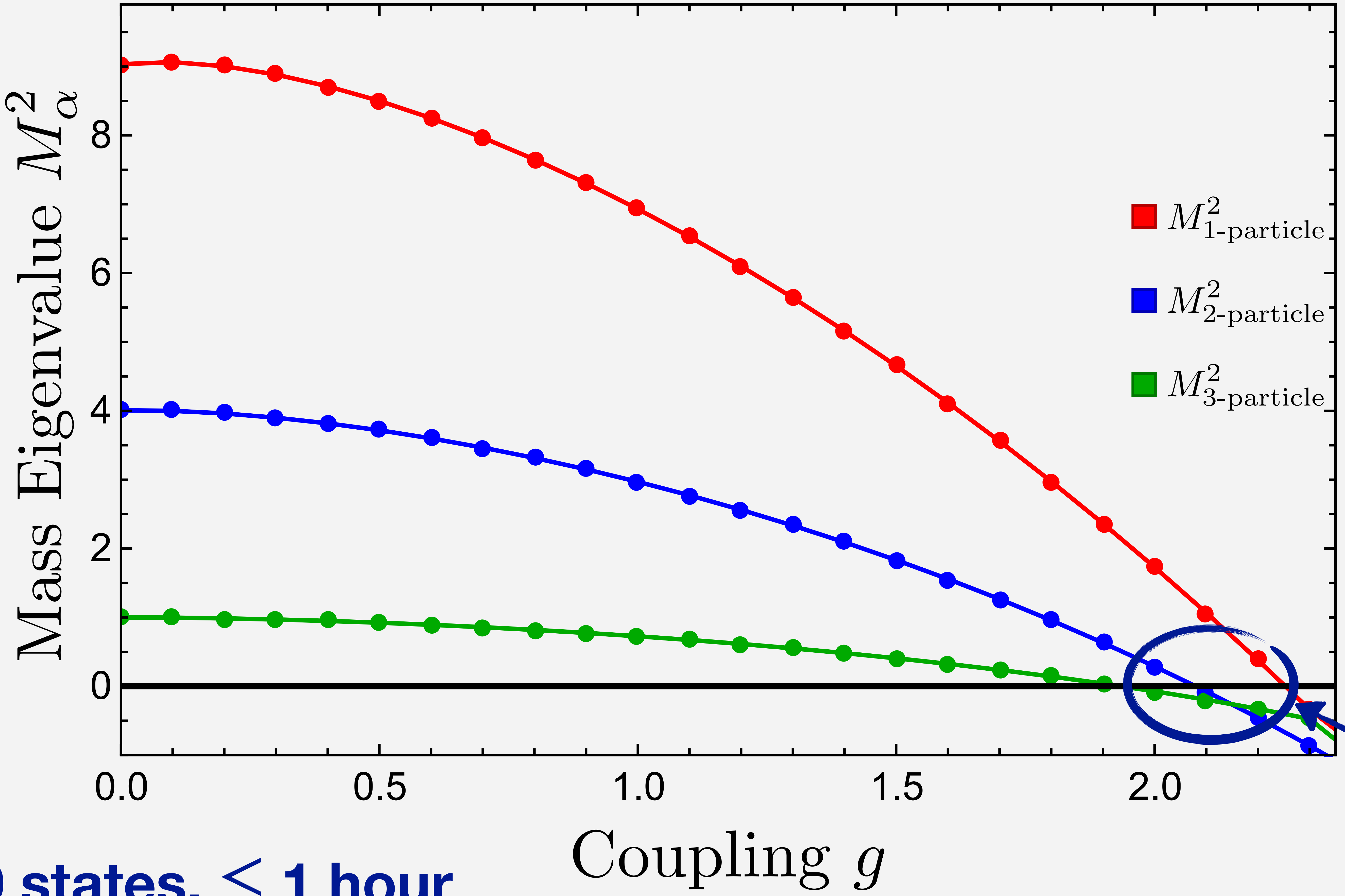



$$|\Psi_{\mathbf{k}}(p)\rangle \Leftrightarrow \mathcal{O} \sim \partial^{k_1}\phi \cdots \partial^{k_n}\phi$$

# Conformal Truncation

$$\mathcal{L} = (\partial\phi)^2 - m^2\phi^2 - g\phi^4$$

3) **Diagonalize:** construct Hamiltonian from Fock space expansion and diagonalize numerically

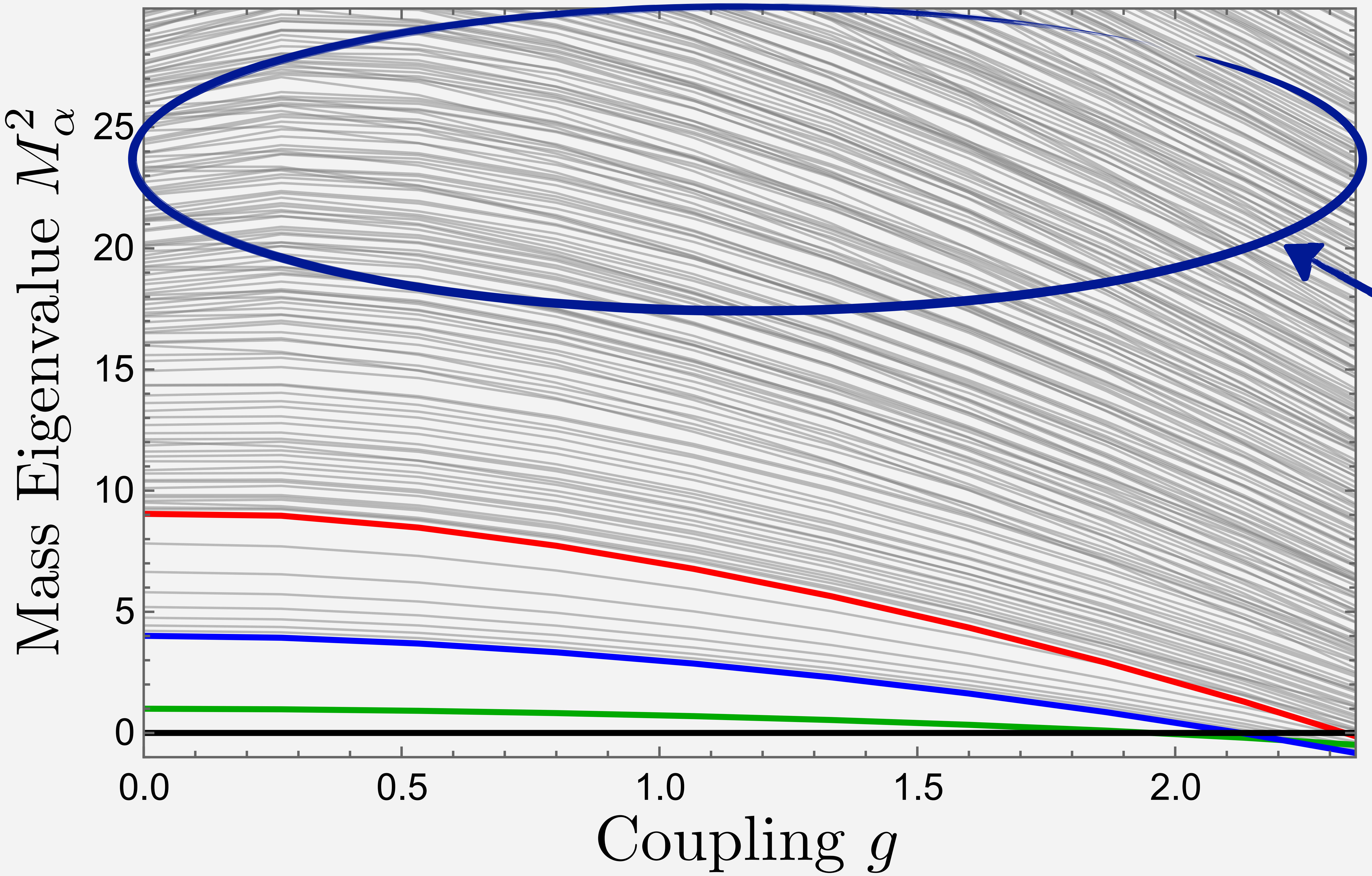



**Critical point**
**→ Ising model**

$M_\alpha^2 \equiv E_\alpha^2$

$\Delta_{\max} = 40 \longrightarrow$ **~20,000 states, $\lesssim$ 1 hour**

# High-Energy Eigenstates



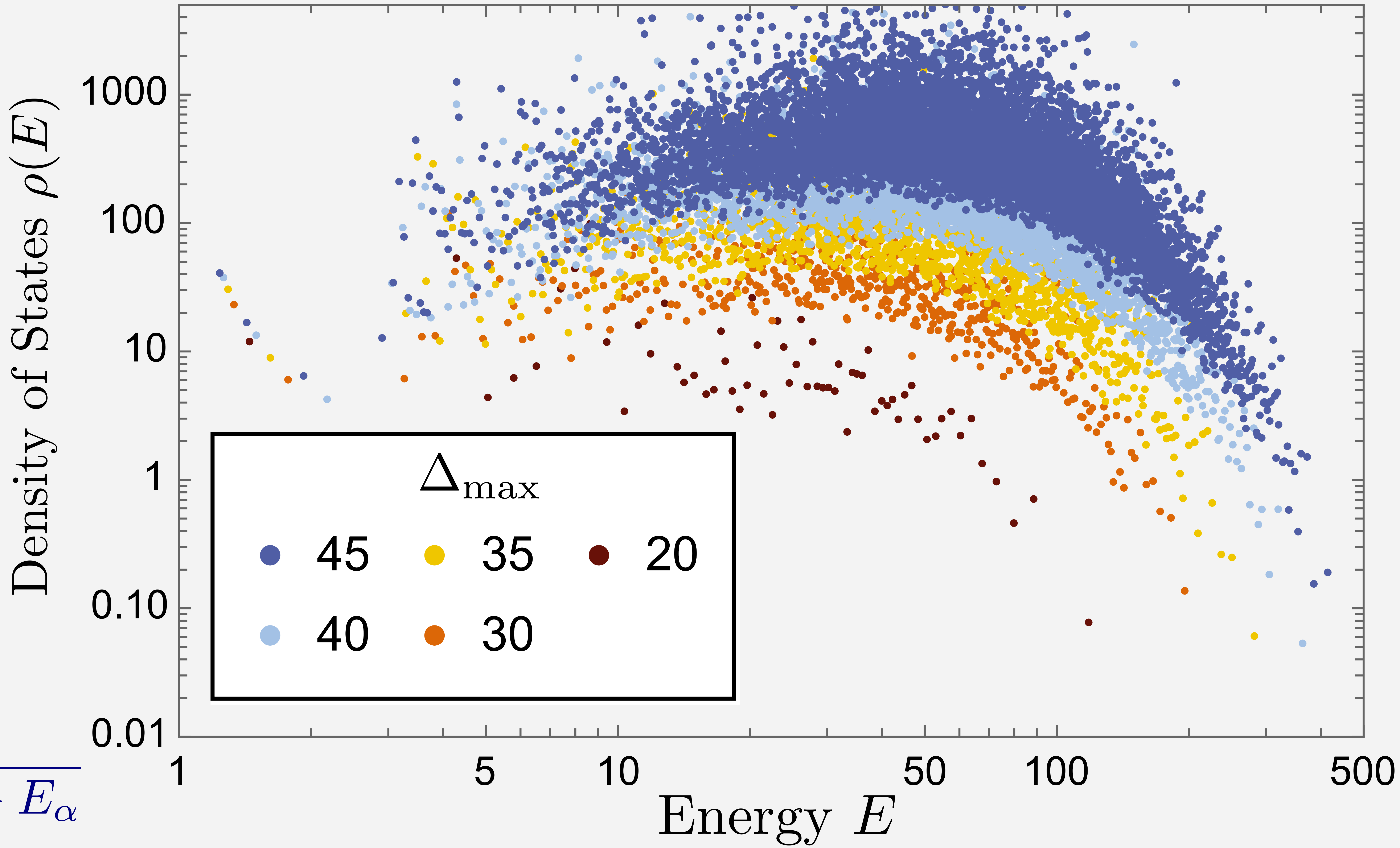

**What about all these states?**

High-energy states are **strongly-coupled**

$$\langle n \rangle \gg \frac{1}{g}$$

**Macroscopic** features match **free** theory, but **microscopic** details are **chaotic**

# Distribution of Eigenstates



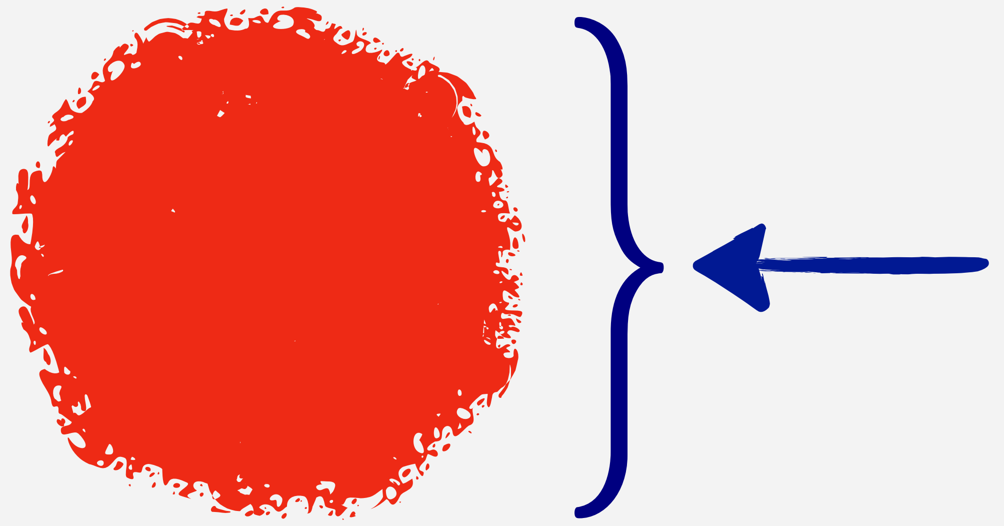

$$\rho(E_\alpha) \equiv \frac{1}{E_{\alpha+1} - E_\alpha}$$

$$g = 1.6$$

# Volume of Eigenstates

- Conformal truncation → **Infinite volume**
- Volume set by **states themselves**

