Bootstrapping Chaos?

Miguel F. Paulos

Workshop on Quantum Chaos
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This Talk:

Systematics of the Bootstrap for
1d Conformal Theories

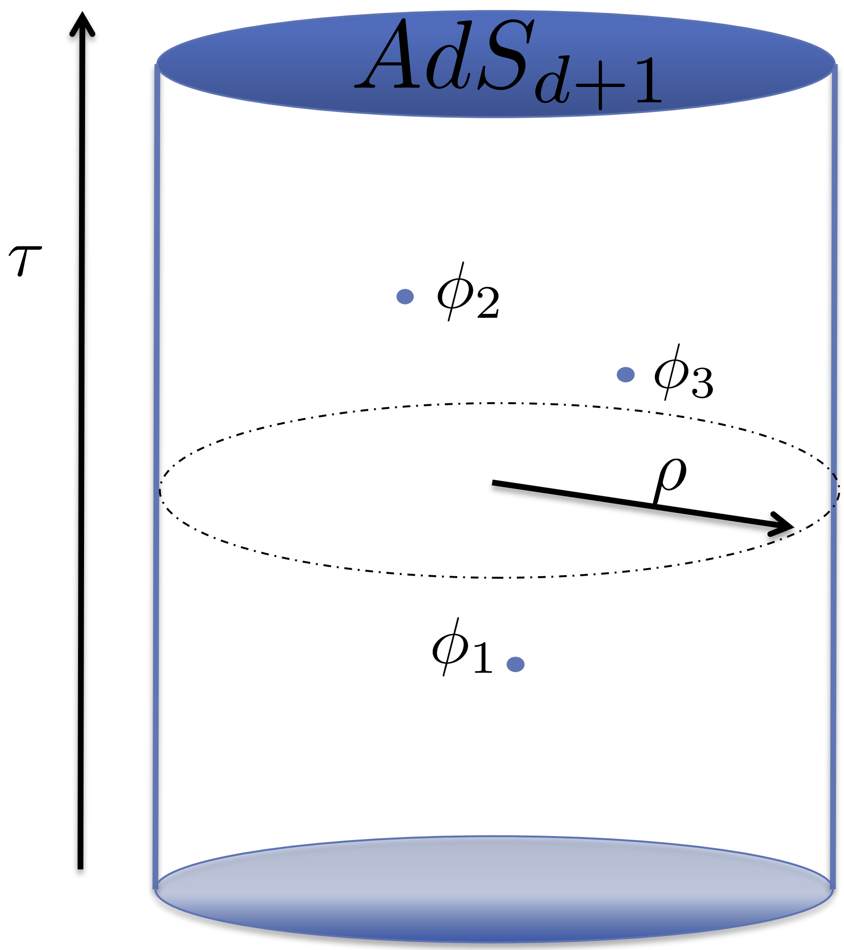
1d CFTs?

- Local 1d CFTs are trivial: topological.
- But many interesting examples still exist: we demand existence of global Ward identities for conformal algebra, but not local (i.e. no “stress tensor”).

- Important Applications:
    - 1D critical phenomena (long range models)
    - Conformal line defects
    - QFT in AdS_2 (including flat space limit)

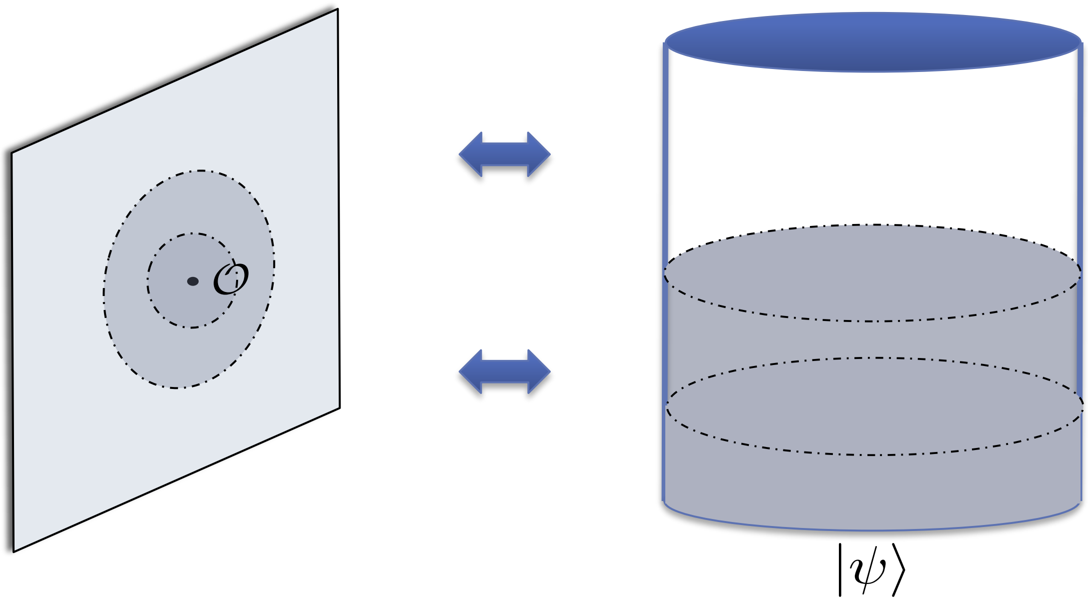
Can we probe interesting, chaotic, quantum systems by studying 1d CFTs?

QFT in AdS



- We put QFT in a box: anti-de Sitter space.
- Poincare symmetry of QFT in d+1 deformed to SO(d,2) – same number of generators.
- These are the symmetries of a conformal field theory in *d* dimensions.
- Pushing bulk operators to the AdS boundary at spatial infinity defines insertions of local boundary operators – CFT operators.

Radial quantization



$$e^{-Dt}\mathcal{O}|0\rangle \quad \Rightarrow \quad e^{-Ht}|\psi\rangle$$

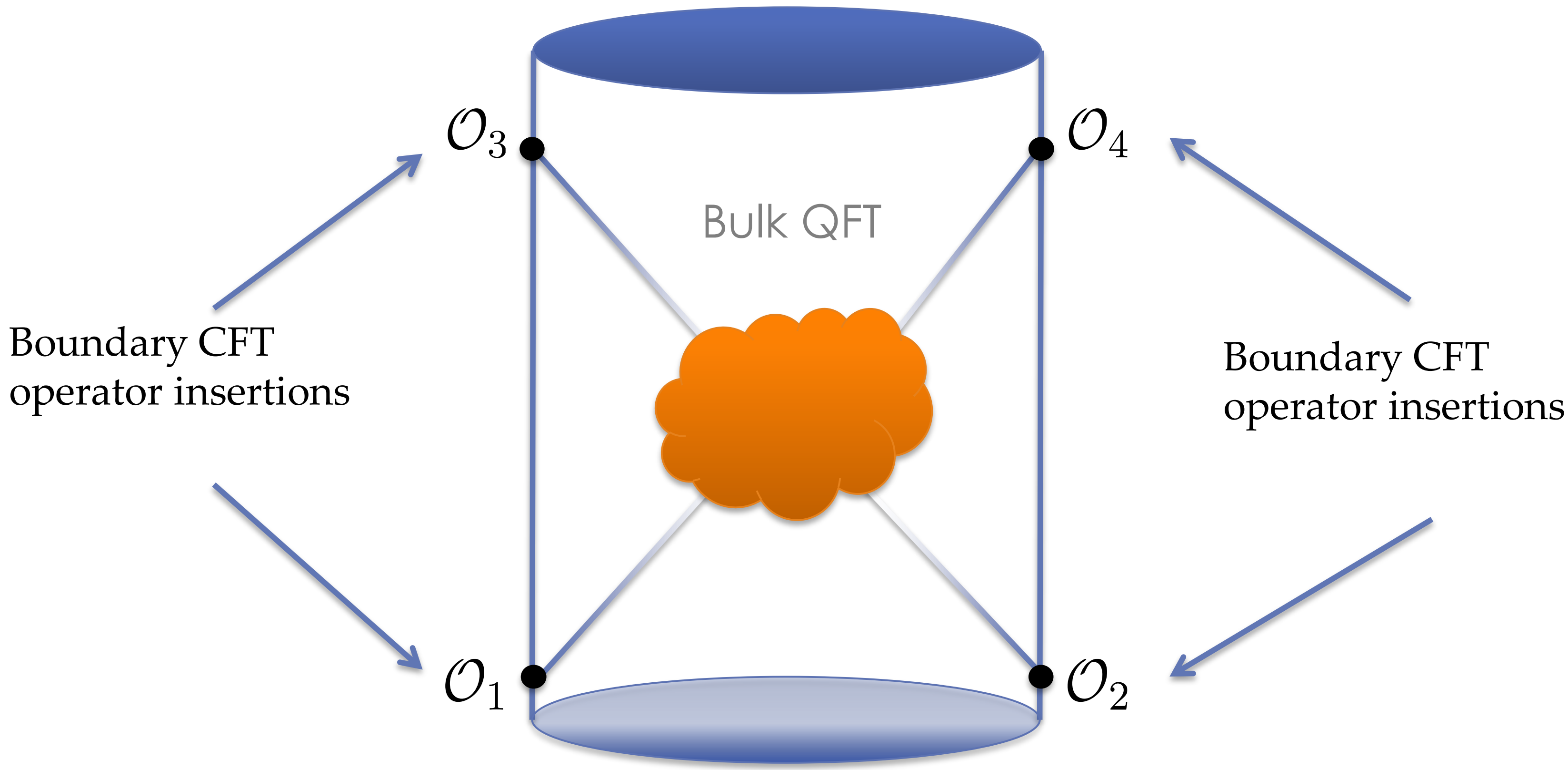
CFT scaling dimension

$$\Delta = E R_{\mathrm{AdS}}$$

Energy in *AdS* units

Scattering

- We can set up a bulk scattering experiment by sourcing with boundary insertions.

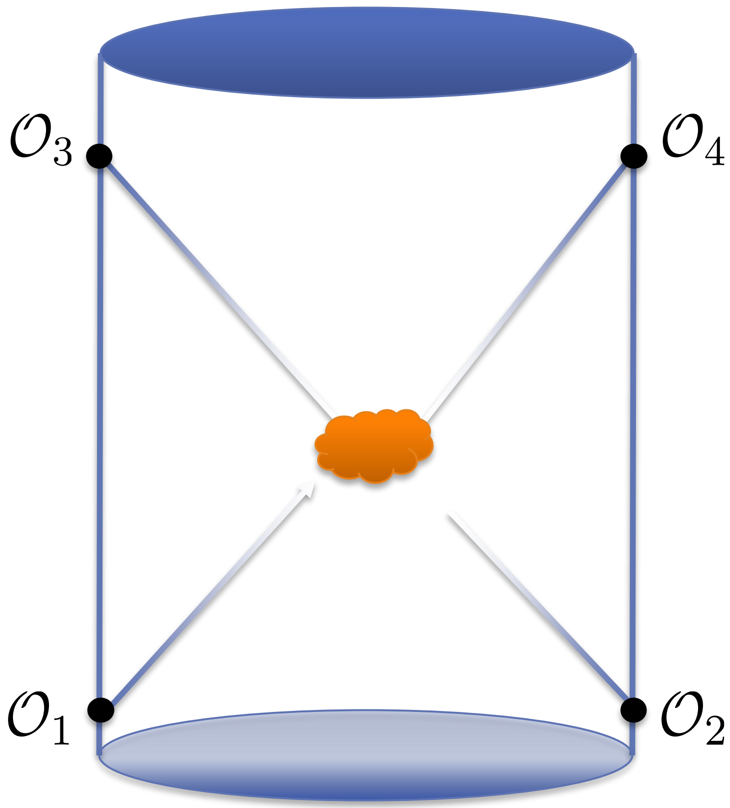


Scattering

- Large AdS radius recovers flat space scattering.

$$\Delta_{\mathcal{O}} = mR_{\text{AdS}}$$

- CFT operators with large scaling dimension!

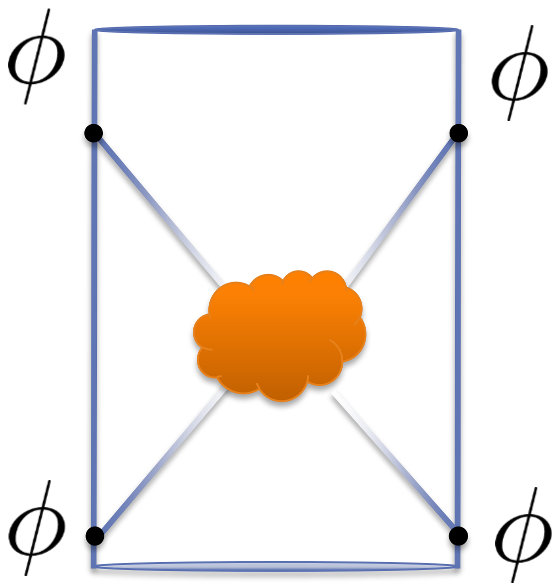


$$\langle \mathcal{O}(x_4)\mathcal{O}(x_3)\mathcal{O}(x_2)\mathcal{O}(x_1)\rangle$$

$$\downarrow$$

$$\langle k_3, k_4 | 1 + i\mathcal{T} | k_1, k_2 \rangle$$

Simple example: free theory



- Free field in AdS2:

$$S = \int_{\mathrm{AdS}_2} \left(\nabla\Phi\nabla\Phi + \Delta_\phi(\Delta_\phi - 1)\Phi^2 \right)$$

$$\phi(x) \sim \lim_{r\to\infty} r^{-\Delta_\phi}\Phi(r,x)$$

- Correlators are those of mean field theory:

$$\langle\phi_1\phi_2\phi_3\phi_4\rangle = \langle\phi_1\phi_2\rangle\langle\phi_3\phi_4\rangle + \ldots$$

$$\langle\phi_1\phi_2\rangle = \frac{1}{x_{12}^{2\Delta_\phi}}$$

Simple example: free theory

- Spectrum of the theory = collection of scaling dimensions
- State operator correspondence gives a complete description of the eigenstates of the Hamiltonian. We merely write all `words' with alphabet:

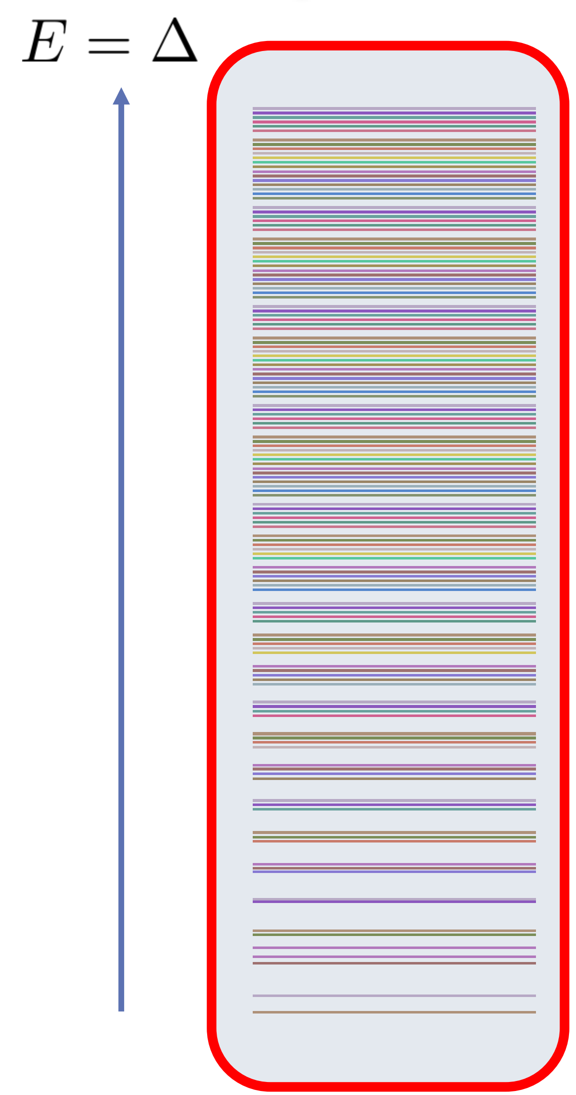
$$\{\phi, \partial\}$$

- To each such word there is an eigenstate. E.g.:

$$|\Psi\rangle = \phi(0)\partial^3\phi(0)\partial\phi(0)|0\rangle$$

$$\Rightarrow \quad H|\Psi\rangle = (3\Delta_\phi + 4)|\Psi$$

- The *total* number of such states grows exponentially with sqrt of energy. But in any correlator only a polynomially growing set of states appears.