$$|E_\alpha\rangle \sim$$

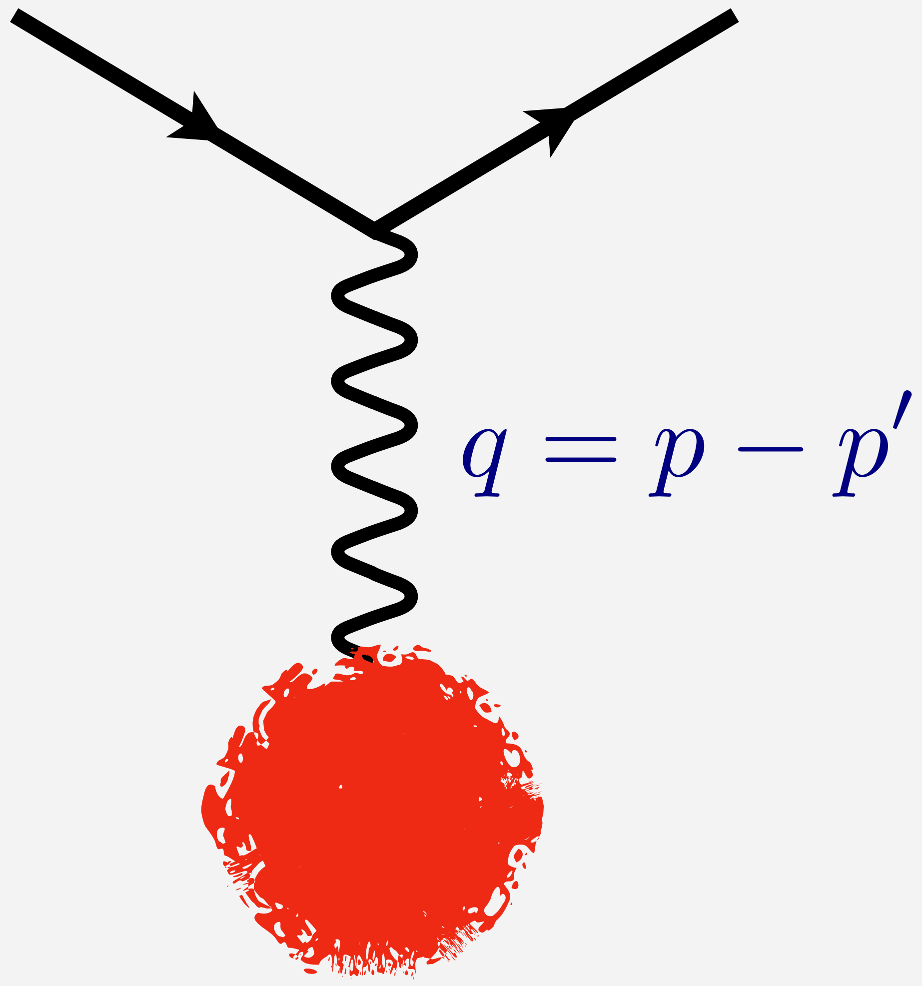



- Measure intrinsic volume with **form factors**

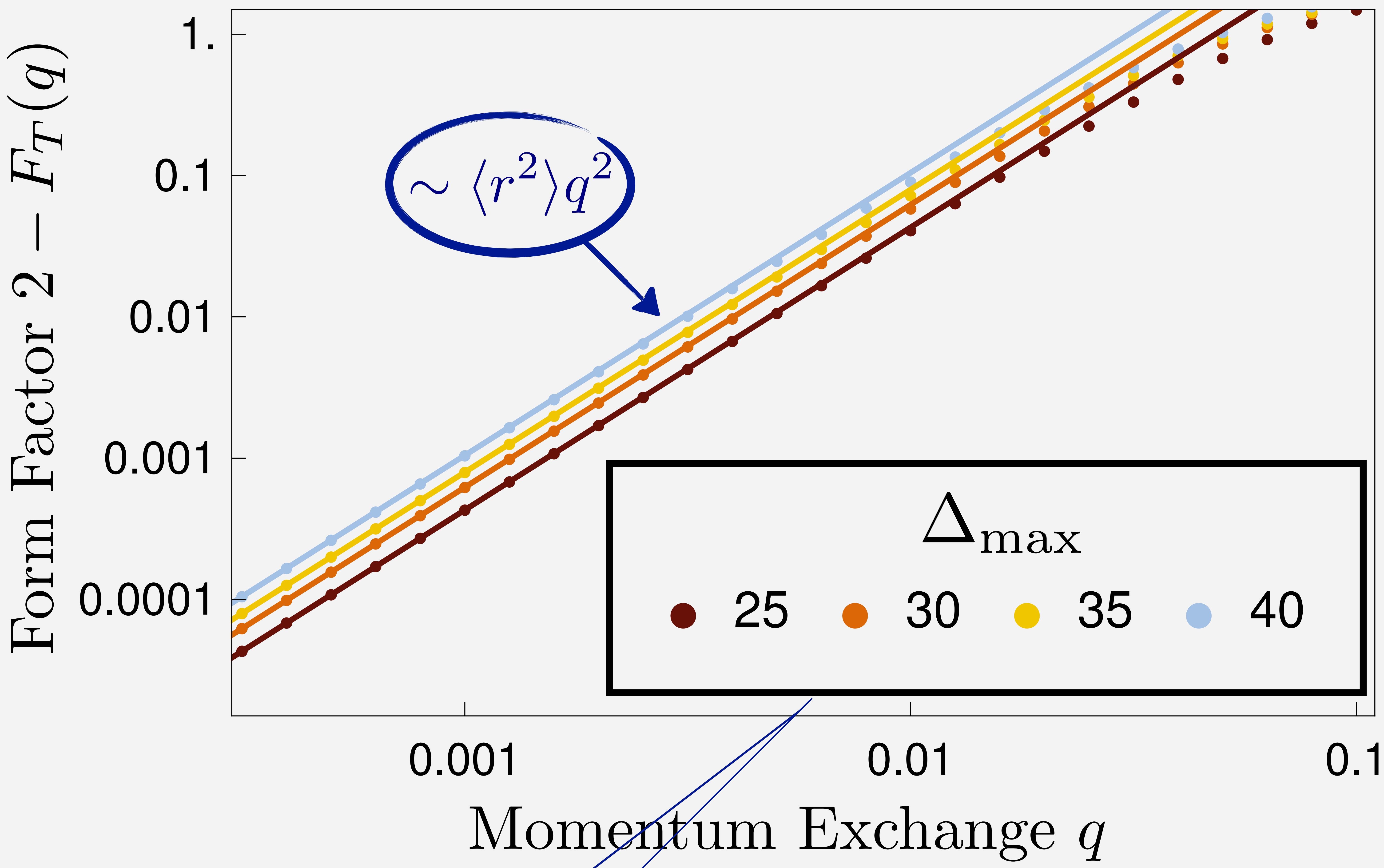
$$\boxed{F_{\mathcal{O}}(q) \equiv \langle E_\alpha(p)|\mathcal{O}(0)|E_\alpha(p')\rangle}$$

$$F_{\mathcal{O}}(q) \sim 1 - \langle r^2 \rangle q^2 + \ldots$$

**Volume** $V \equiv \sqrt{\langle r^2 \rangle}$



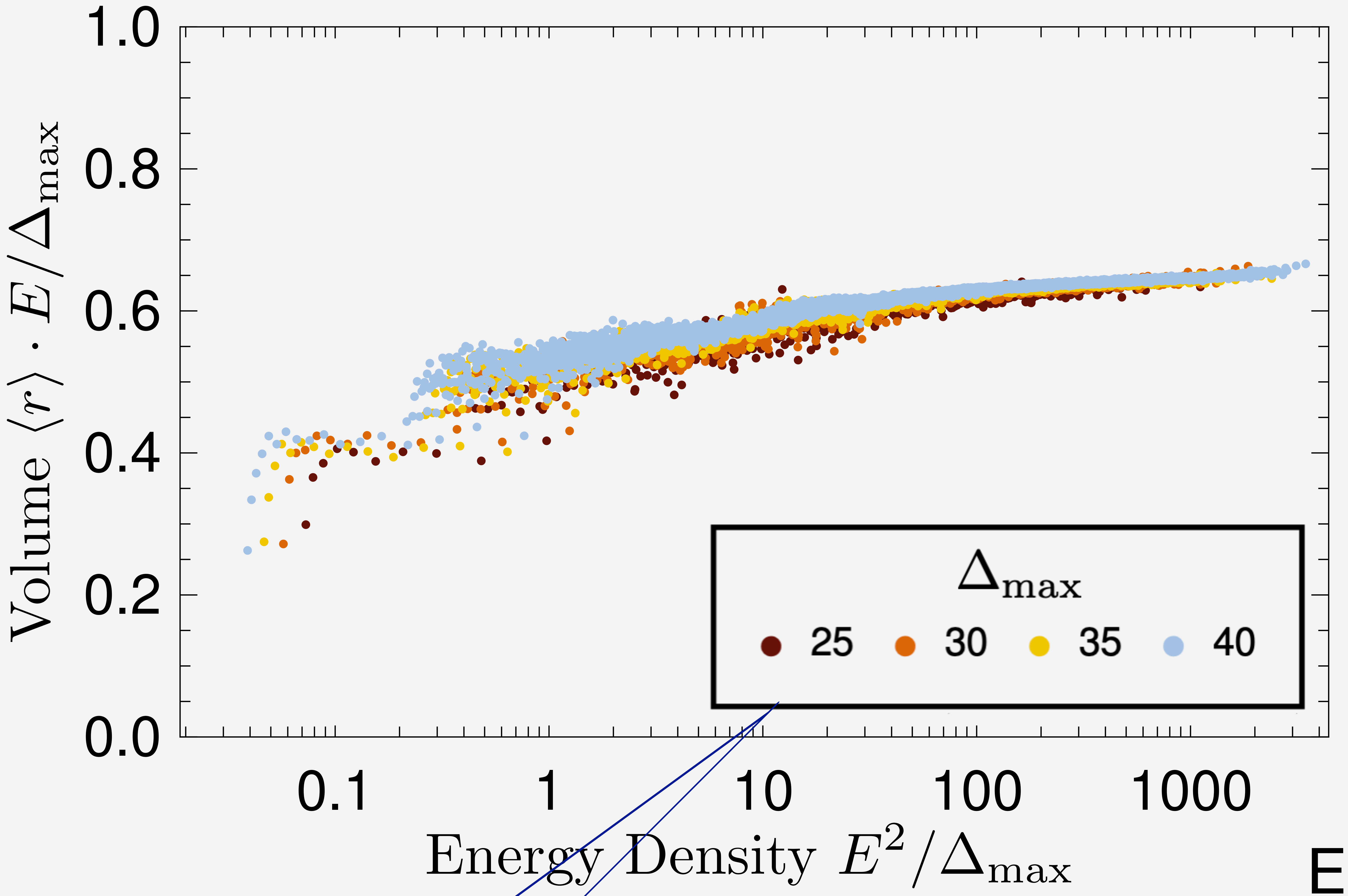
# Volume from Form Factor




**Truncation**-dependent volume: $V \equiv \sqrt{\langle r^2 \rangle} \sim \Delta_{\max}$ $\qquad \phi \overleftrightarrow{\partial}^k \phi \Rightarrow V \sim k$

$g = 1.6$

# Volume from Form Factor



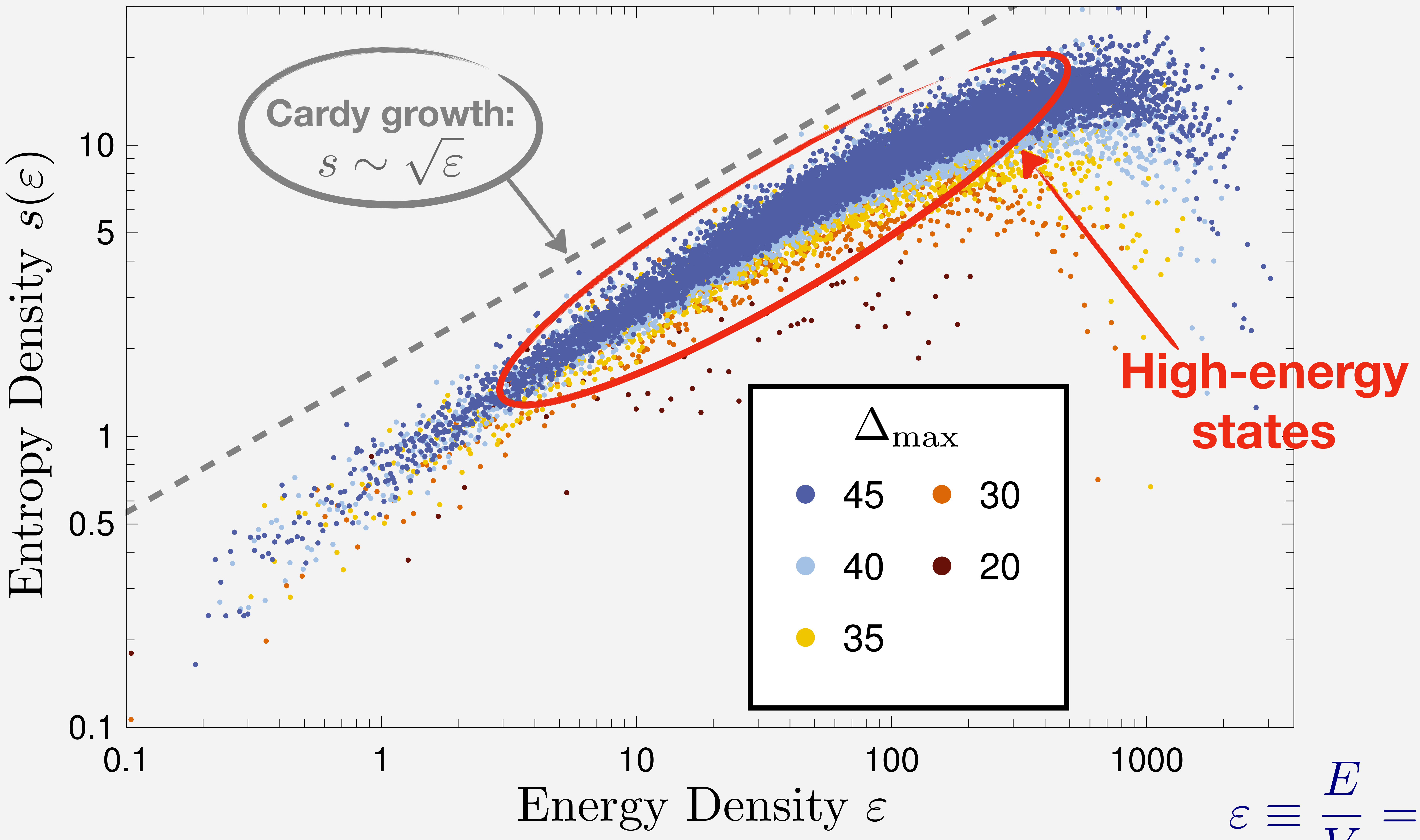

Energy density:

$$\varepsilon \equiv \frac{E}{V} \sim \frac{E^2}{\Delta_{\max}}$$

**Energy**-dependent volume: $V \equiv \sqrt{\langle r^2 \rangle} \sim \frac{\Delta_{\max}}{E}$

$g = 1.6$

# Density of States

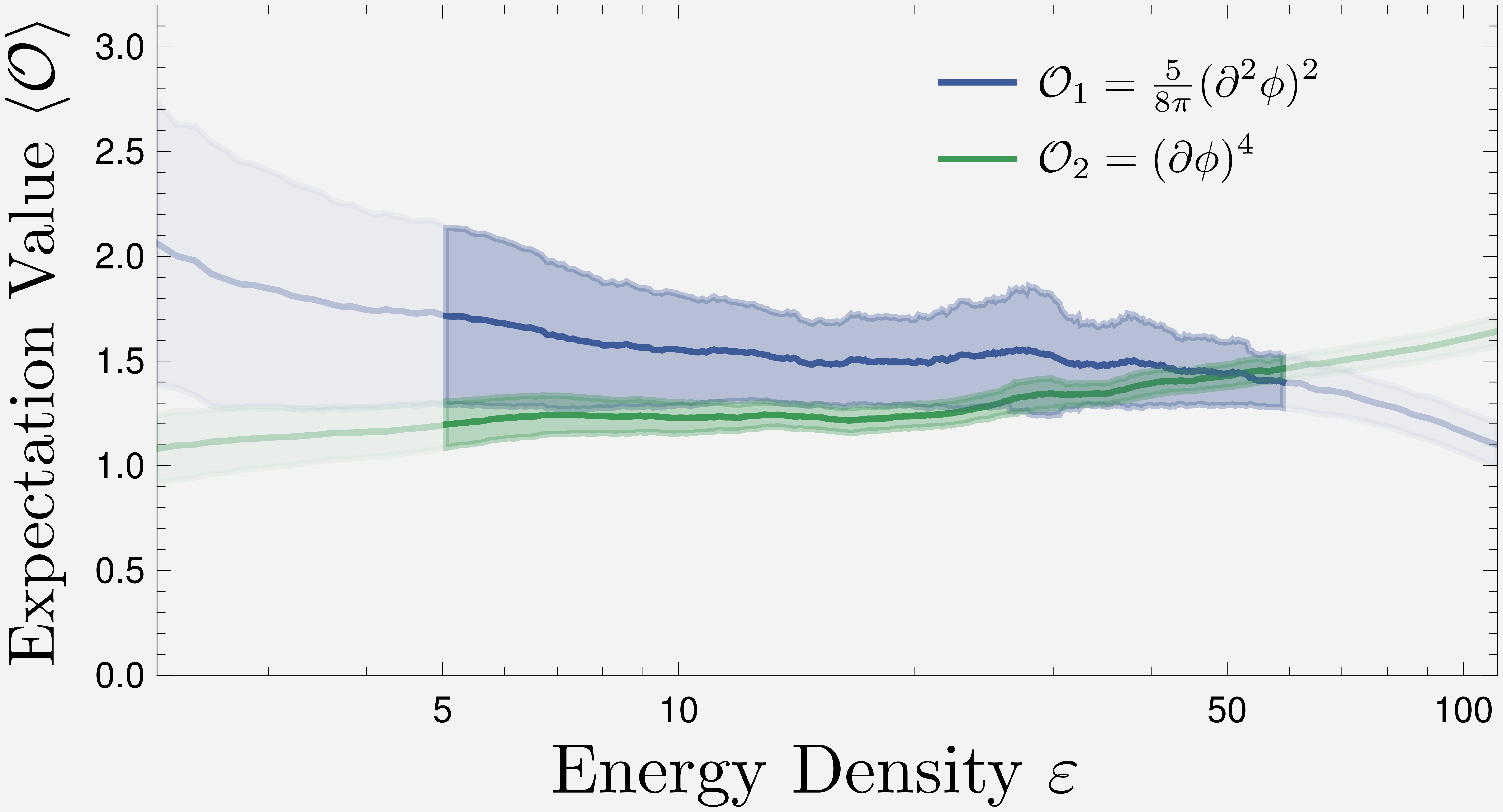



$$s \equiv \frac{1}{V} \log \rho$$

$$g = 1.6$$

$$\varepsilon \equiv \frac{E}{V} = \frac{E^2}{\Delta_{\max}}$$

# Expectation Values



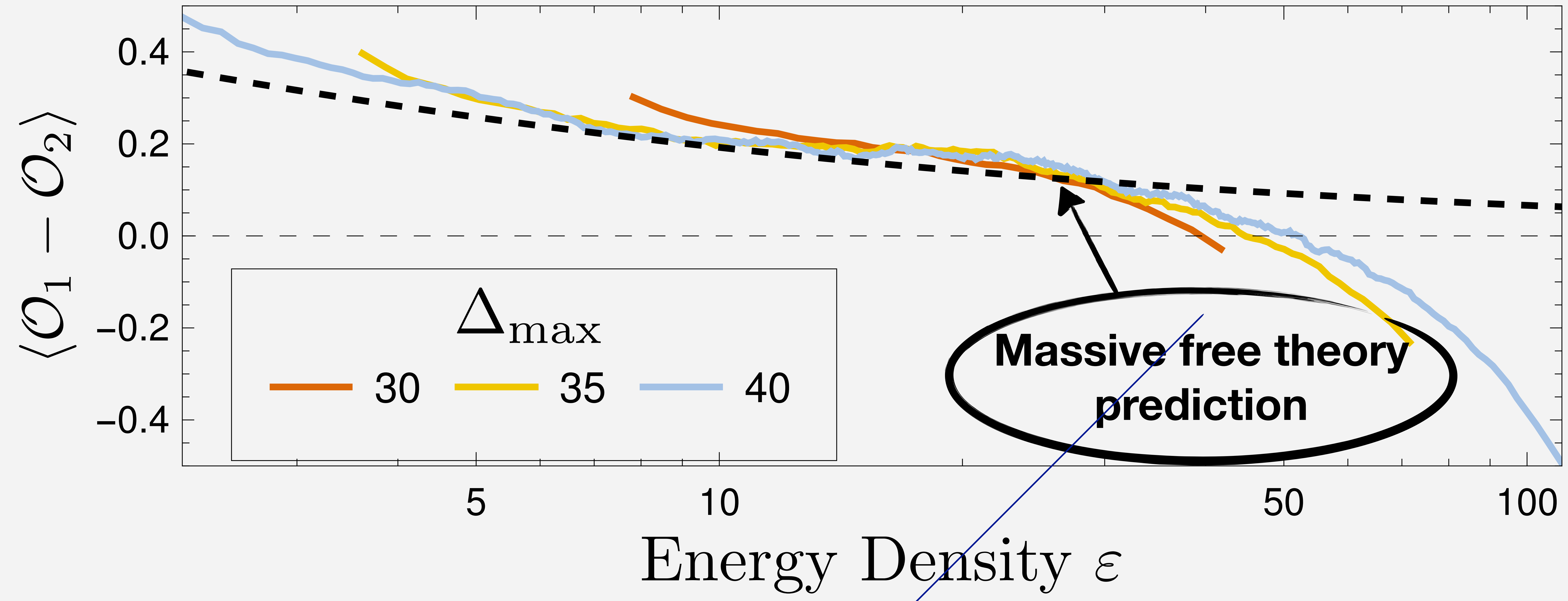

$\mathcal{V} \equiv \mathcal{O}_1 - \mathcal{O}_2 =$ Virasoro primary

CFT prediction: $\langle\mathcal{V}\rangle_\beta = 0$

$g = 1.6$

**Macroscopic** features match **free theory** as $\varepsilon \to \infty$

# Expectation Values



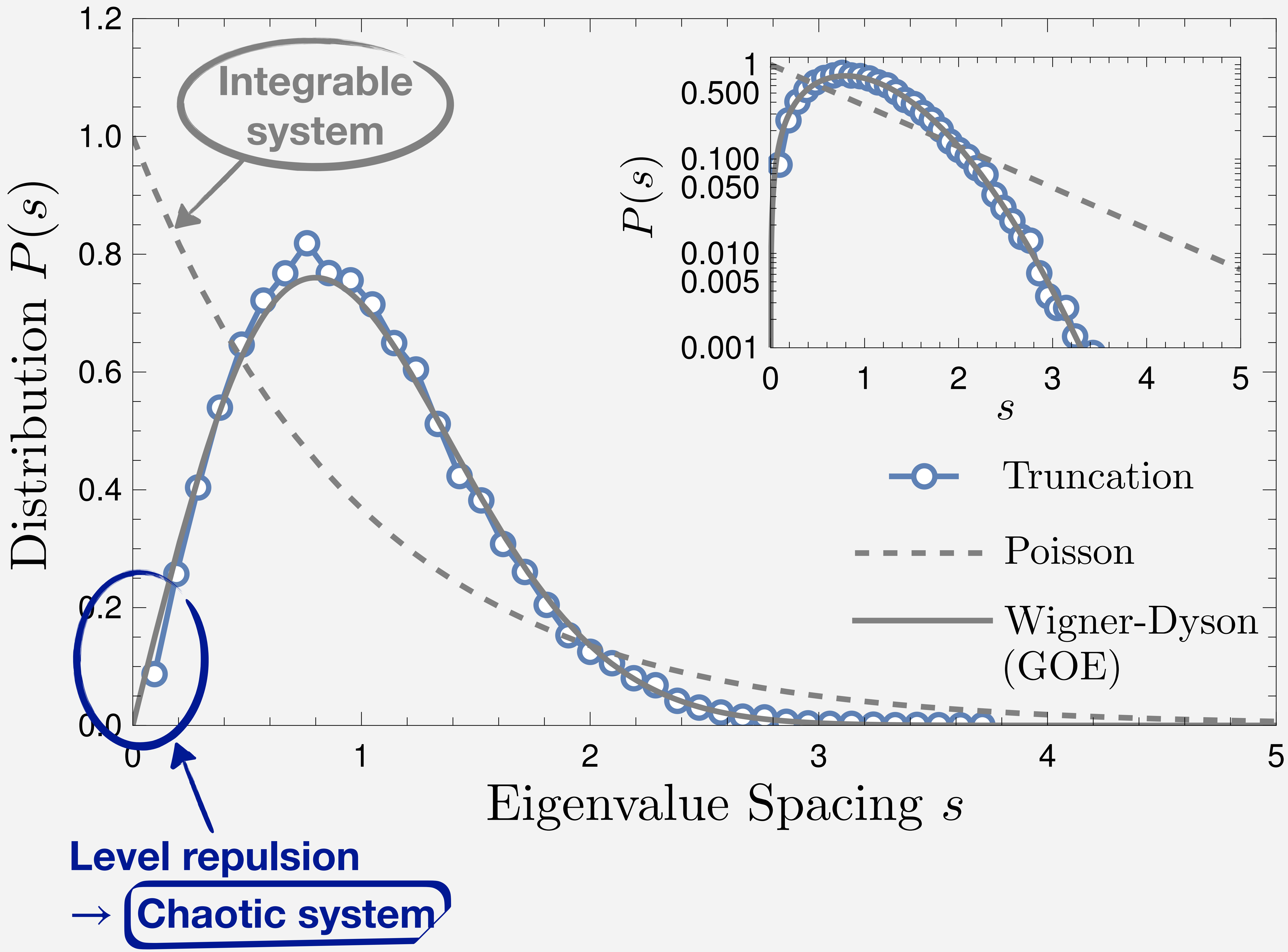

High-energy eigenstates satisfy **ETH**: $\langle E_\alpha | \mathcal{O} | E_\alpha \rangle \to \langle \mathcal{O} \rangle_\beta$

$\mathcal{O}_1 = \frac{5}{8\pi}(\partial^2\phi)^2$

$\mathcal{O}_2 = (\partial\phi)^4$

$g = 1.6$

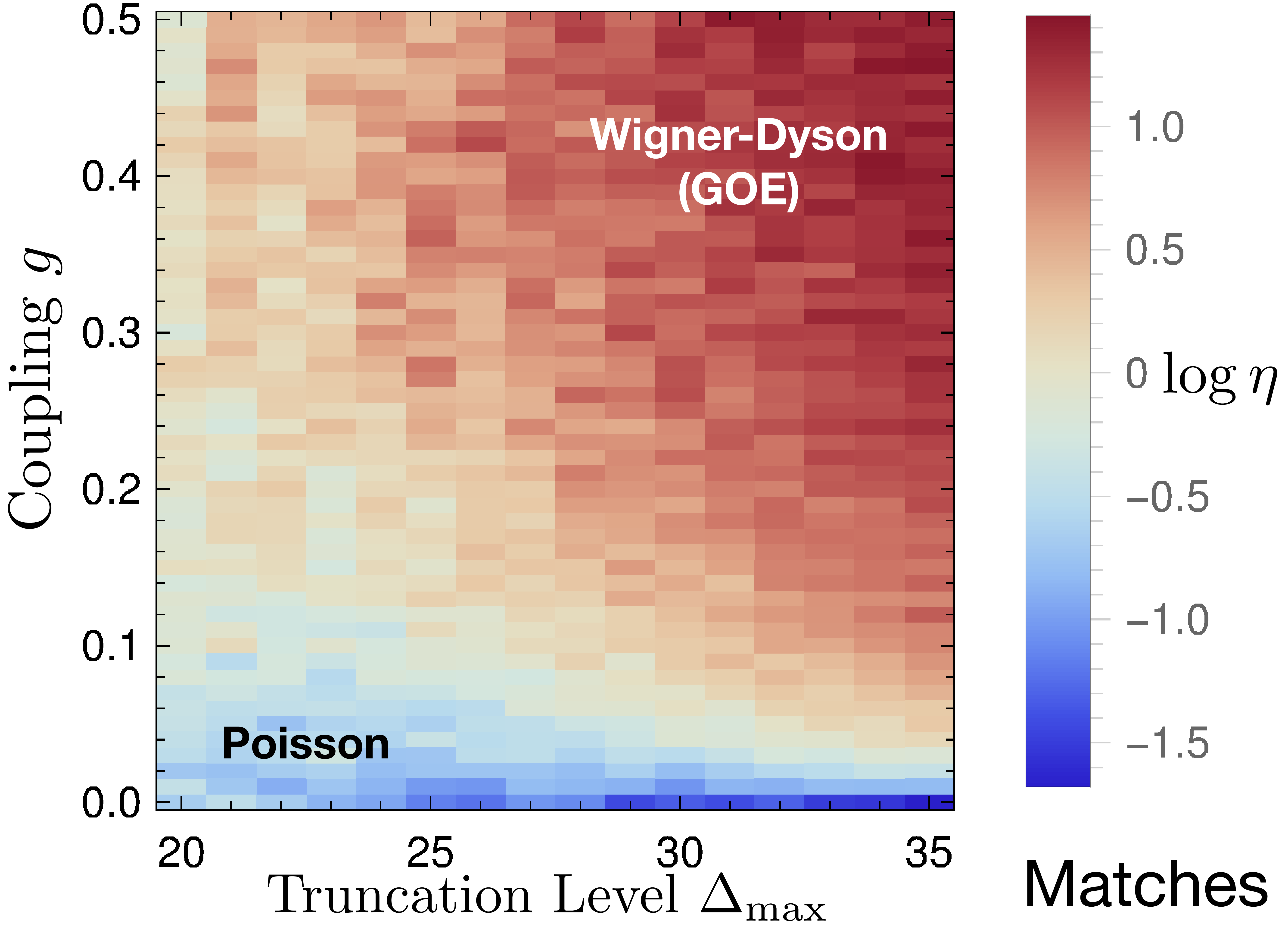
# Eigenvalue Spacing




$s_\alpha \equiv E_{\alpha+1} - E_\alpha$

$g = 1.6$

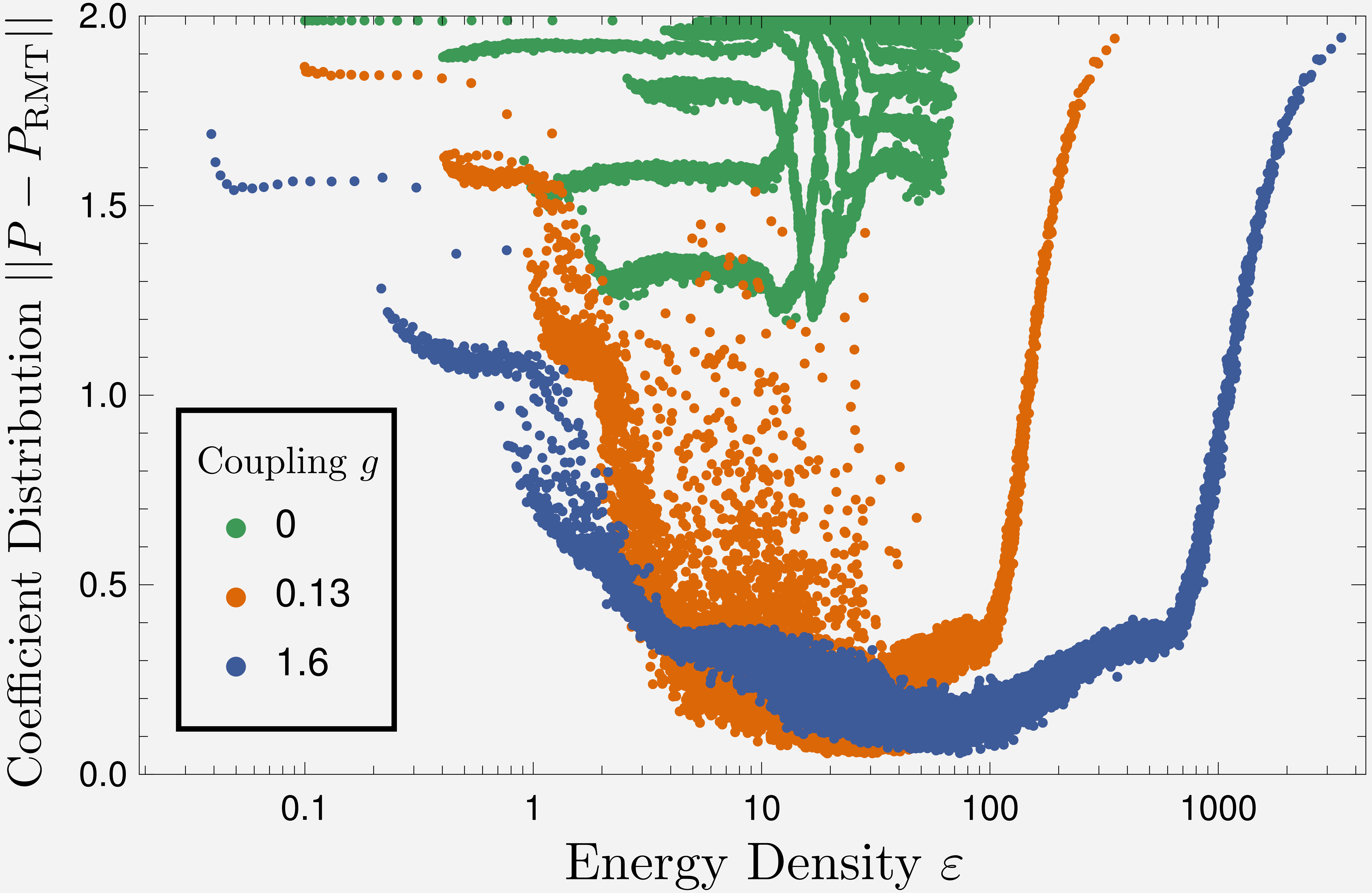
# Eigenvalue Statistics




$$\eta \equiv \frac{||P - P_{\text{Poisson}}||}{||P - P_{\text{Wigner-Dyson}}||}$$