Bootstrapping locality

- We will motivate the 1d CFT bootstrap in a slightly unusual way.
- Consider an (infinite dimensional) Hilbert space containing two sites. Formally we demand there exists P commuting with Hamiltonian,

$$P^2 = 1\,, \qquad [H, P] = 0$$

- A *local operator* on the Hilbert space is a pair:

$$\phi_L, \phi_R: \qquad P\phi_L P = \phi_R$$

such that the *locality condition* holds: $(H|E_i\rangle = E_i|E_i\rangle)$

$$\langle E_i|[\phi_L(t), \phi_R(0)]|E_j\rangle = 0\,, \quad t \in [0, T]$$

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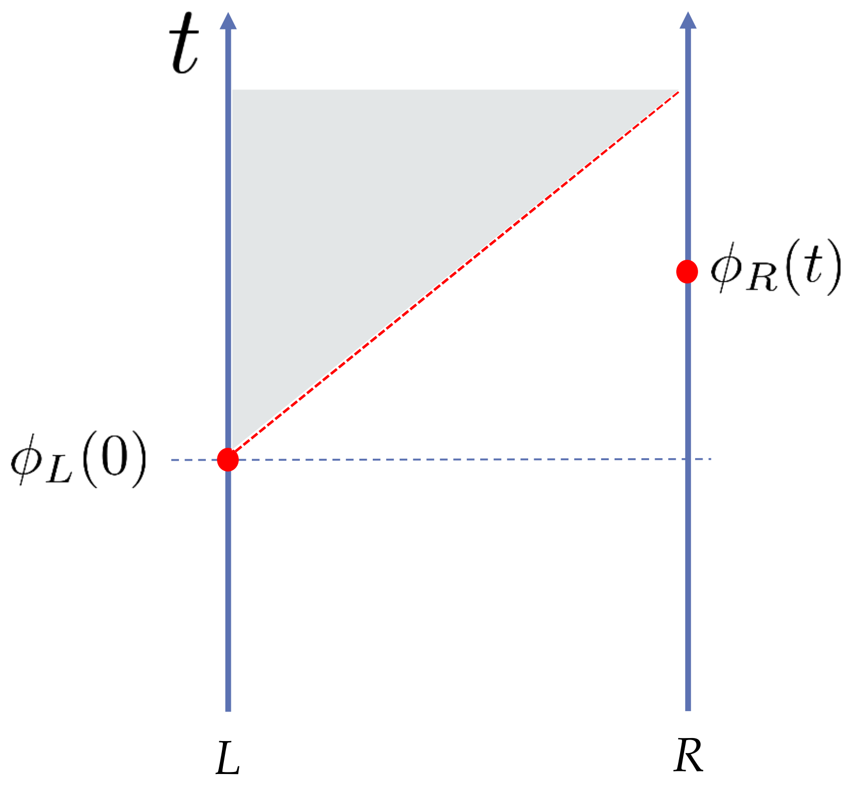
$$\sum_{E'} |\langle E|\phi_L(0)|E'\rangle|^2 (-1)^{P_{E'}} \sin[t(E-E')] = 0, \quad t \in [0, T]$$

Bootstrapping locality

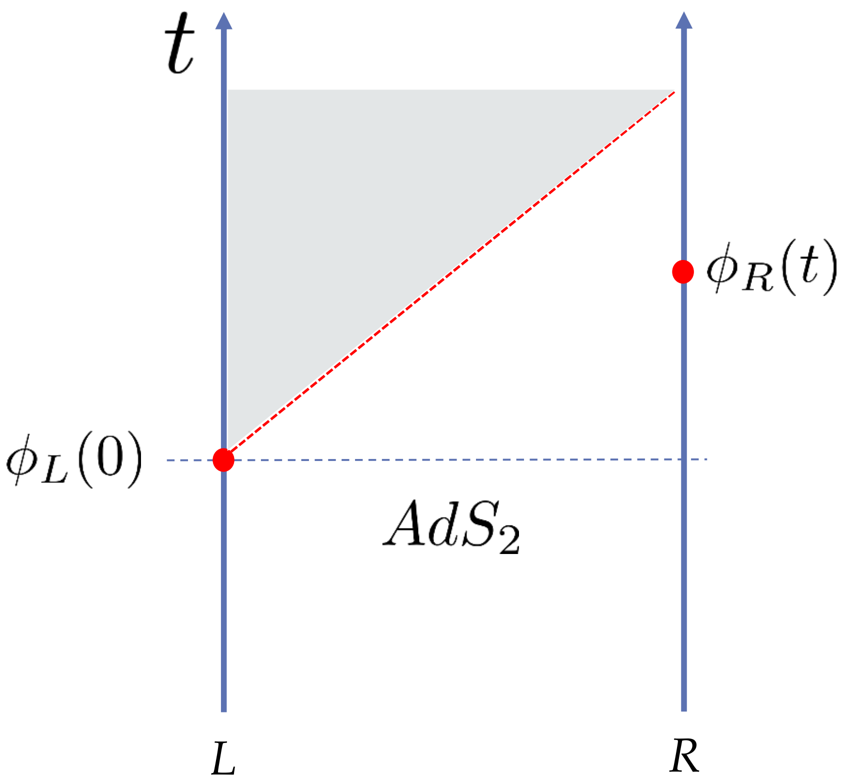
$$\langle E_i|[\phi_L(t), \phi_R(0)]|E_j\rangle = 0\,, \quad t \in [0, T]$$

- Intuition: there exists Hamiltonians for which the two sites are `far away'.
- The above equation constrains both allowed (i.e. local) Hamiltonians and operators matrix elements.
- General solution: `fill in' the gap between the two sites with a local QFT.



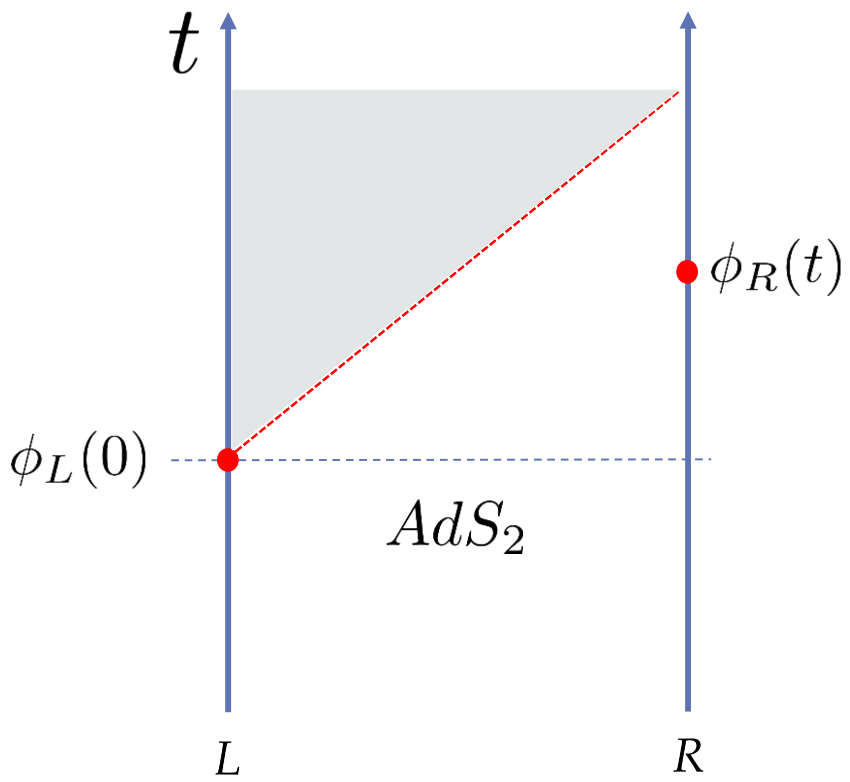
Bootstrapping locality

- For a 1D CFT we `fill in' the space with a QFT in AdS2.
- Thanks to state operator correspondence, the above locality equation can be shown to be equivalent to associativity of the OPE: i.e. the crossing equation.
- The *conformal bootstrap* is thus this locality problem in the special case where the Hilbert space has additional (conformal) symmetry.
- It constructs simultaneously Hamiltonian and operators satisfying locality.



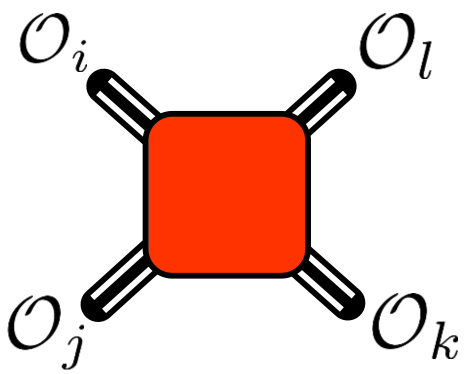
Bootstrapping locality

- How does locality/crossing imply (or not) chaotic spectra?



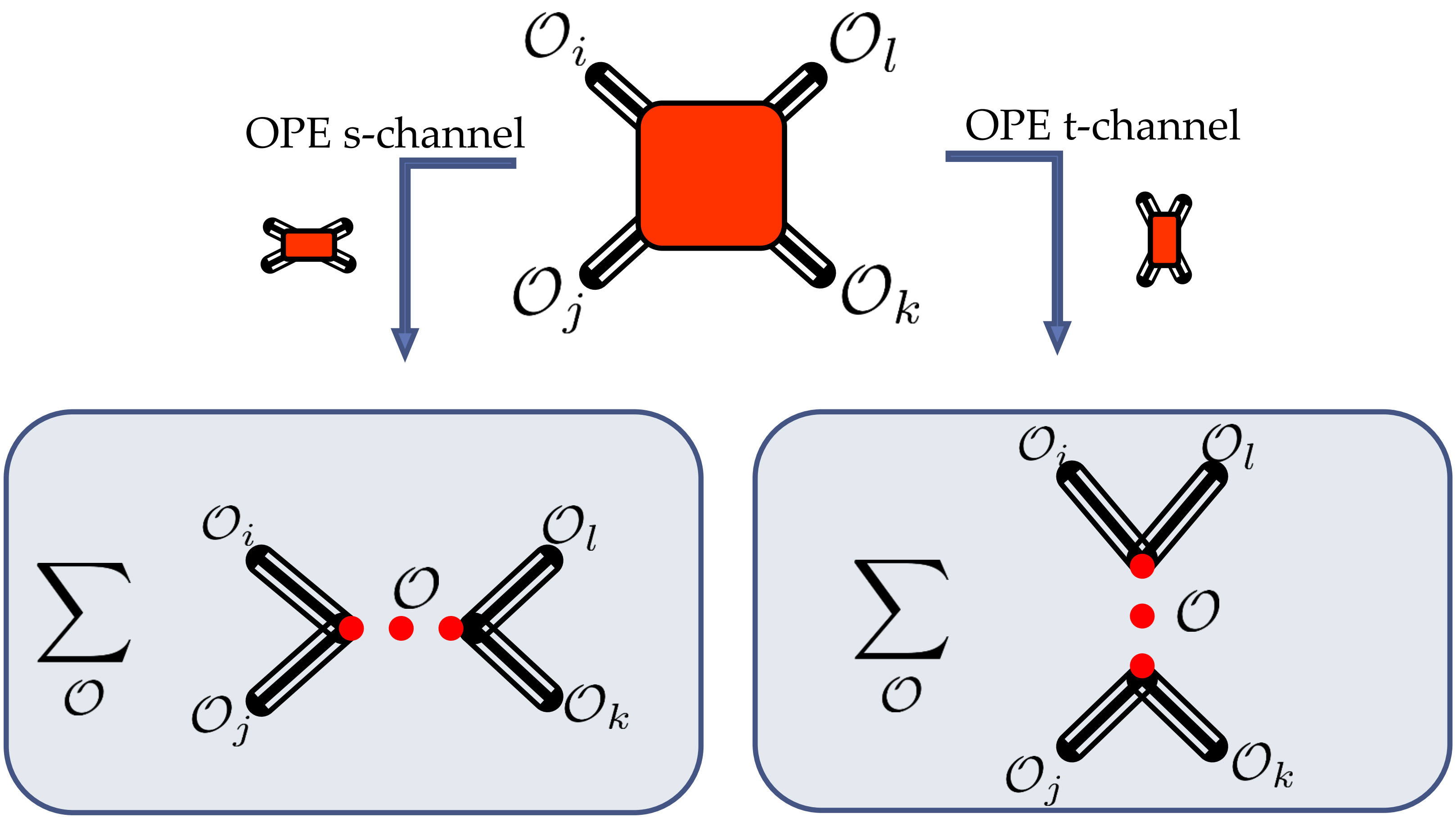
Correlator Bootstrap

- Goal: determine correlators of local operators.