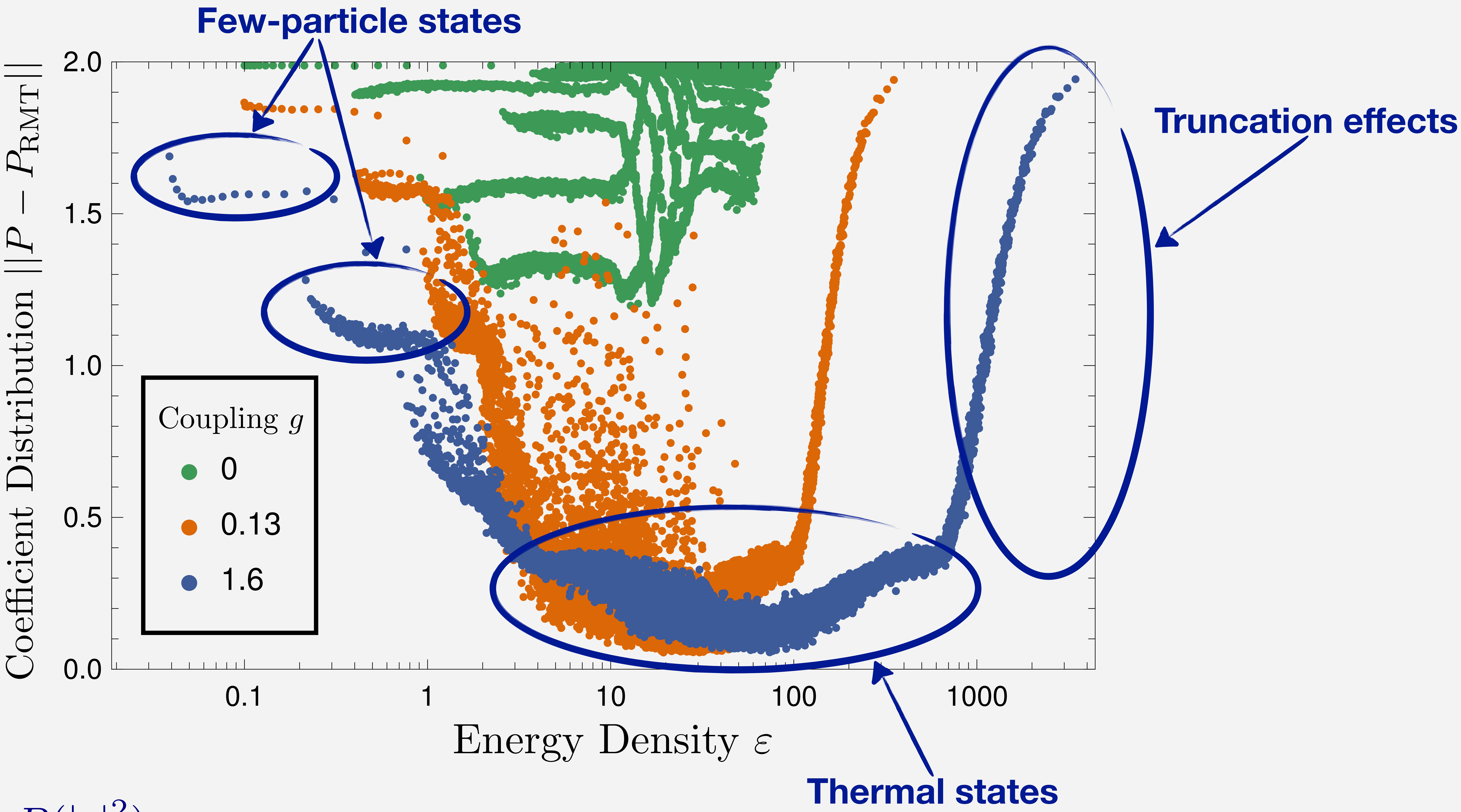
Matches RMT for **any nonzero coupling**

# Eigenvector Coefficients




$c_{iE} \equiv \langle i|E\rangle \longrightarrow P(|c|^2)$

# Eigenvector Coefficients

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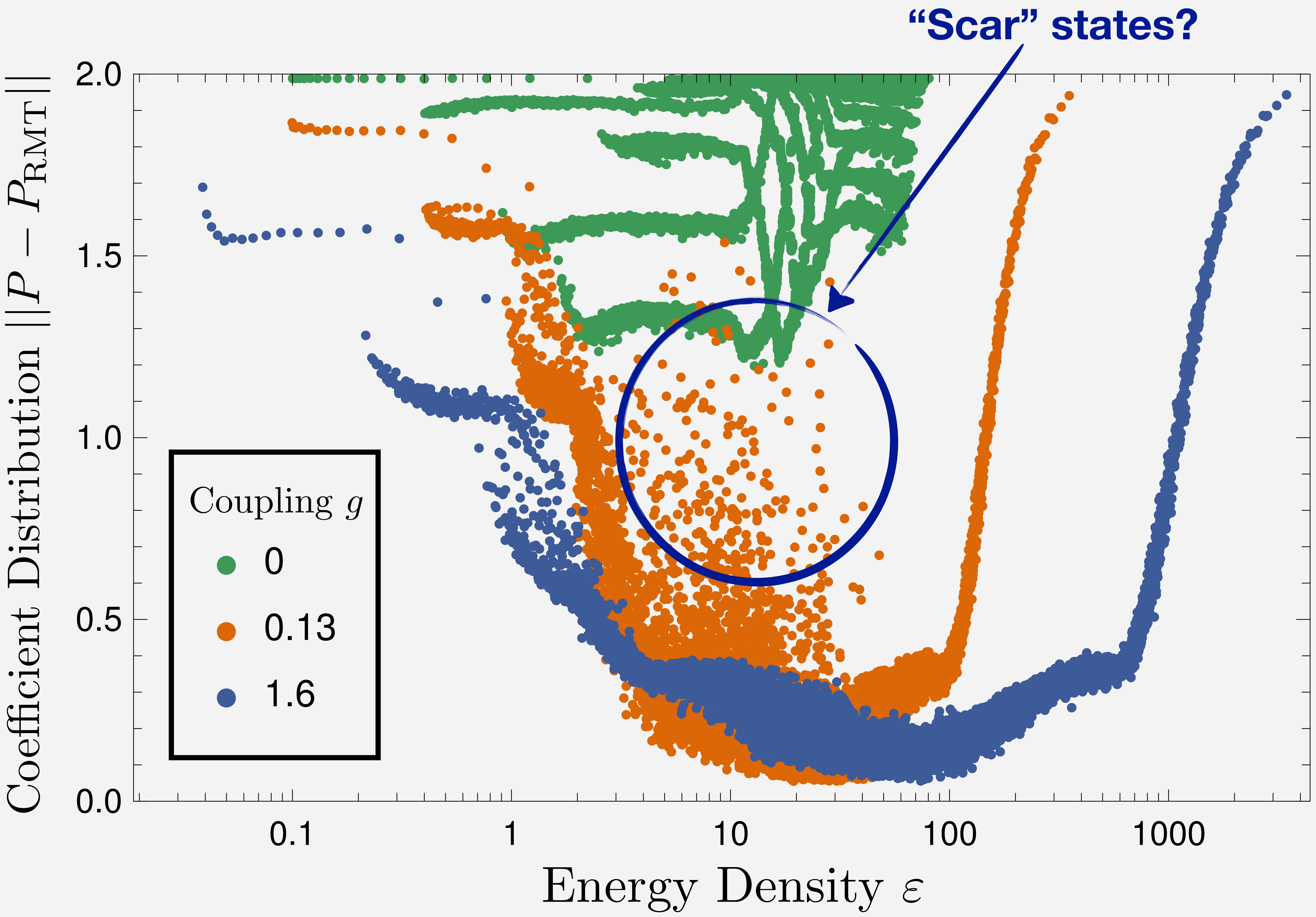


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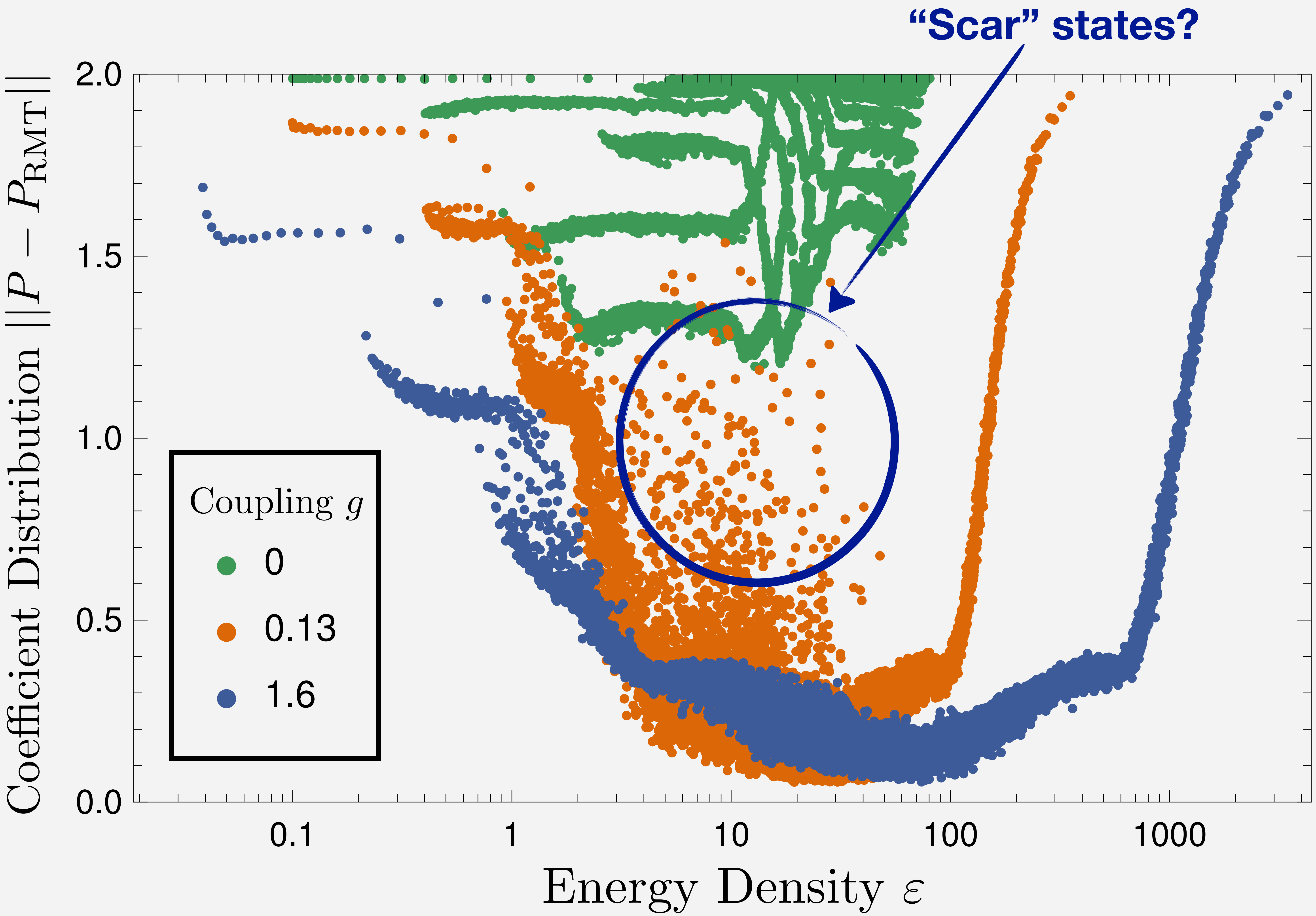
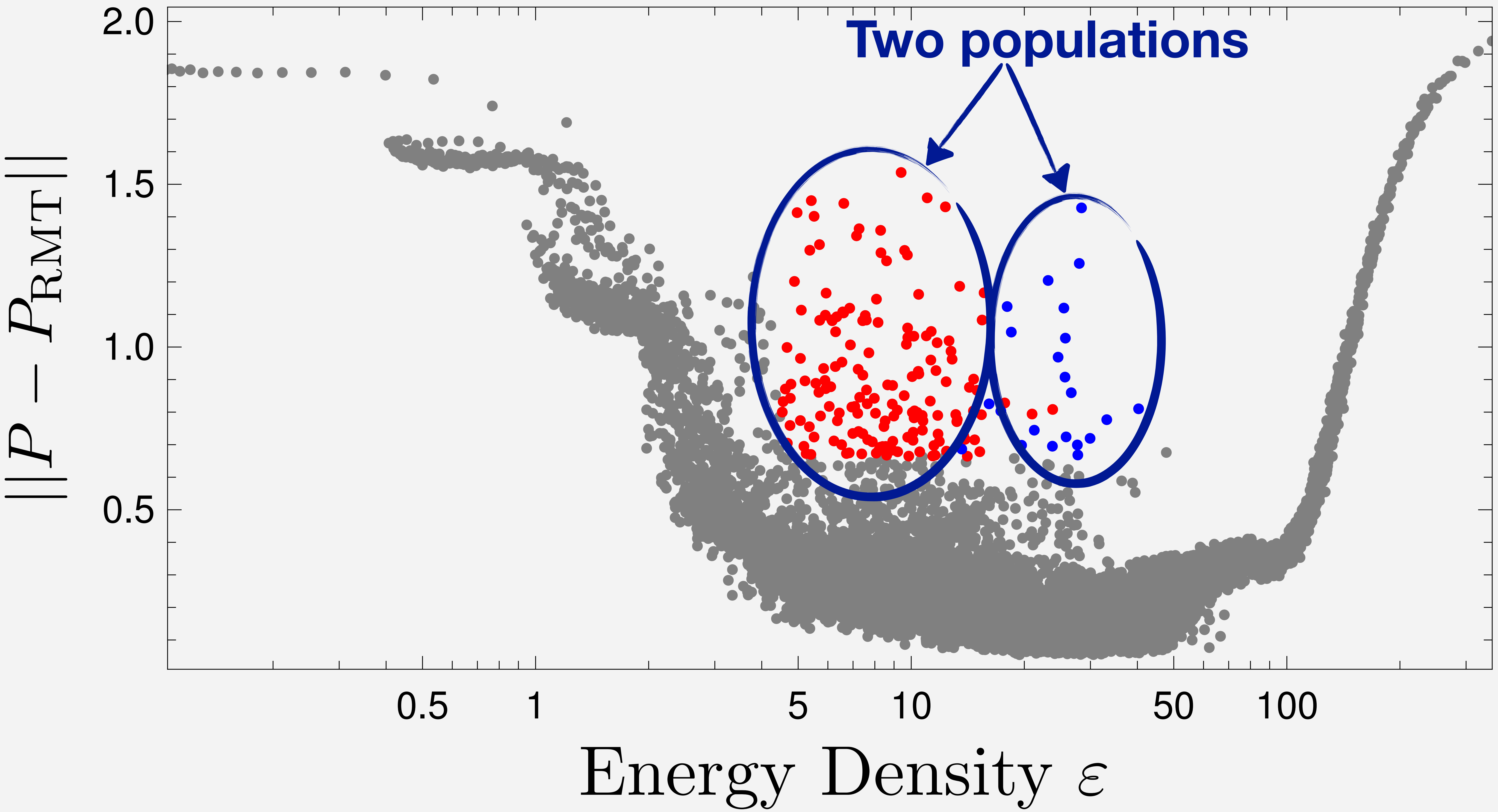