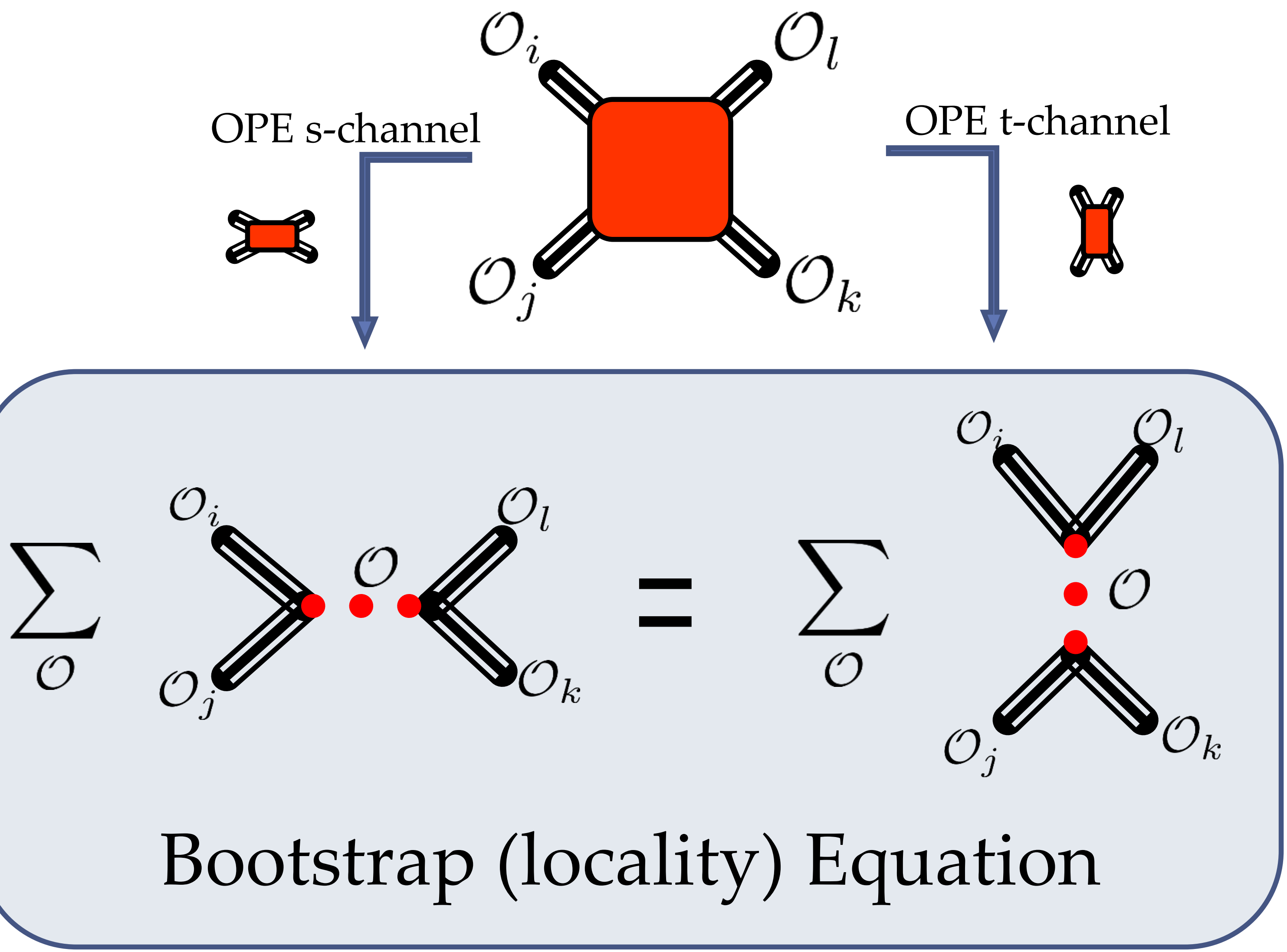
$\mathcal{O}_i$ generic operator in the CFT

OPE associativity

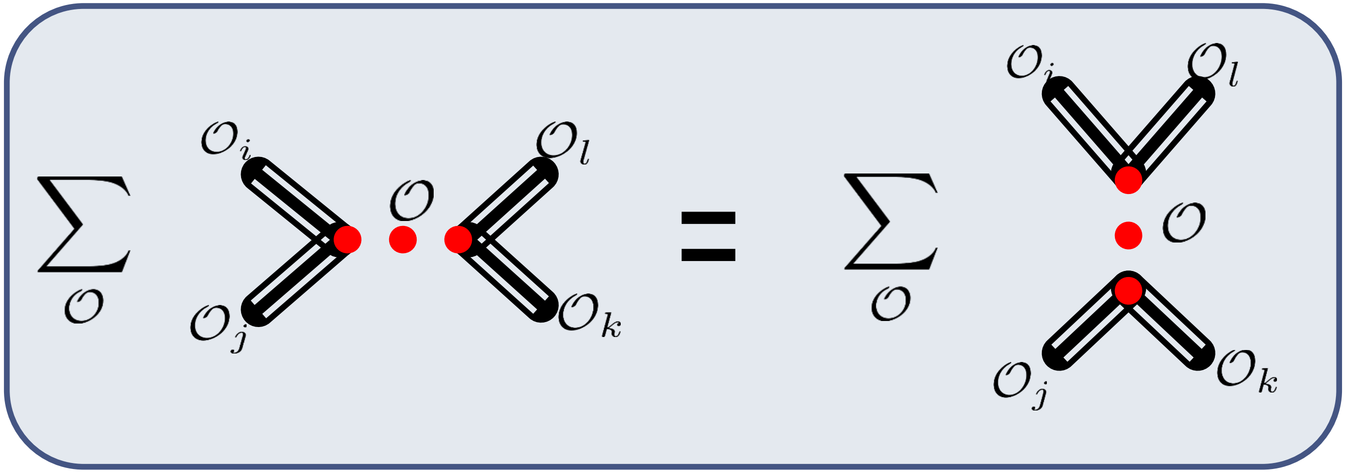


$$\mathcal{O}_i \times \mathcal{O}_j = \sum_{\mathcal{O}} \lambda_{12,\mathcal{O}} \mathcal{O}$$

OPE associativity



The Bootstrap Program



- Proposal: above determine all allowed sets of sphere CFT data *Polyakov, '74* *Ferrara, Grillo, '73*
- Sufficient to consider 4 pt functions - but must consider *all* of them (∞)
- Bootstrap: extracting maximal amount of information from these equations.

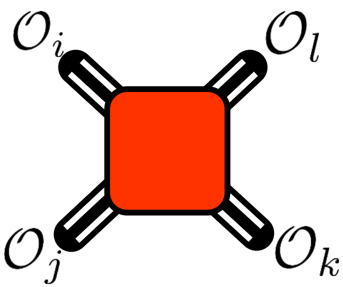
Two truncations

Bootstrap methods involve two kinds of truncations:

- **Fundamental**
    - Truncate set of OPEs to consider
- **Technical**
    - Truncate constraints arising from a fixed set of OPEs
- Both truncations imply that some information is lost.

Fundamental truncation

- We can consider only a *finite* set of correlators and associated bootstrap equations



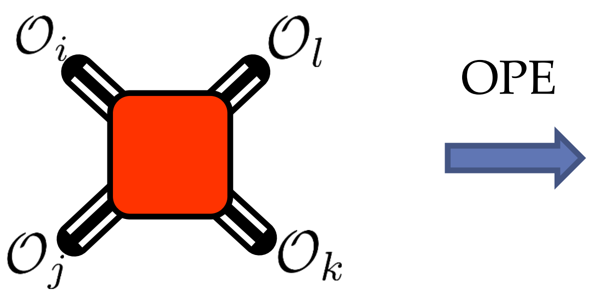
$$i, j, k, l = 1, \ldots, N$$

- Are such equations sufficient for determining the correlators*?

under ***finite* set of assumptions – enough to specify theory of interest*

Fundamental truncation

- If true, *generically* it would tell us partial OPE information but *full* CFT spectrum* (!)

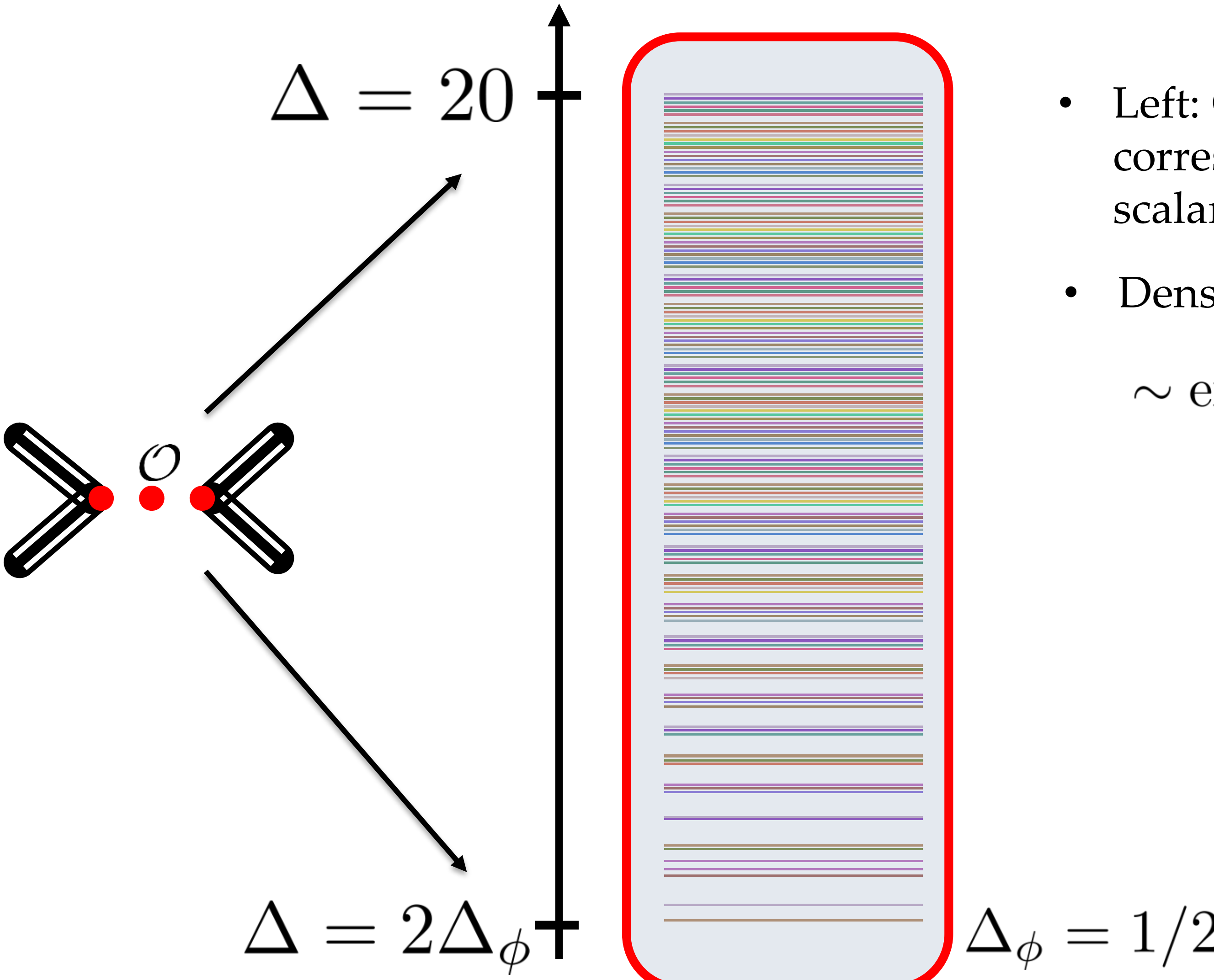


$$\left\{ \Delta_{\mathcal{O}}, \lambda_{\mathcal{O}}^{i,j}, \lambda_{\mathcal{O}}^{i,l}, \lambda_{\mathcal{O}}^{i,k} \right\}$$

- That is, even a single correlation function generically knowns about the full spectrum of the theory…

**modulo symmetry selection rules*

Fundamental truncation



- Left: CFT spectrum* corresponding to single scalar field in AdS_2
- Density of states

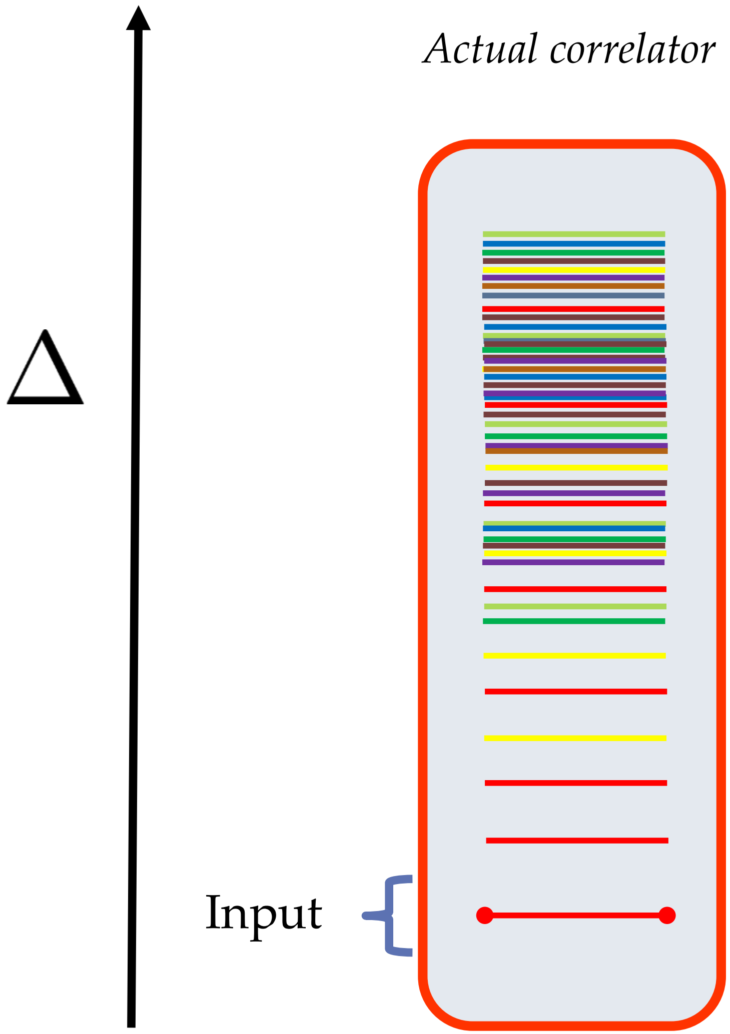
$$\sim \exp\left(c\sqrt{\Delta}\right)$$

primary operators perturbed with random anomalous dimensions to lift degeneracies

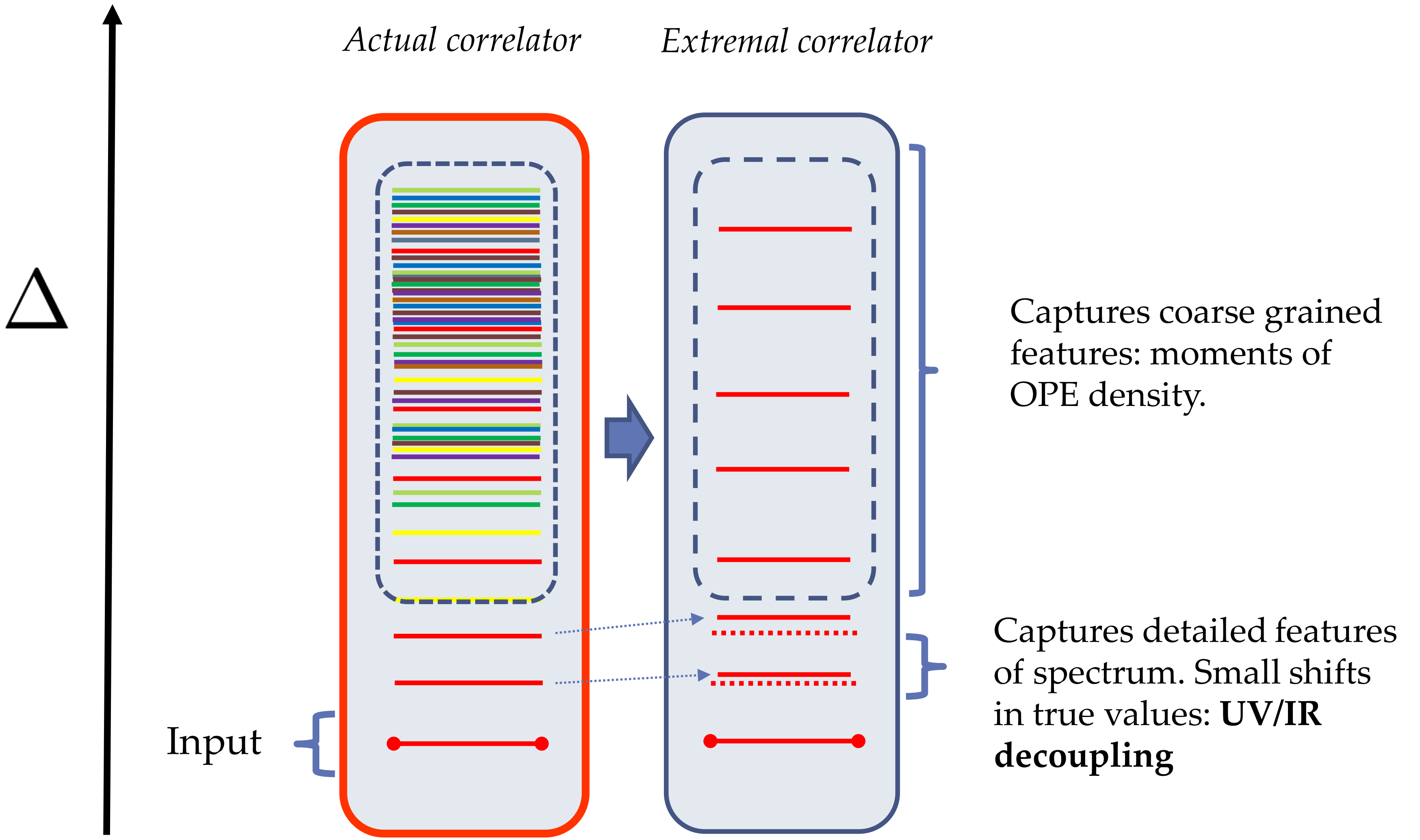
Fundamental truncation

- In fact, bootstrap equations have insufficient resolving power to uniquely determine full OPE (under finite assumptions).
- Constraints are sufficient only to uniquely solve for special solutions to the bootstrap equations we call **extremal**.
- Such solutions are uniquely determined by being the sparsest possible consistent with finite assumptions.
- In particular, they have a **constant density** of operators above some scale and hence are *generically* **unphysical** (i.e. cannot be embedded into full-fledged CFTs).
- *Bootstrap works only if actual CFT correlators of interest are sufficiently close to extremal ones.*

Fundamental truncation

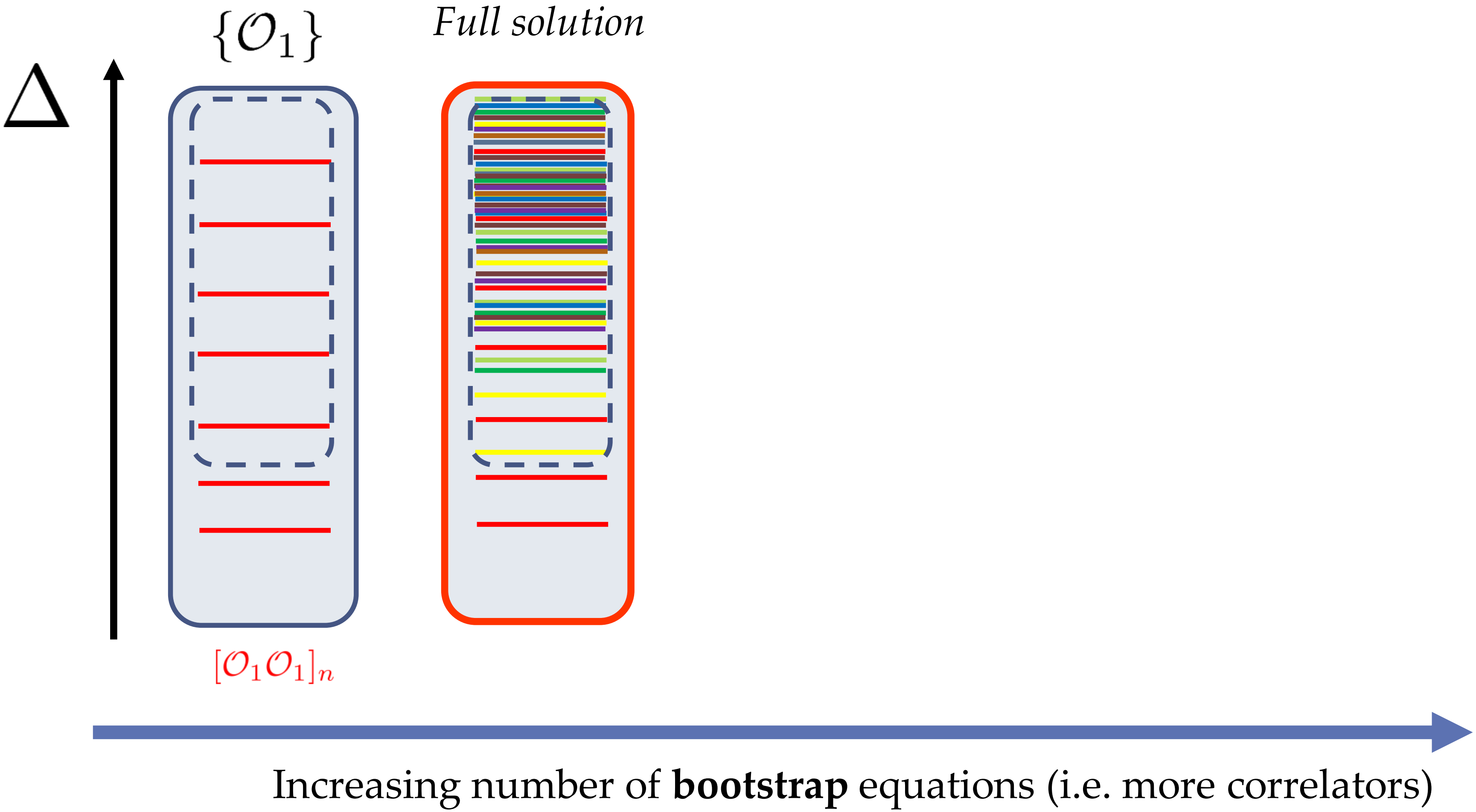


Fundamental truncation



Fundamental truncation

- Adding bootstrap equations leads to denser extremal solutions: better approximations to the actual OPE.



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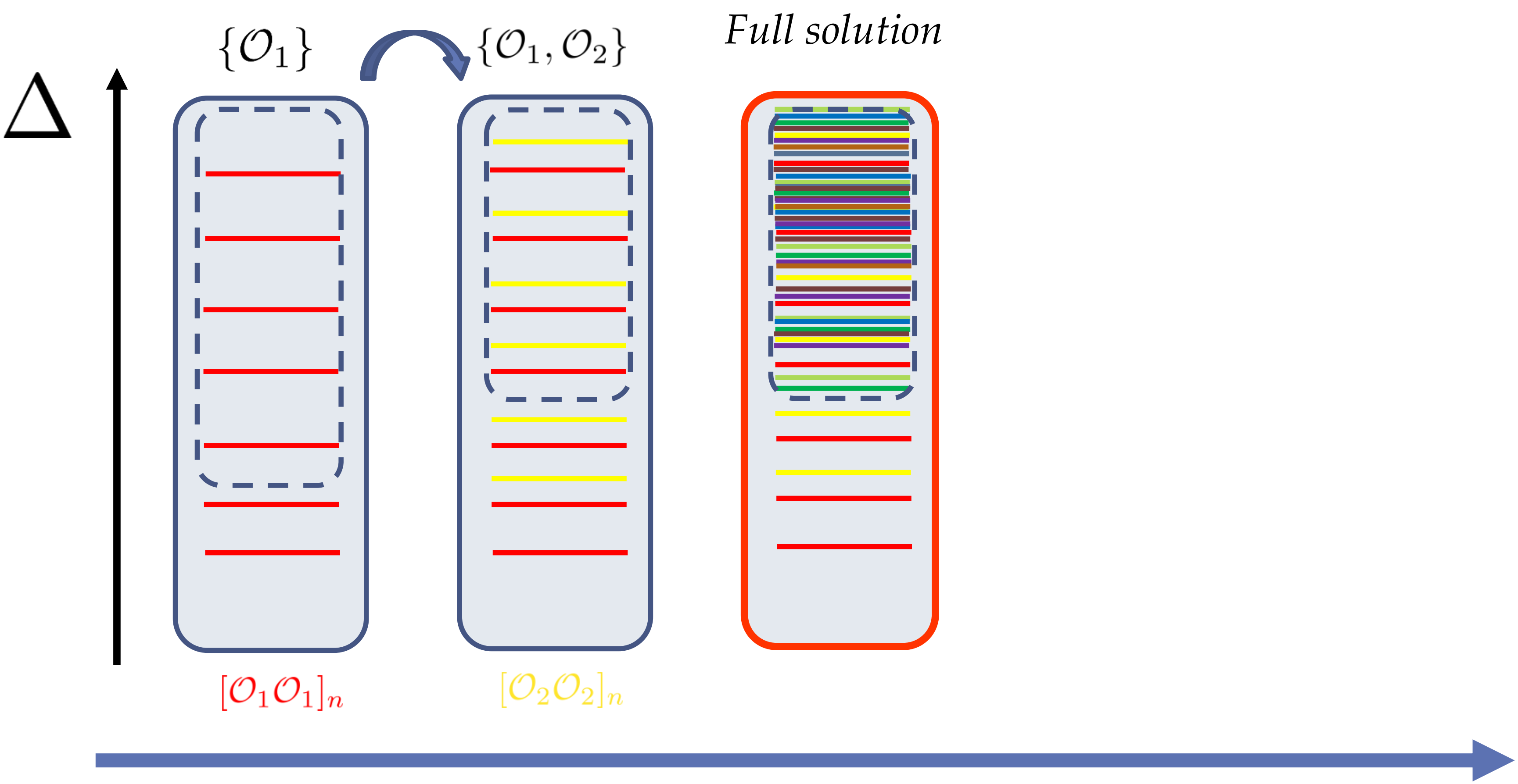

Increasing number of **bootstrap** equations (i.e. more correlators)

Fundamental truncation

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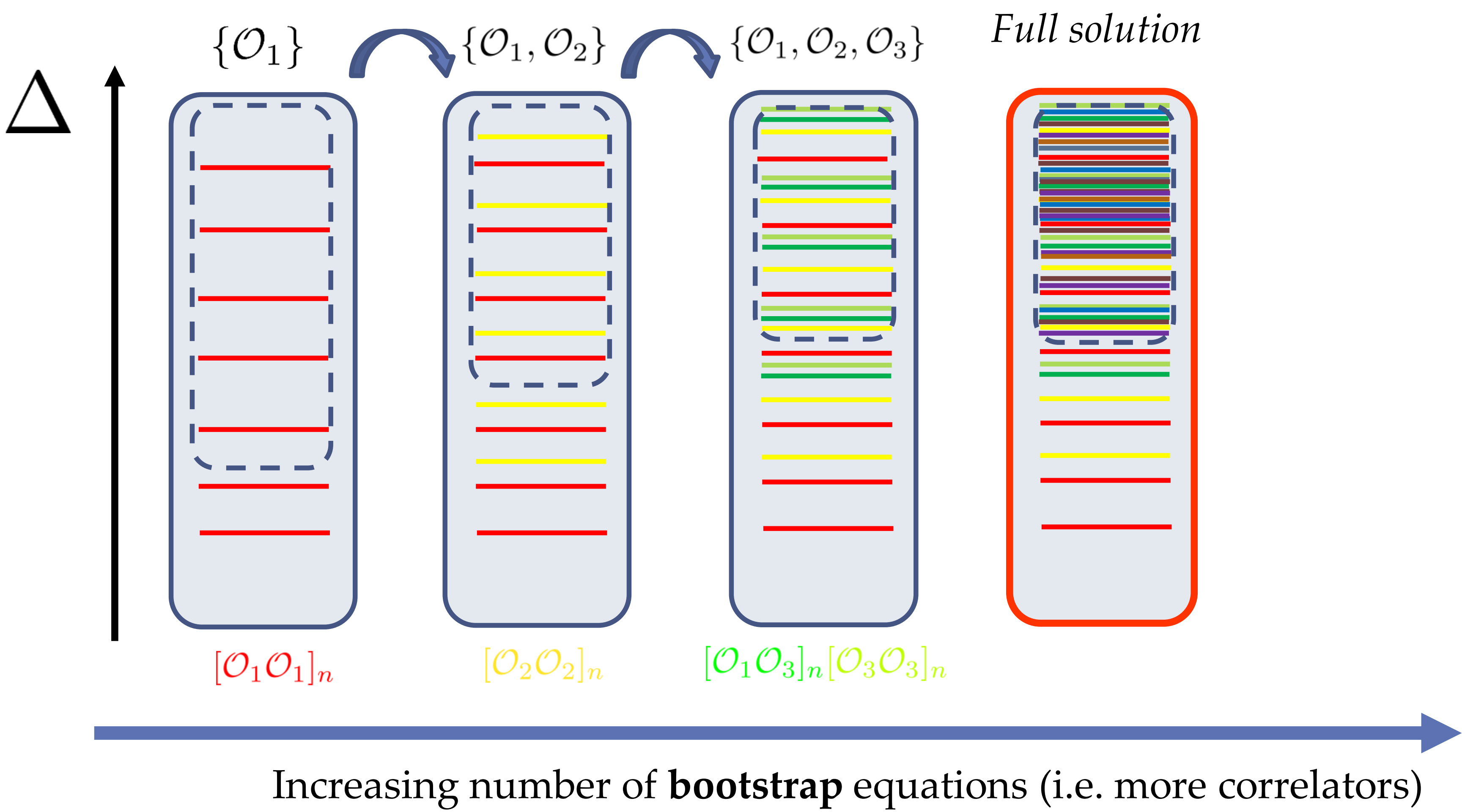