**No evidence** of scars at strong coupling

$$c_{iE} \equiv \langle i|E\rangle \longrightarrow P(|c|^2)$$

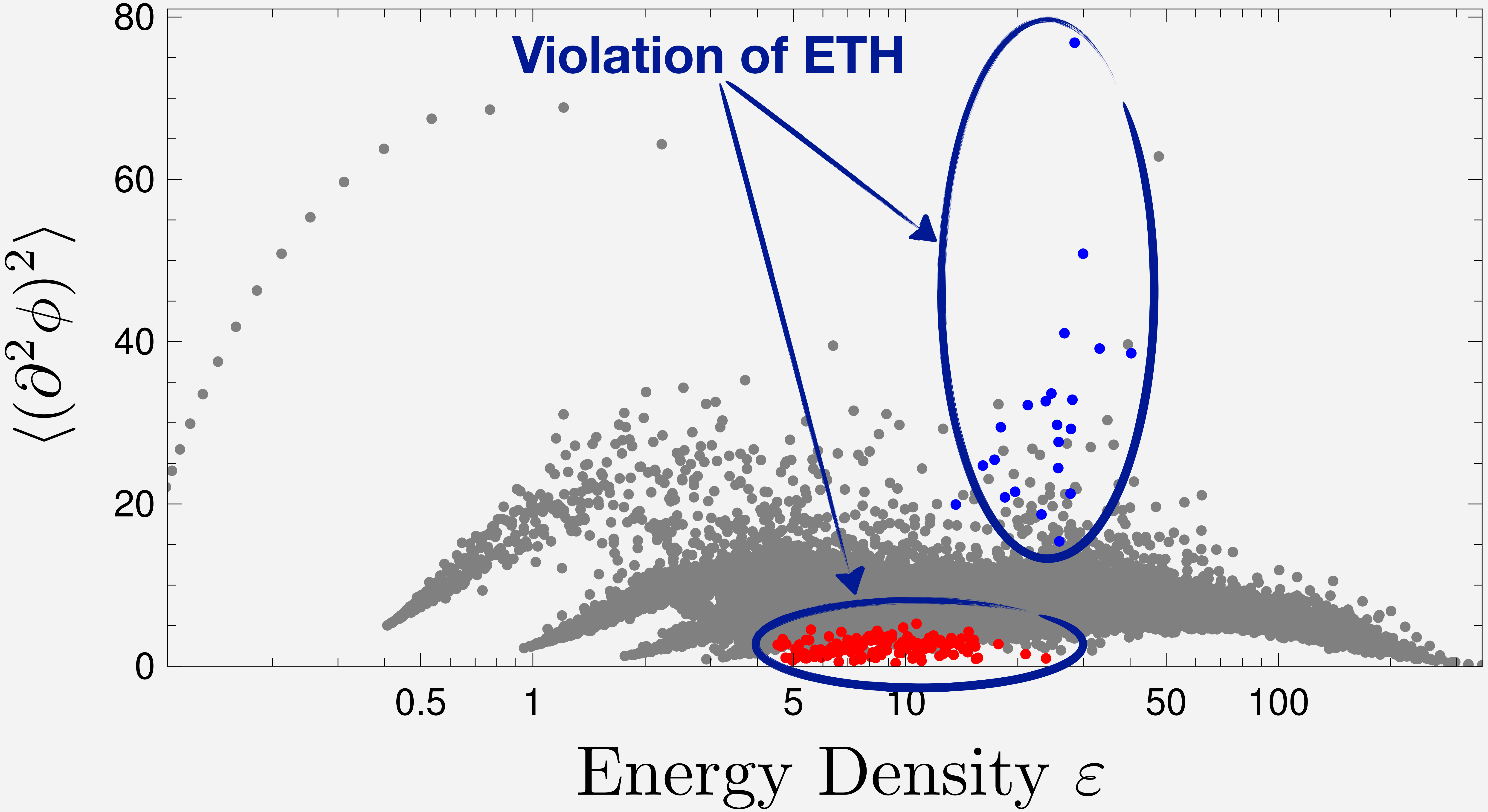
# Scar States




What are these **non-RMT** states?

$g = 0.13$

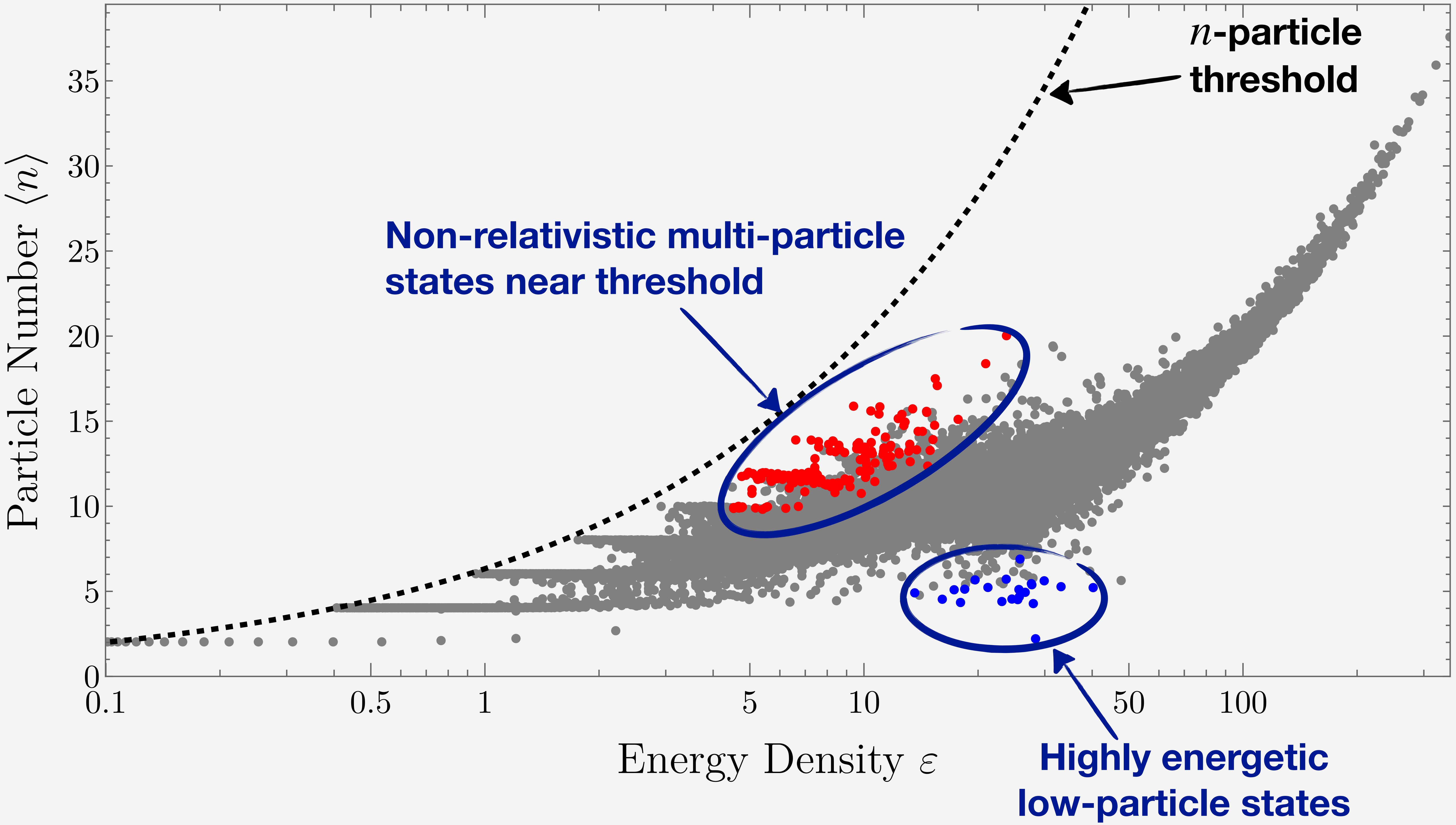
# Scar States




Can categorize by **expectation values**

$g = 0.13$

# Scar States




$g = 0.13$

# Summary

- QFTs are **chaotic** (even at weak coupling)
- High-energy dynamics governed by **ETH** and **RMT**
- High-energy states obtained numerically are “healthy”
  $\rightarrow$ average behavior **converges quickly** and **matches ETH**
- Existence of **scar states** near threshold at **weak coupling**
- **No evidence** of scars at **strong coupling**

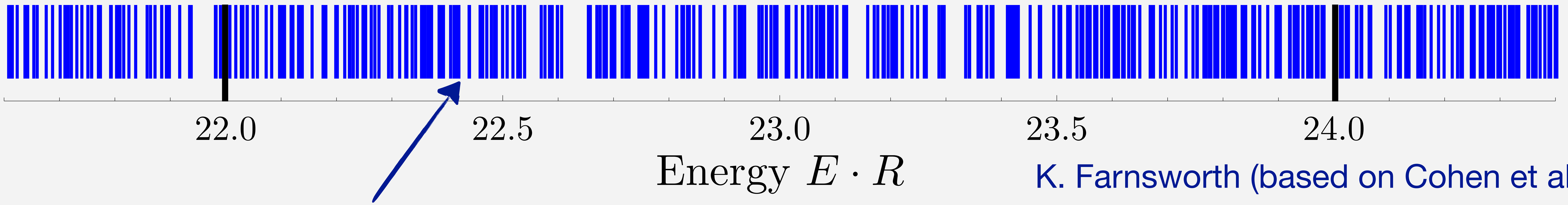
# Future Directions

- Further study ETH (**off-diagonal** matrix elements)
- Extract **hydrodynamic** features from SFF
  Winer, Swingle '20
- Continue to thermal 2-pt functions → **Hydrodynamics**, **transport**
- Extend to other (**higher-dimensional**) QFTs
- Connect near-threshold scar states to **semiclassical predictions**
  Rubakov '95, Son '95
- Implications for "**approximate CFTs**"?
  Belin et al '23
- **You** tell me!