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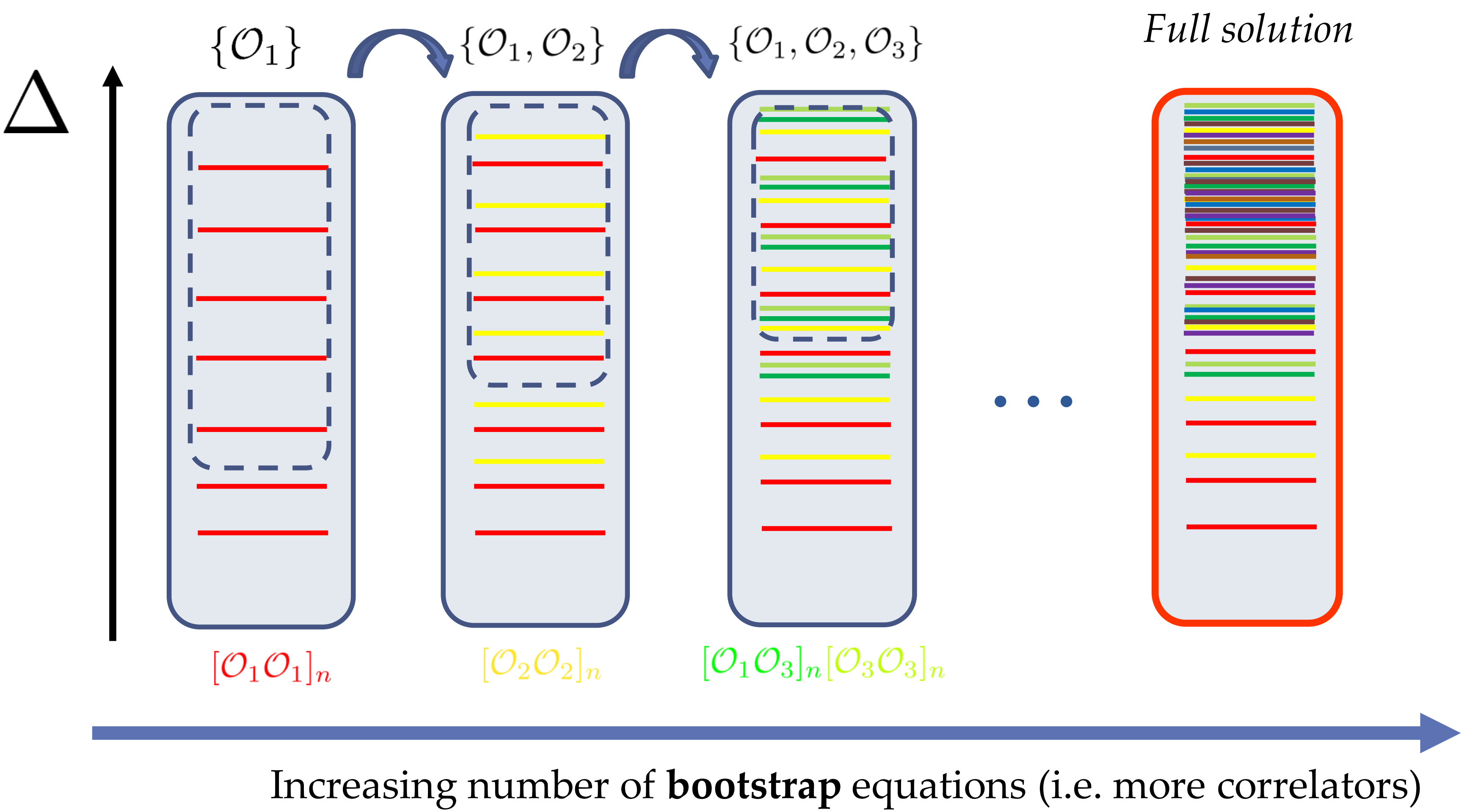


Fundamental Truncation

In summary:

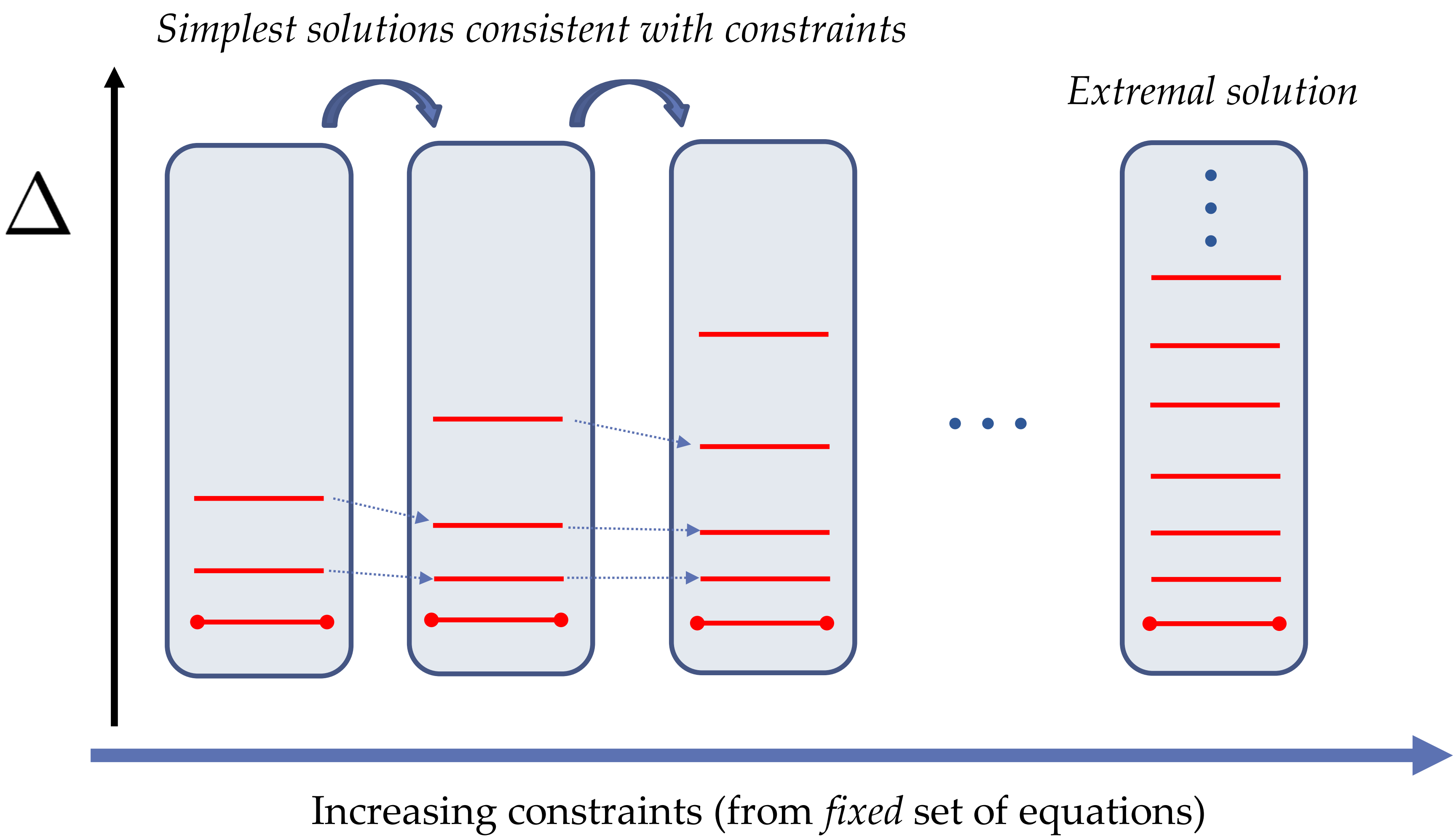
- **Physical correlators:**
  - OPE content consistent with *infinite set* of bootstrap equations
- **Extremal correlators:**
  - OPE content consistent with chosen *finite set* of bootstrap equations
  - Sparsest possible under finite number of assumptions
  - Approximate some desired CFT
  - Determine rigorous bounds (which they saturate) on general CFTs (as we will see)

How to construct them?

Technical truncation

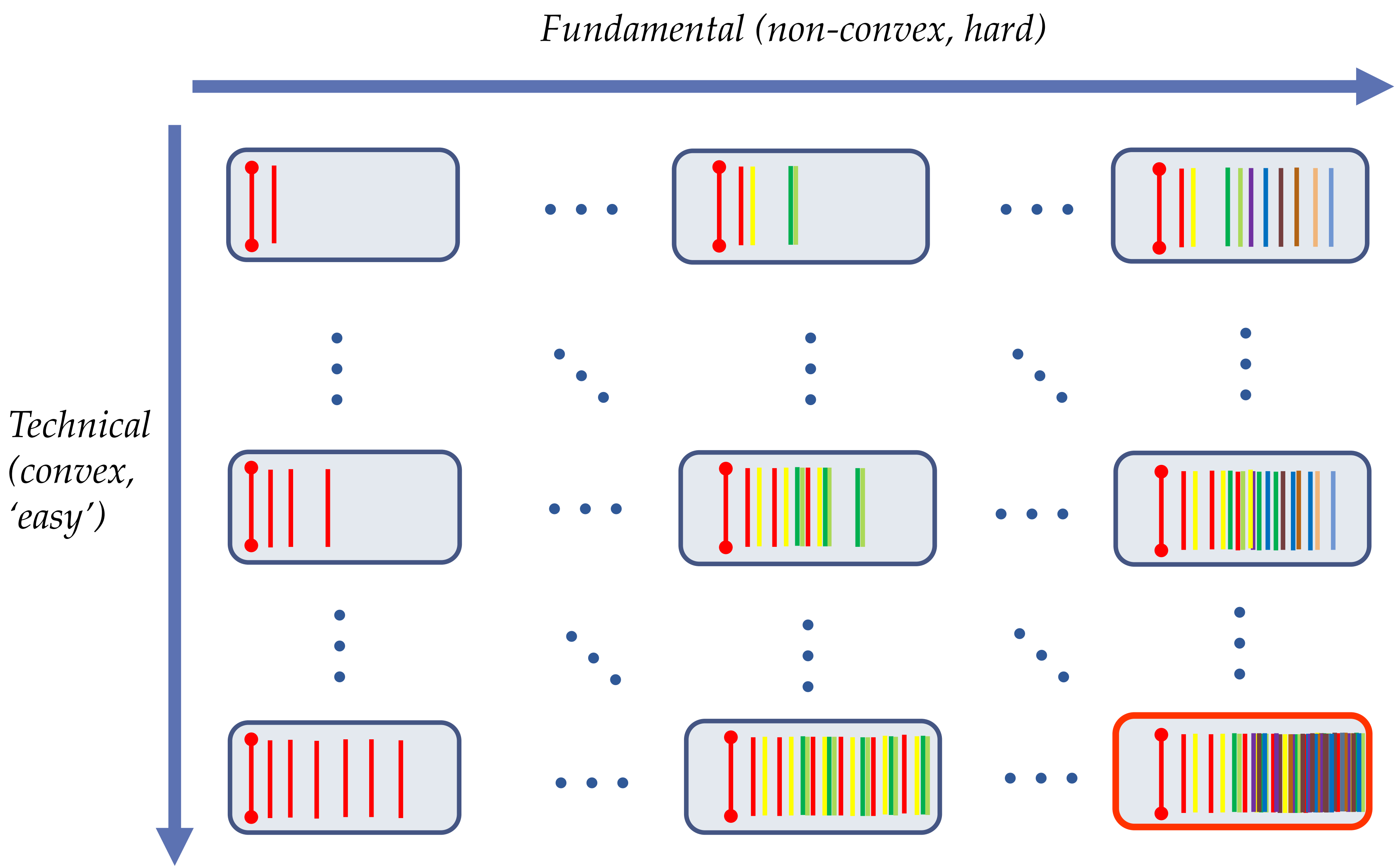
- Each individual bootstrap equation imposes infinitely many constraints on an infinite set of data.
- We must solve these constraints to determine the extremal solution.
- In practice, generically only a finite but large subset of constraints can be solved.
- *Choice of constraint basis is* ***critical:*** a good choice of basis allows for an efficient, and arbitrarily good determination of the *full* extremal correlator.

Technical truncation



Approach to infinity strongly dependent on choice of constraint basis

The extremal bootstrap program



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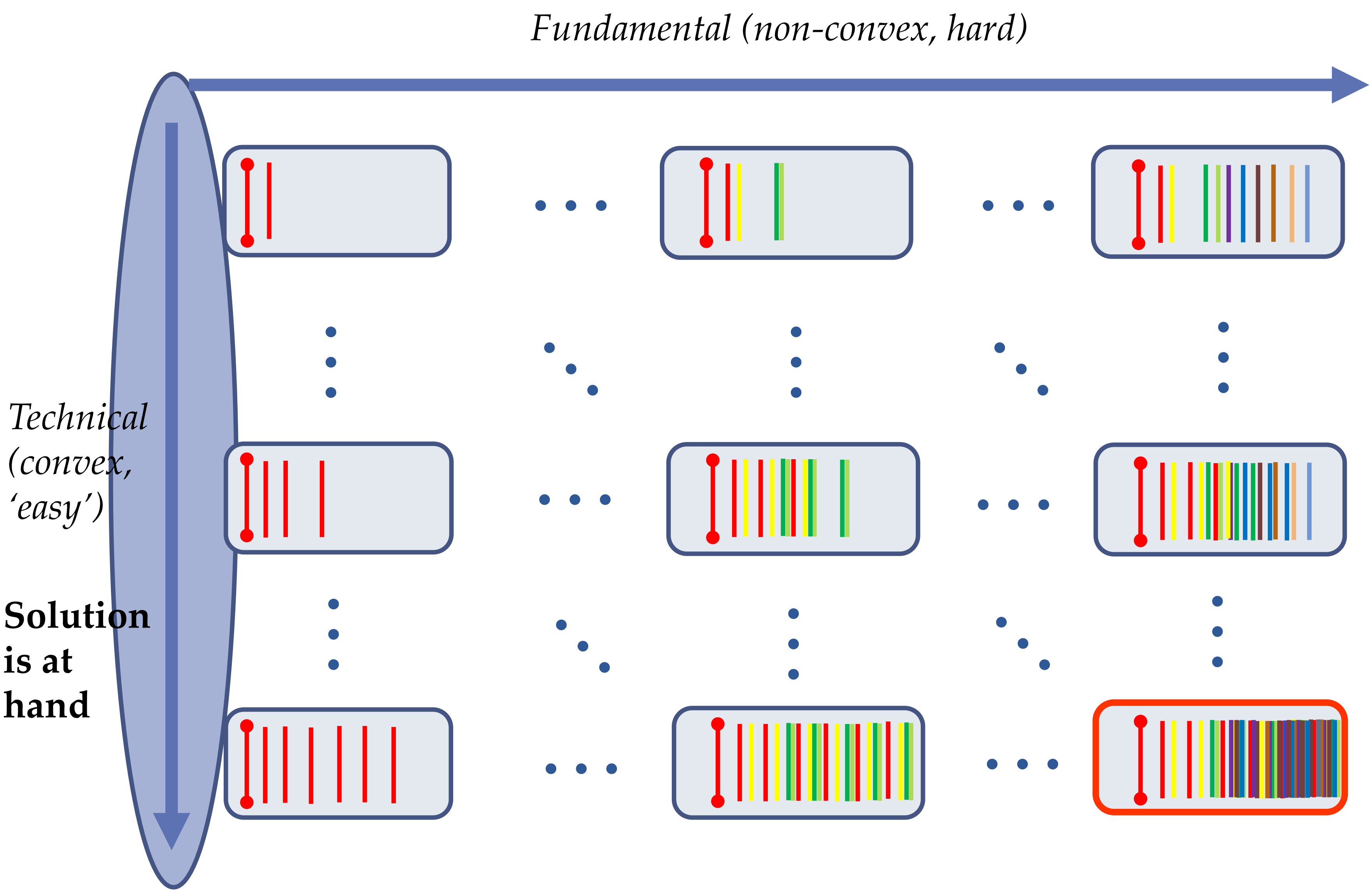
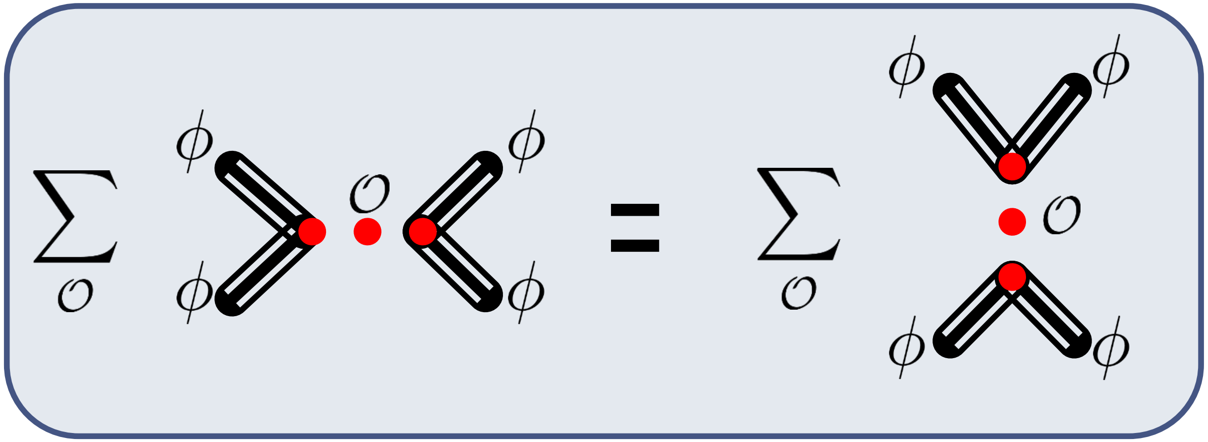


Single equation



$$\sum_{\Delta} a_{\Delta} F_{\Delta}(z) = 0$$

$$(a_{\Delta} := \lambda^2_{\phi\phi\mathcal{O}_{\Delta}} \geq 0)$$

- Linear constraints..

Introducing bases

$$\sum_{\Delta} a_{\Delta} F_{\Delta}(z) = 0$$

- Uncountable constraints on uncountable variables..
- Introduce countable basis:

$$F_{\Delta}(z) = \sum_{n=0}^{\infty} \omega_n^B[F_{\Delta}] B_n(z) \qquad \omega_n^B[B_m] = \delta_{n,m}$$

Linear functionals dual to *basis elements*

Introducing bases

$$\sum_{\Delta} a_{\Delta}\, F_{\Delta}(z) = 0$$

- Uncountable constraints on uncountable variables..
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$$F_{\Delta}(z) = \sum_{n=0}^{\infty} \omega_n^B[F_{\Delta}] B_n(z) \qquad \omega_n^B[B_m] = \delta_{n,m}$$

- Paradigmatic example: Taylor series around chosen point

$$B_n(z) = (z - z_0)^n\,, \qquad \omega_n^B = \frac{1}{n!}\partial_z^n\Big|_{z=z_0}$$

Rattazzi, Rychkov, Tonni, Vichi '08

Functional Bootstrap Equations

$$F_\Delta(z) = \sum_{n=0}^{\infty} \omega_n^B[F_\Delta] B_n(z)$$

$$\sum_\Delta a_\Delta \, F_\Delta(z) = 0$$

$$\Updownarrow$$

$$\sum_\Delta a_\Delta \omega_n^B [F_\Delta] = 0 \, , \quad n = 0, 1, \ldots$$

Extremal bases

- Starting from general decomposition,

$$F_\Delta(z) = \sum_{n=0}^{\infty} \omega_n^B[F_\Delta] B_n(z) \qquad\qquad \omega_n^B[B_m] = \delta_{n,m}$$

we choose:

$$\left[\begin{array}{l} \{B_n\} = \{F_{\Delta_n}\} \cup \{\partial_\Delta F_{\Delta_n}\} \\ \{\omega_n^B\} = \{\alpha_n\} \cup \{\beta_n\} \end{array}\right]$$

... with $\{\Delta_n\}$ *to be determined*

$$\omega(\Delta) \equiv \omega[F_\Delta]$$

$$F_\Delta(z) = \sum_{n=0}^{\infty} [\alpha_n(\Delta) F_{\Delta_n}(z) + \beta_n(\Delta) \partial_\Delta F_{\Delta_n}(z)]$$

Mazac, MP '18

Extremal bases

- Assume that bases of the form below exist:

$$F_\Delta(z) = \sum_{n=0}^{\infty} [\alpha_n(\Delta) F_{\Delta_n}(z) + \beta_n(\Delta) \partial_\Delta F_{\Delta_n}(z)]$$

… with $\{\Delta_n\}$ *to be determined*

- The duality conditions become:

$$\omega_n^B[B_m] = \delta_{n,m} \quad \Rightarrow \quad \begin{array}{ll} \alpha_n(\Delta_m) = \delta_{n,m}\,, & \partial_\Delta \alpha_n(\Delta_m) = 0 \\ \beta_n(\Delta_m) = 0\,, & \partial_\Delta \beta_n(\Delta_m) = \delta_{n,m} \end{array}$$

- Thus duality for such bases has direct consequences on the shape of the functional actions (double zeros at specific dimensions).

Mazac, MP '18

Extremal bases

- Assume that bases of the form below exist:

$$F_\Delta(z) = \sum_{n=0}^{\infty} [\alpha_n(\Delta) F_{\Delta_n}(z) + \beta_n(\Delta)\partial_\Delta F_{\Delta_n}(z)]$$

… with $\{\Delta_n\}$ *to be determined*

- **Functional Bootstrap** equations:

$$\sum_\Delta a_\Delta F_\Delta(z) = 0 \quad \Longleftrightarrow \quad \begin{array}{l} \sum_\Delta a_\Delta \alpha_n(\Delta) = 0 \\ \sum_\Delta a_\Delta \beta_n(\Delta) = 0 \end{array} \qquad n = 0, 1, \ldots$$

… with duality

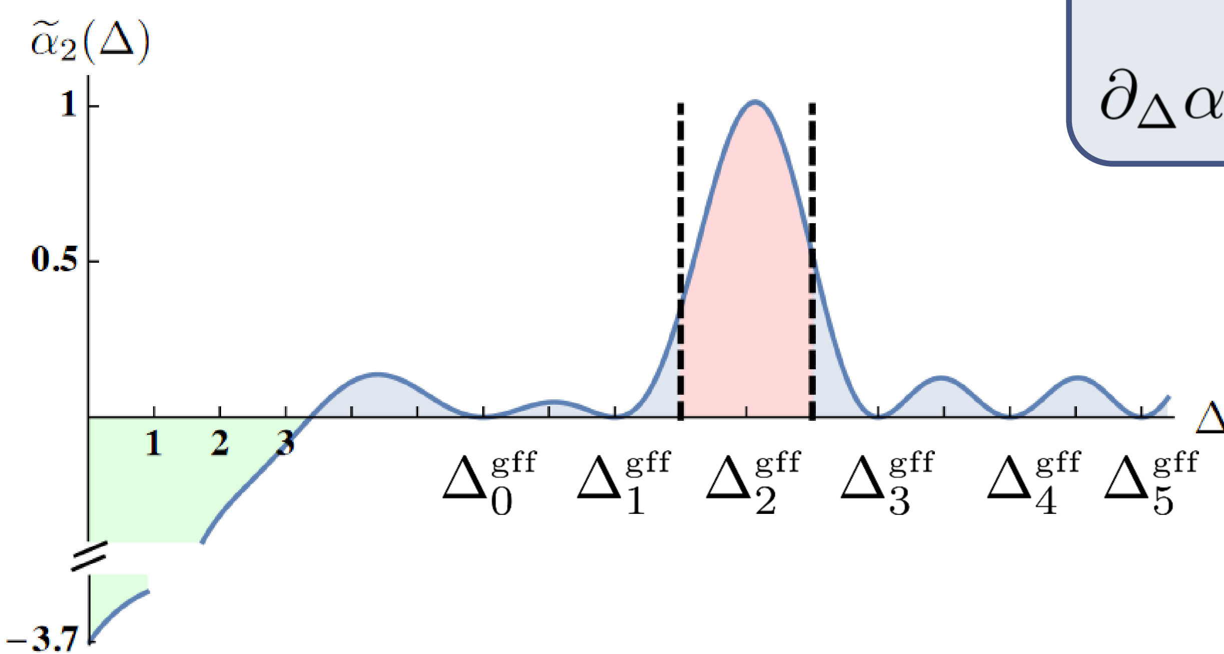
$$\alpha_n(\Delta_m) = \delta_{n,m}\,, \qquad \partial_\Delta \alpha_n(\Delta_m) = 0$$

$$\beta_n(\Delta_m) = 0\,, \qquad \partial_\Delta \beta_n(\Delta_m) = \delta_{n,m}$$

Bounds and extremality

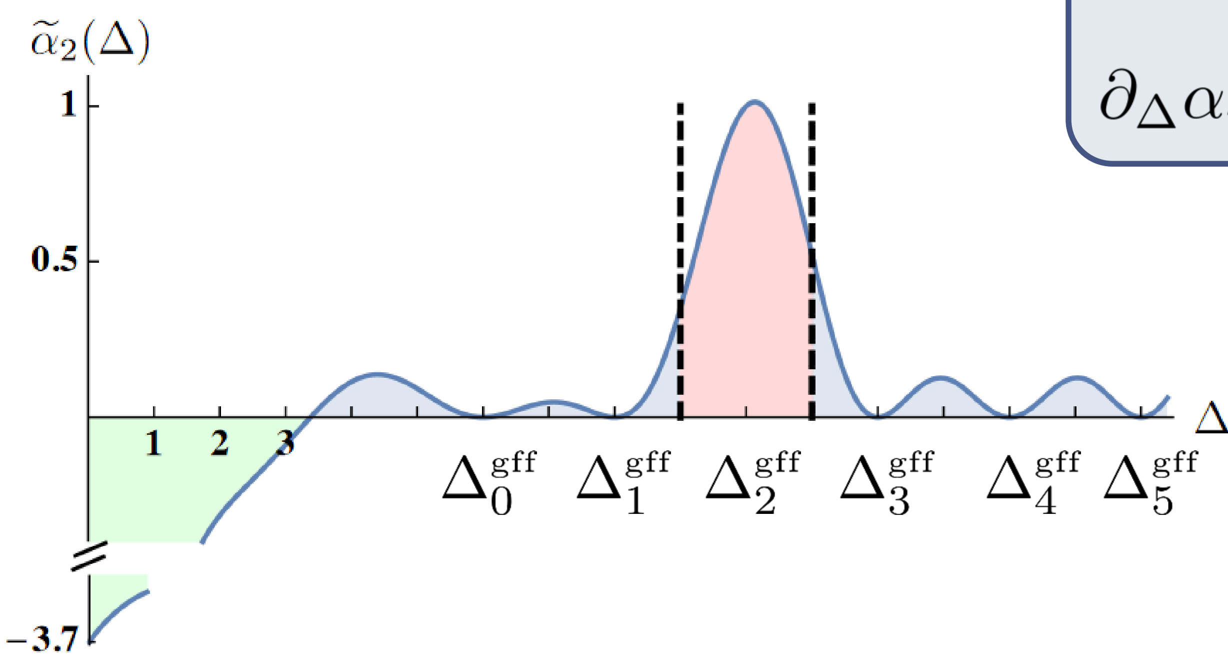
- Duality conditions naturally lead to bounds

$$\alpha_n(\Delta_m) = \delta_{n,m}$$
$$\partial_\Delta \alpha_n(\Delta_m) = 0$$



$$\sum_\Delta a_\Delta \alpha_n(\Delta) = 0 \quad \Rightarrow \sum_\Delta a_\Delta \alpha_n(\Delta) \leq \sum_\Delta a_\Delta(-\alpha_n(\Delta))$$