# BACKUP SLIDES

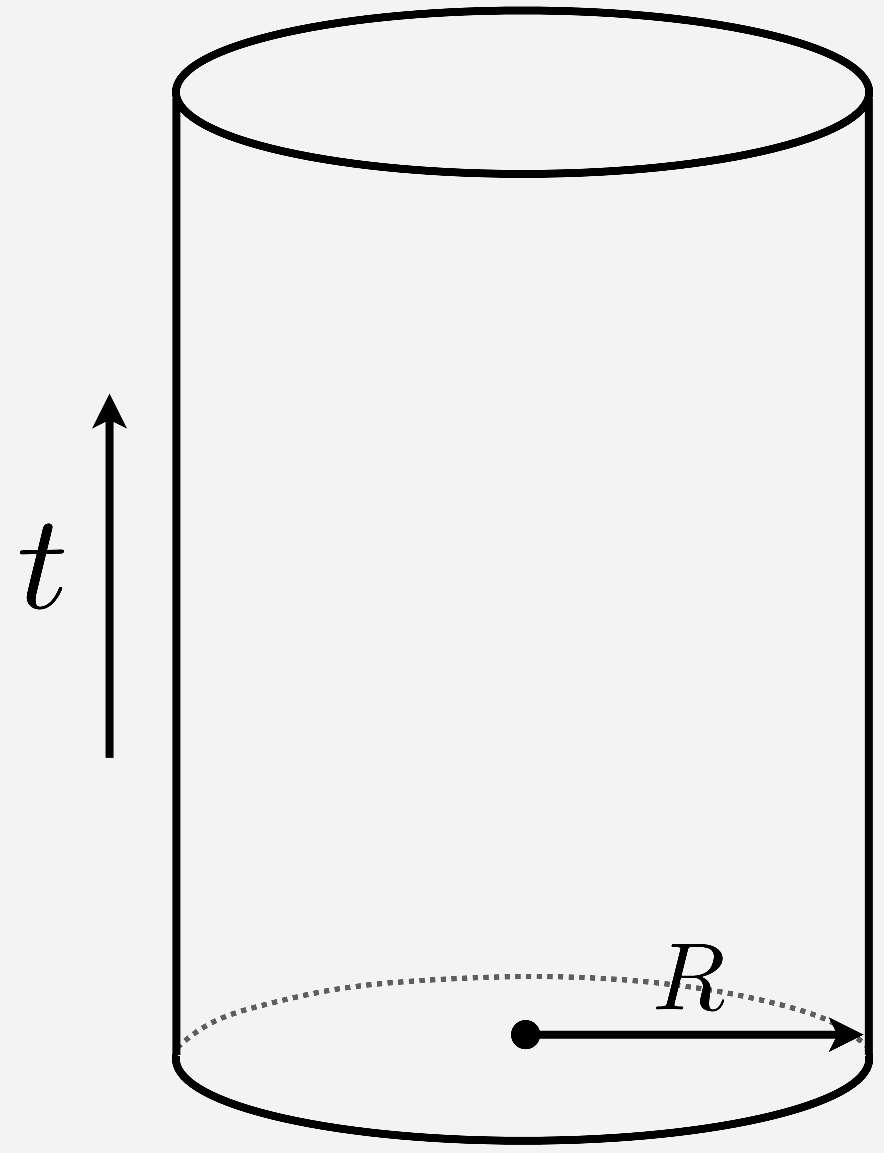
# Finite-Volume Spectrum

$$\mathcal{L} = (\partial\phi)^2 - m^2\phi^2 - g\phi^4$$



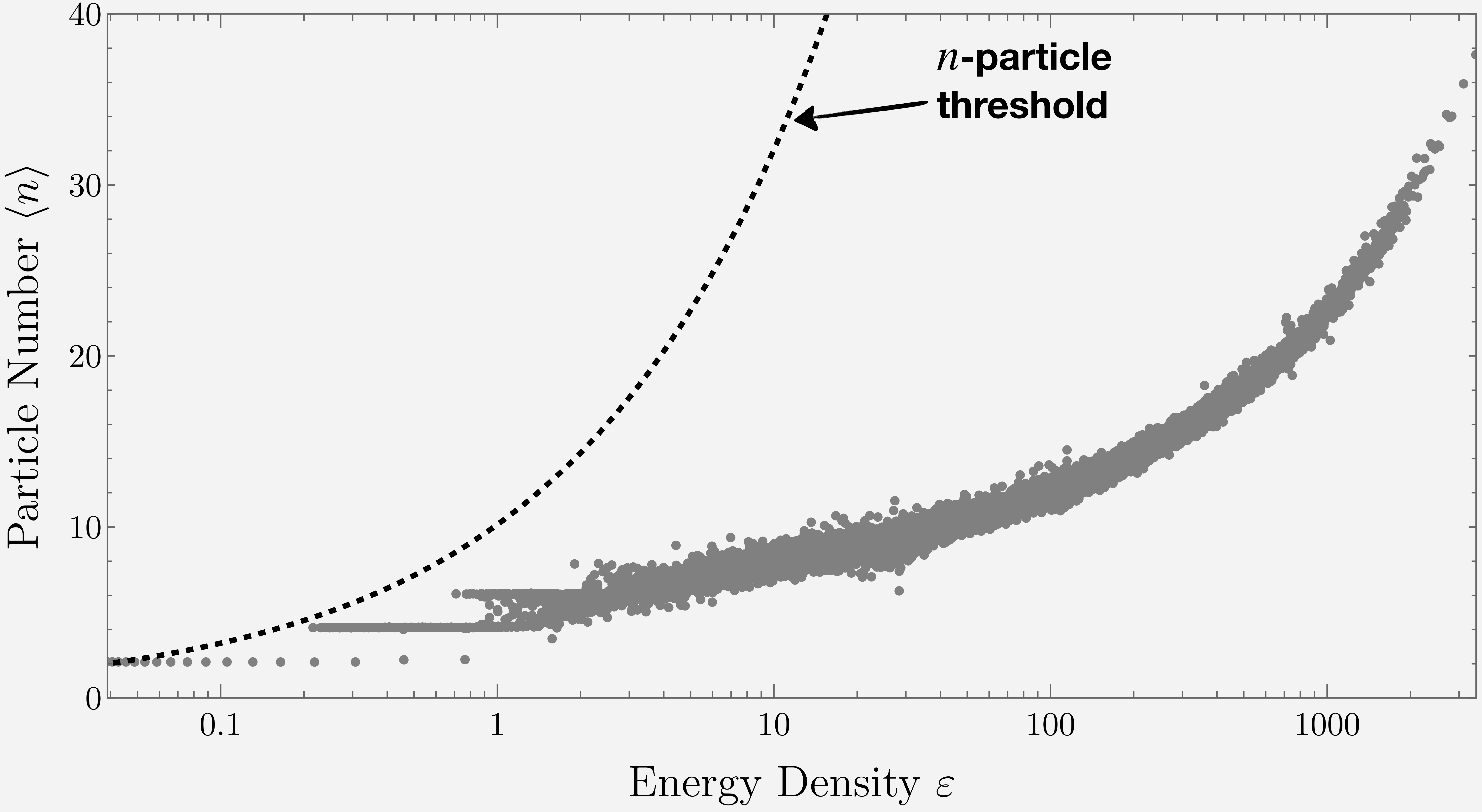

K. Farnsworth (based on Cohen et al ’21)

**High-energy spectrum is chaotic at finite coupling ($\neq$ UV CFT)**



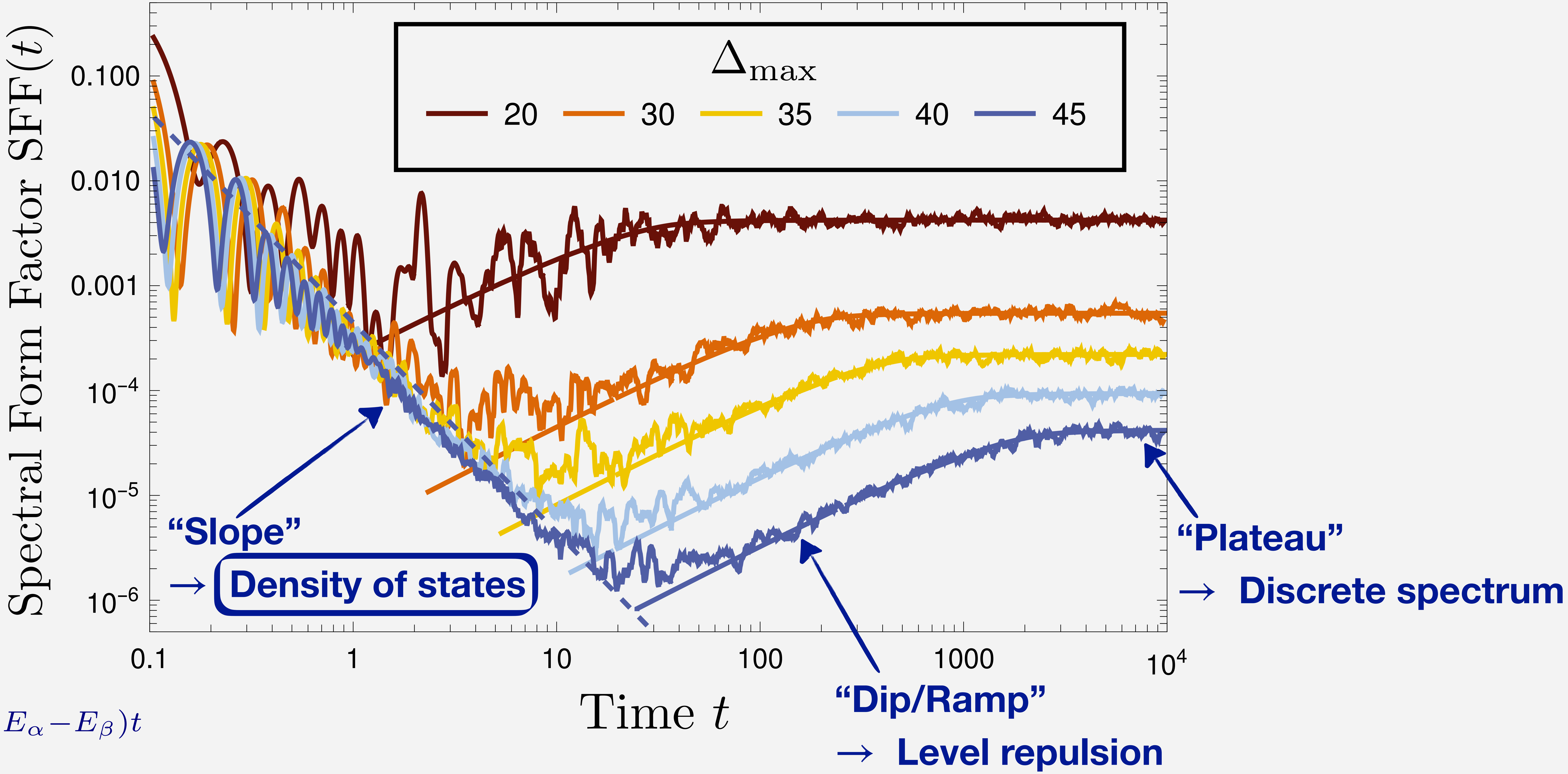

$g = 4 \quad E_{\text{max}} = 19 \quad 2\pi R = 10$

# No Visible Scars at Strong Coupling




$g = 1.6$

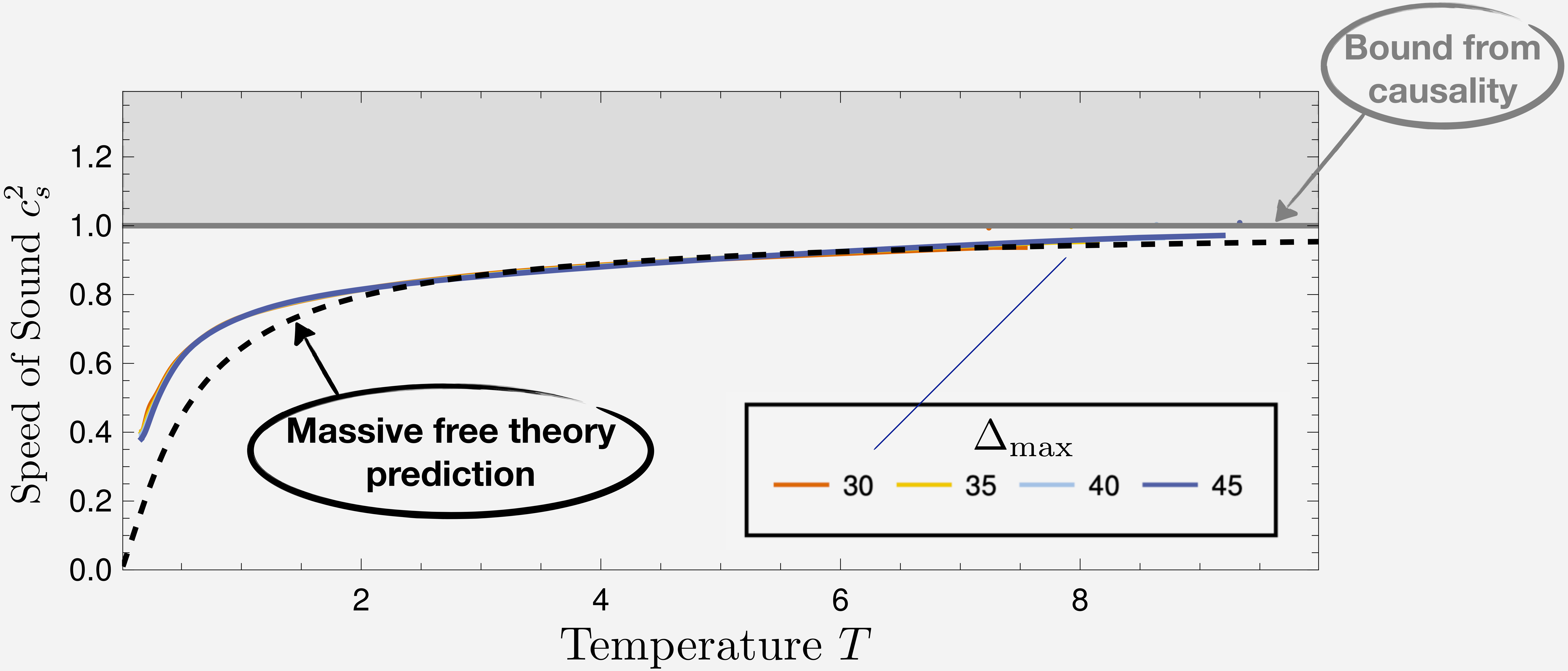
# Spectral Form Factor




$$\mathrm{SFF}(t) \equiv \frac{1}{N^2} \sum_{\alpha,\beta} e^{i(E_\alpha - E_\beta)t}$$