Bounds and extremality



$$\alpha_n(\Delta_m) = \delta_{n,m}$$

$$\partial_\Delta \alpha_n(\Delta_m) = 0$$

- This picture also makes clear that the crossing equation has `finite resolution': cannot resolve OPE density inside O(1) sized-window

Extremal bases

- **Motivation**:
  - These bases can diagonalize bootstrap equations, allowing us to construct *extremal solutions*.

$$F_0 + \sum_{i=1}^{K} \hat{a}_i F_{\hat{\Delta}_i} + \sum_{m=0}^{\infty} a_m F_{\Delta_m} = 0$$

Extremal bases

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$$F_0 + \sum_{i=1}^{K} \hat{a}_i F_{\hat{\Delta}_i} + \sum_{m=0}^{\infty} a_m F_{\Delta_m} = 0$$

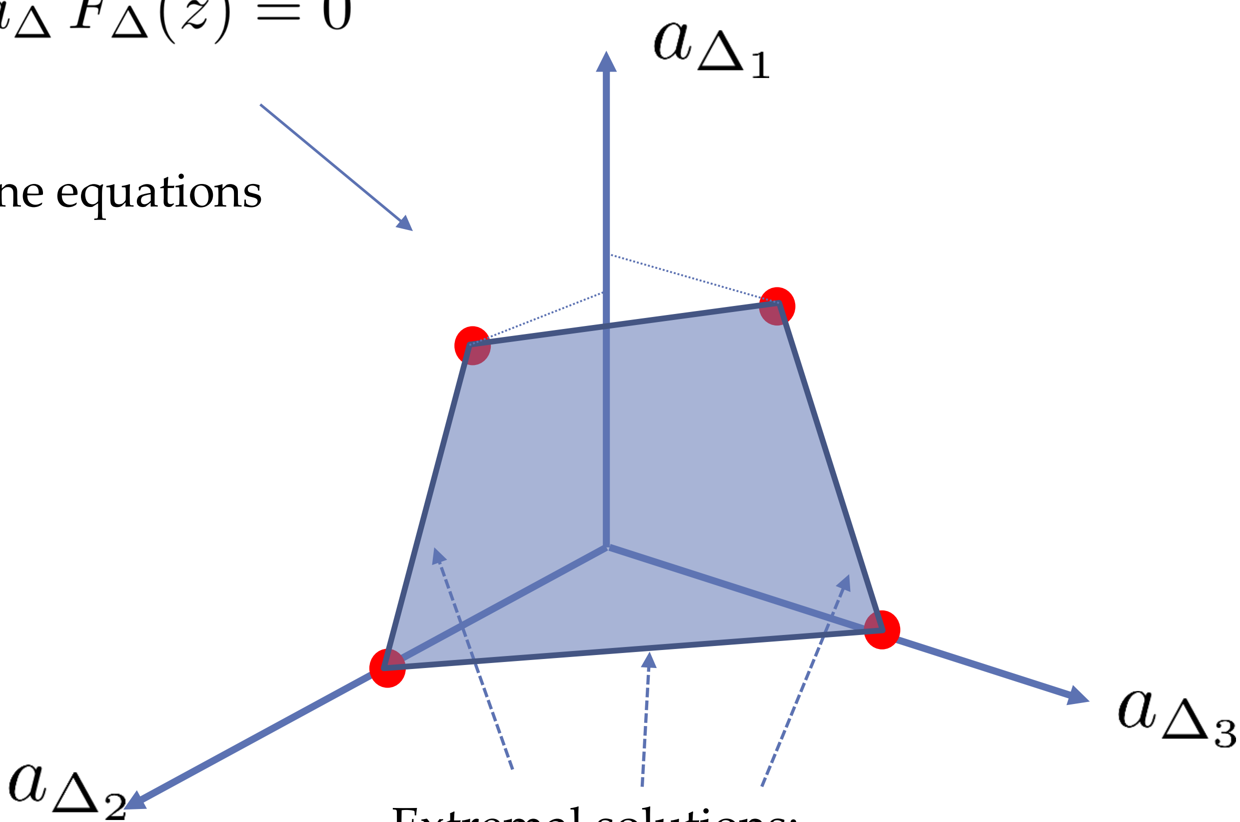
Inputs, label solution/basis

Dimensions appearing in the basis, determined implicitly by inputs

Bounds and extremality

$$\sum_{\Delta} a_{\Delta} F_{\Delta}(z) = 0$$

Hyperplane equations



Extremal solutions:

- sparser
- maximize OPE (=> bounds)

Example: the GFF bases

- Example: generalized free *fermion*. In this case input is identity.

$$\alpha_n^f(\Delta_m^f) = \delta_{n,m}\,, \qquad \partial_\Delta \alpha_n^f(\Delta_m^f) = 0$$

$$\beta_n^f(\Delta_m^f) = 0\,, \qquad \partial_\Delta \beta_n^f(\Delta_m^f) = \delta_{n,m}$$

$$\Delta_n = \Delta_n^f := 1 + 2\Delta_\phi + 2n$$

$$\langle\phi|\phi(1)\phi(z)|\phi\rangle = \frac{1}{z^{2\Delta_\phi}} + \frac{1}{(1-z)^{2\Delta_\phi}} - 1$$

- Simple enough choice that functionals can be explicitly determined.
- Any other basis can be expressed in terms of this one.

Mazac, Paulos '18

Example: the GFF bases

- Example: generalized free *boson*. In this case inputs are identity and an operator with undetermined OPE but dimension $\Delta_0^b = 2\Delta_\phi$

$$\alpha_n^f(\Delta_m^f) = \delta_{n,m}\,, \qquad \partial_\Delta \alpha_n^f(\Delta_m^f) = 0$$

$$\beta_n^f(\Delta_m^f) = 0\,, \qquad \partial_\Delta \beta_n^f(\Delta_m^f) = \delta_{n,m}$$

$\Rightarrow$

$$\Delta_n = \Delta_n^b := 2\Delta_\phi + 2n$$

$$\langle\phi|\phi(1)\phi(z)|\phi\rangle = \frac{1}{z^{2\Delta_\phi}} + \frac{1}{(1-z)^{2\Delta_\phi}} + 1$$

- Simple enough choice that functionals can be explicitly determined.
- Any other basis can be expressed in terms of this one.

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Hybrid Bootstrap

- These bases can be used as building blocks to express **any other** extremal solution efficiently by a hybrid numerical/analytic algorithm.

- The algorithm is possible because **extremal spectra asymptote to a boson or a fermion** beyond a certain scale Δ^* (problem dependent).

- The result is a characterization of extremal solutions:

  1. Excluding inputs, **the number of states is the same as for a free correlator.**

  2. Anomalous dimensions of those states decay polynomially with energy (typically 1/E^2)

  3. The solution asymptotes to a free fermion or boson depending on essentially the number of fixed inputs.

Girault, MP, Suchel, WiP

Multiple correlation functions.

- The construction of general extremal solutions is greatly simplified by knowledge of one such solution and the associated basis of constraints.

- The strategy for a mixed system of correlators is to again construct such a basis.

- The basis is dual to a system of correlators in the tensor product theory of N generalized free fields.

- Sum rules most easily derived in the Polyakov bootstrap picture

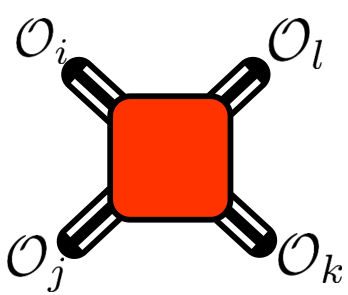
Ghosh, Kaviraj, MP '23

Polyakov '74
Dey, Kaviraj, Ghosh,Gopakumar, Sinha, Sen..

Mixed Polyakov Bootstrap

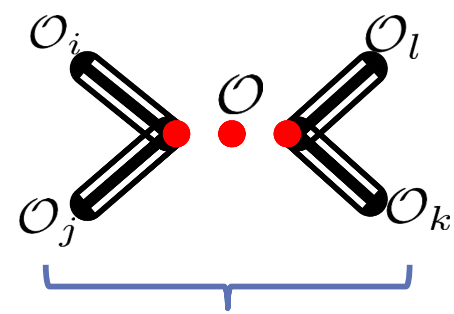
- Kinematics: think of operators as having *extra quantum numbers* describing orientation in OPE space:

$$\lambda_{\mathcal{O}}^{i,j} \equiv \sqrt{\lambda_{\mathcal{O}}^2}\, r_{i,j}^{\mathcal{O}}$$



$i, j, k, l = 1, \ldots, N$

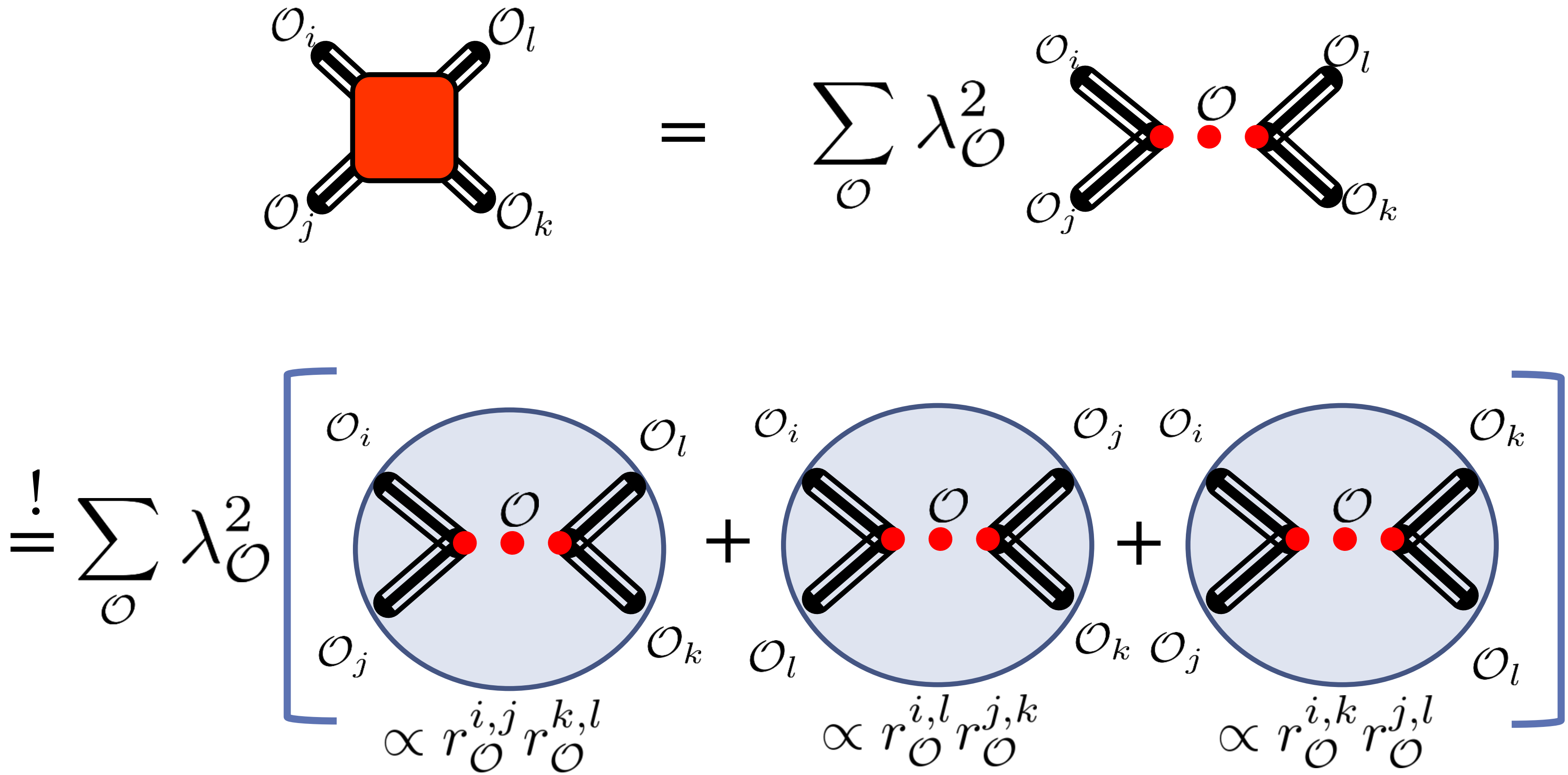
$$= \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2$$



Conformal block:

$$\propto r_{\mathcal{O}}^{i,j} r_{\mathcal{O}}^{k,l}$$

Mixed Polyakov Bootstrap



$+$ Relevant AdS contact interactions

Mixed Polyakov Bootstrap

- Using the OPE on this expansion leads to an infinite set of sum rules satisfied by general CFTs

$$\sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 \, \alpha_{(ij)_n}^{ij,kl}(\Delta_{\mathcal{O}}, r_{\mathcal{O}}) = 0$$

$$\sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 \, \alpha_{(kl)_n}^{ij,kl}(\Delta_{\mathcal{O}}, r_{\mathcal{O}}) = 0$$

Functionals can also be written as matrices contracted with $\lambda_{\mathcal{O}}^{i,j}$

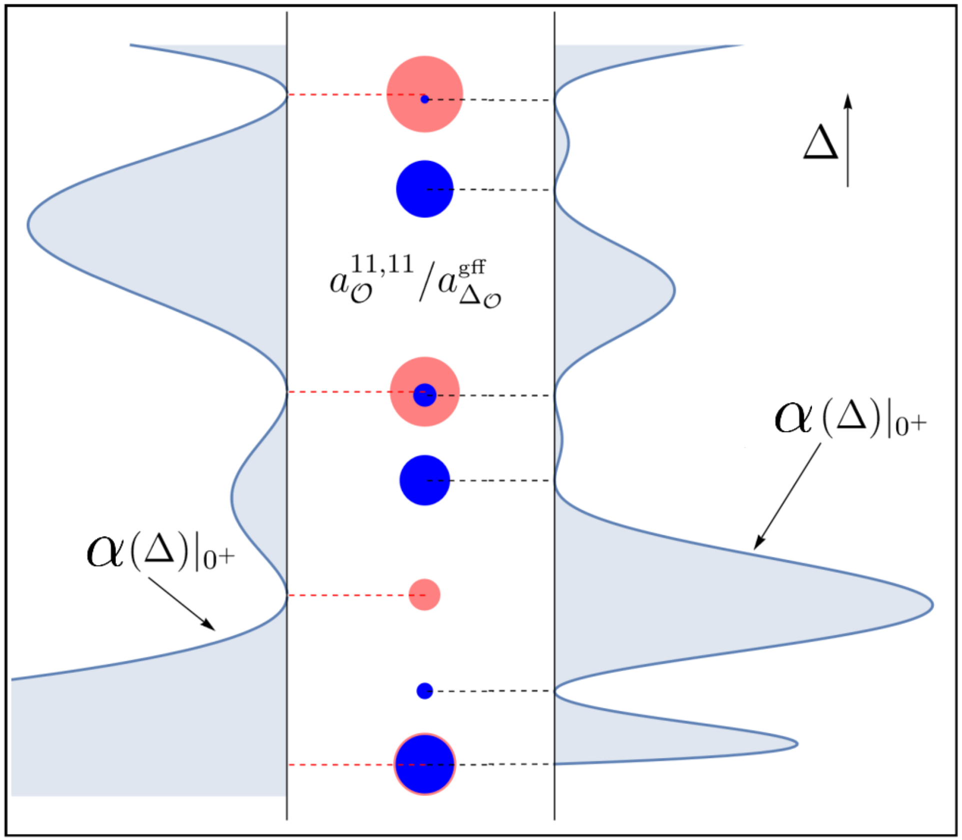
- Functional actions above satisfy duality conditions (which now involve dimension *and* OPE orientation)

- Sum rules diagonalized by tensor product theory solution, determining all CFT data there.