$g = 1.6$

# Thermodynamics

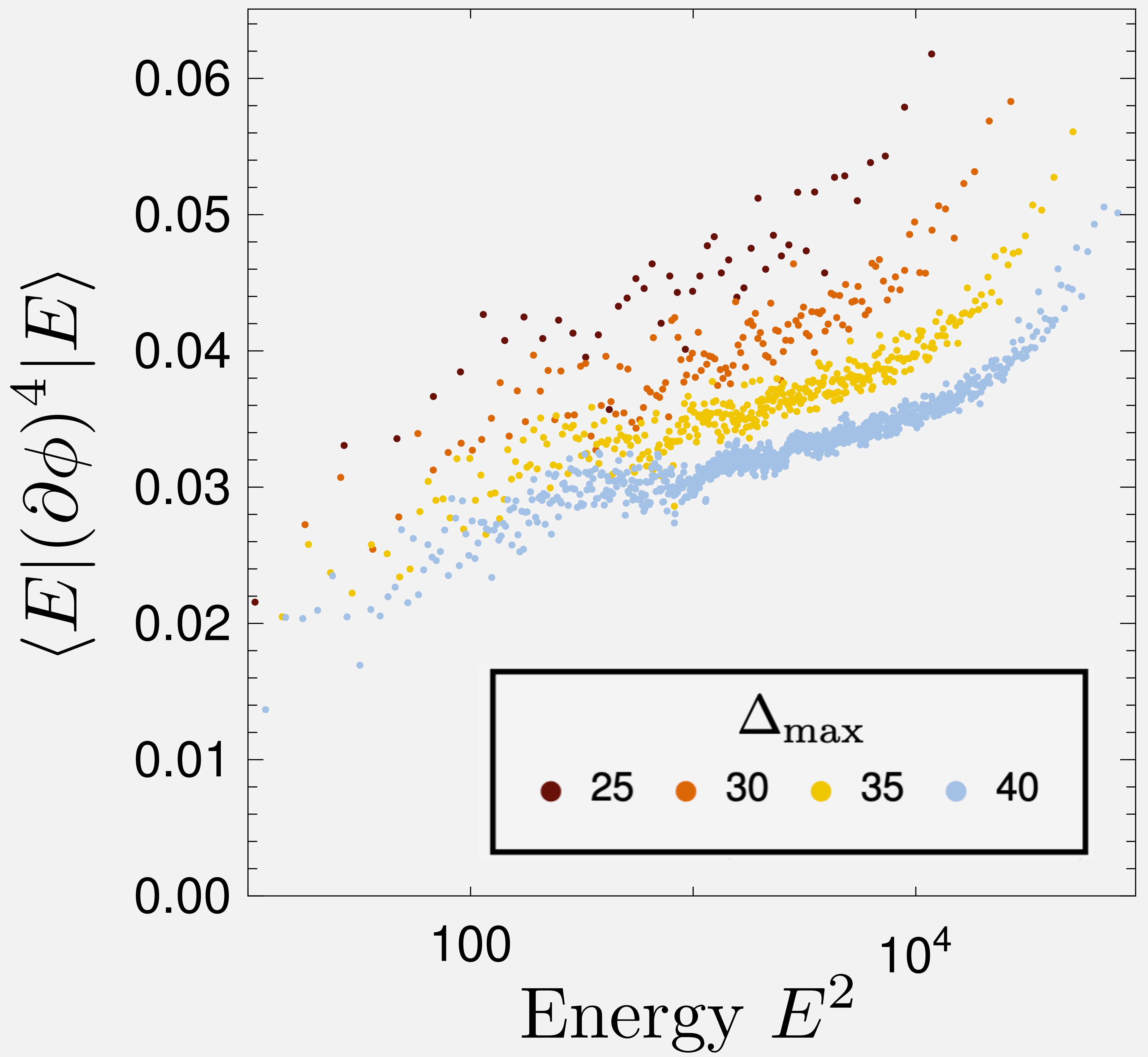


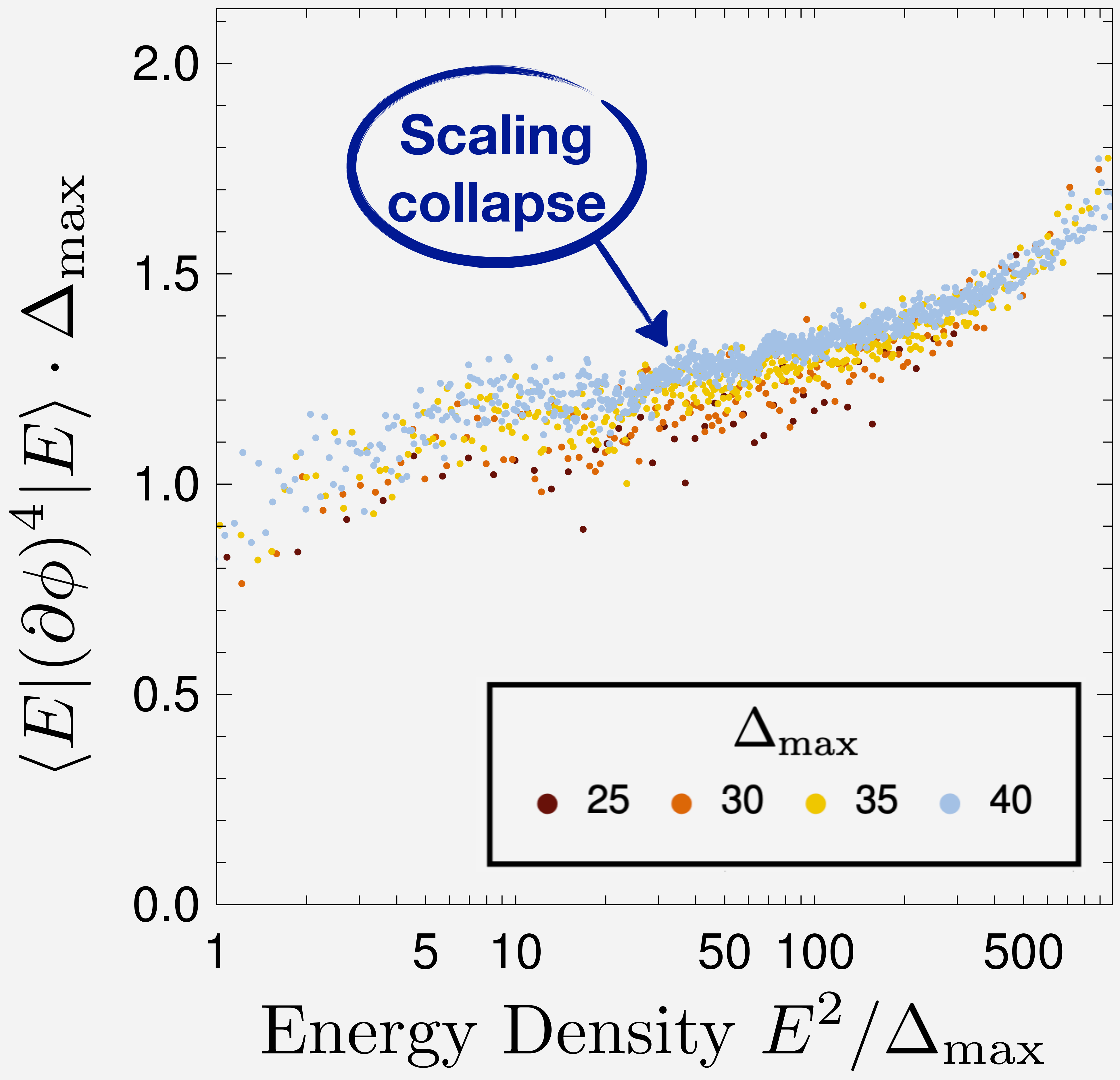

**Canonical ensemble:** $Z = \sum_{\alpha} e^{-\beta E} \longrightarrow \varepsilon \sim \frac{1}{V}\partial_\beta \log Z,\ P \sim \frac{1}{\beta}\partial_V \log Z \longrightarrow c_s^2 \equiv \left(\frac{\partial P}{\partial \varepsilon}\right)_V$

$g = 1.6$

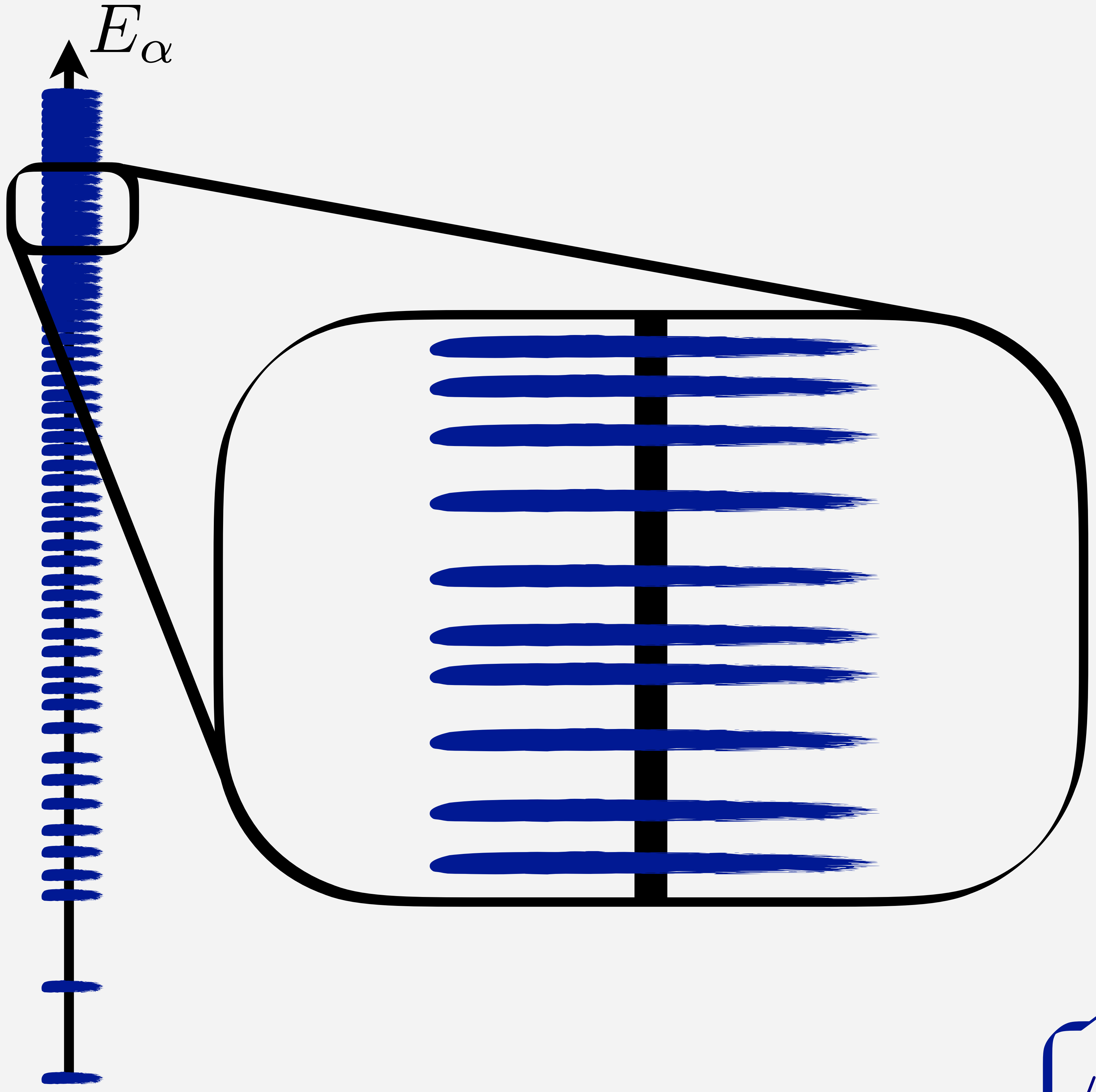
# Expectation Values

ETH: $\langle E|\mathcal{O}|E\rangle \sim \langle \mathcal{O}\rangle_\beta(\varepsilon)$



# High-Energy Eigenstates




- **Random matrix theory:**
  Eigenvalues in a small energy window essentially match those of a random matrix

- **Eigenstate thermalization hypothesis:**
  Expectation values are smooth functions of energy with small random variations

$$\langle E_\alpha|\mathcal{O}|E_\beta\rangle = \mathcal{O}(\bar{E})\delta_{\alpha\beta} + e^{-S(\bar{E})/2} g_{\mathcal{O}}(\bar{E},\Delta E) R_{\alpha\beta}$$

**Thermal expectation** $\langle \mathcal{O} \rangle_\beta$

**Random matrix**