Extremal solutions, mixed correlators

Z_2 even, parity even sector (0^+)

$\langle \phi\phi\phi\phi \rangle$



$\langle \phi\phi\phi\phi \rangle$

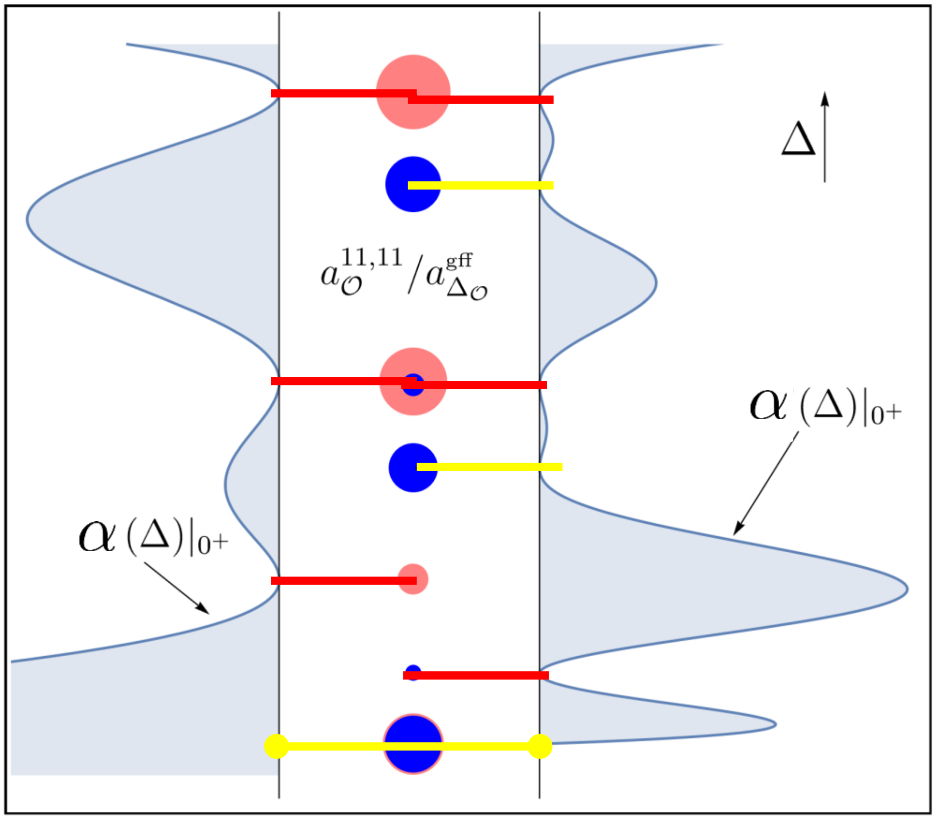
$\langle \phi\phi^2\phi\phi^2 \rangle$

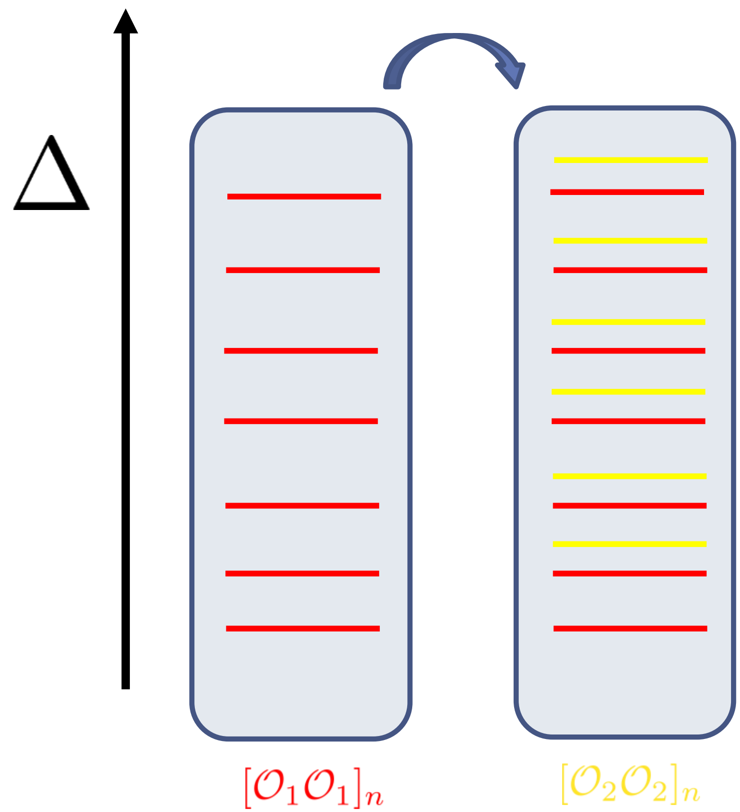
$\langle \phi^2\phi^2\phi^2\phi^2 \rangle$

Example: maximize OPE of ϕ^4, theory with Z_2 symmetry

Extremal solutions, mixed correlators

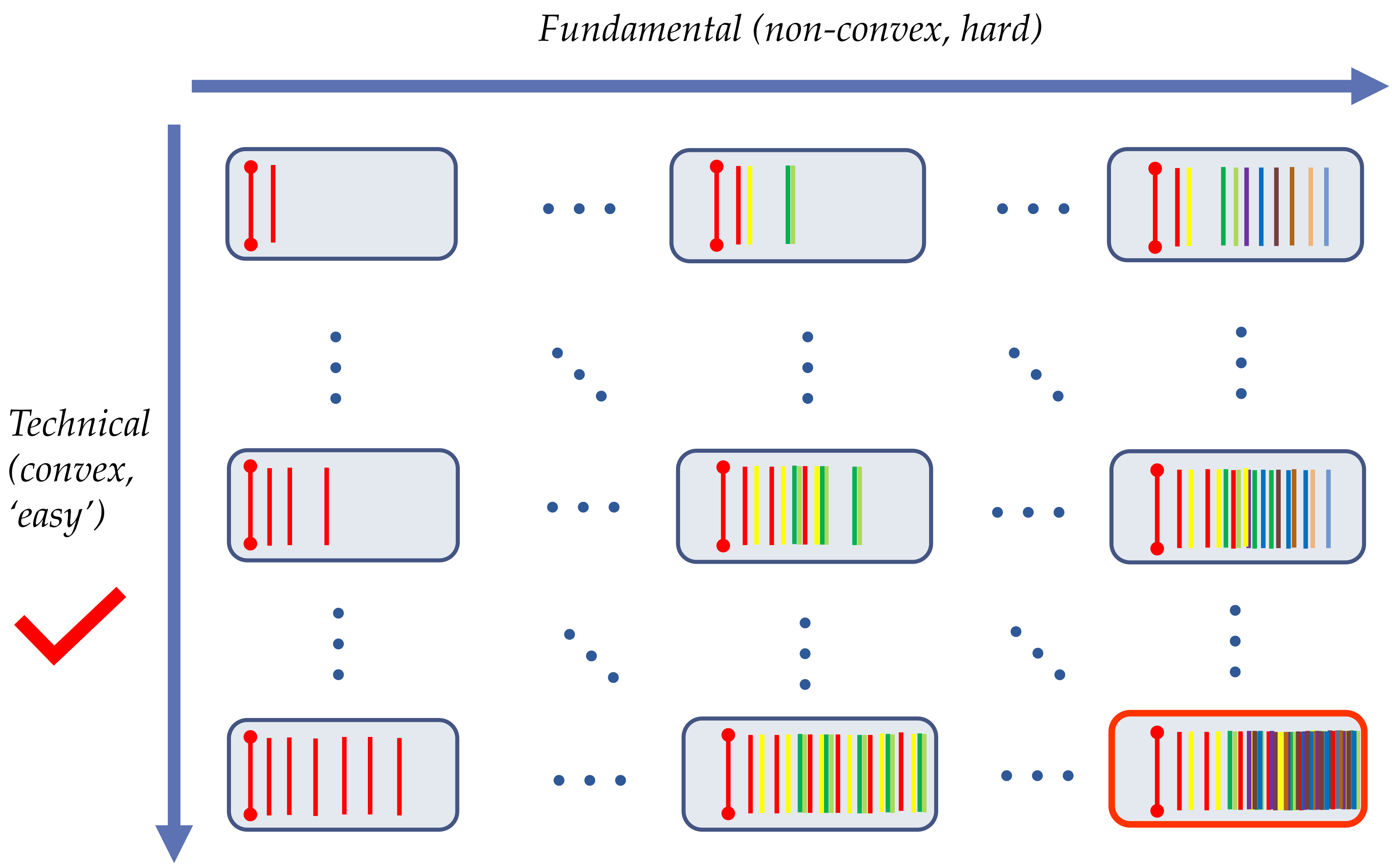
Z_2 even, parity even sector (0^+)





Comments and questions…

The extremal bootstrap program



Systematics

- The bootstrap is the construction of a Hamiltonian supporting local operators, and the operators themselves.

- Locality is implemented gradually as we include more correlation functions, recall:

$$\langle E_i | [\phi_L(t), \phi_R(0)] | E_j \rangle = 0\,, \quad t \in [0, T]$$

- This requires fine-grained cancellations between many terms.

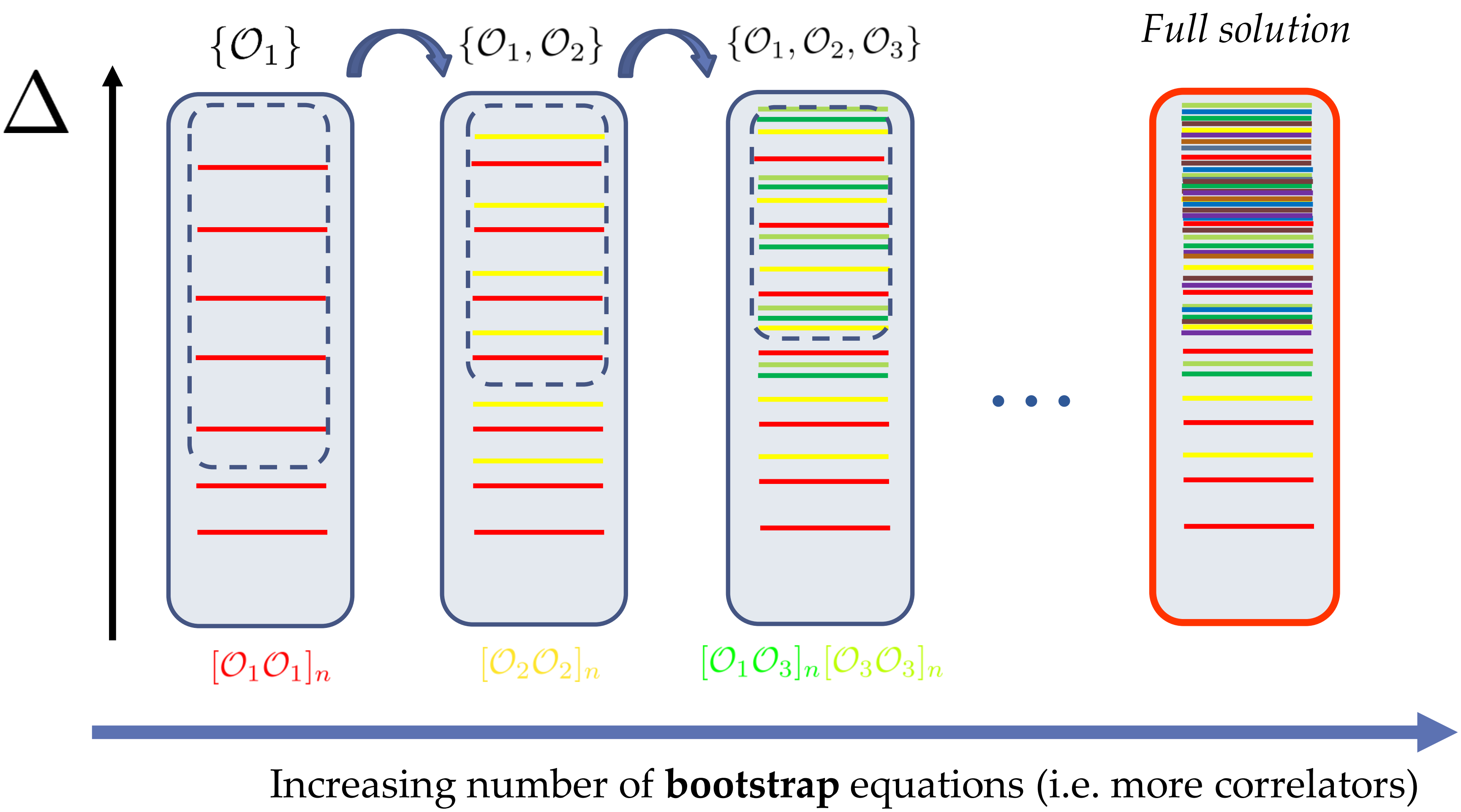
- From this perspective, it seems miraculous that local Hamiltonians can be chaotic..

Systematics

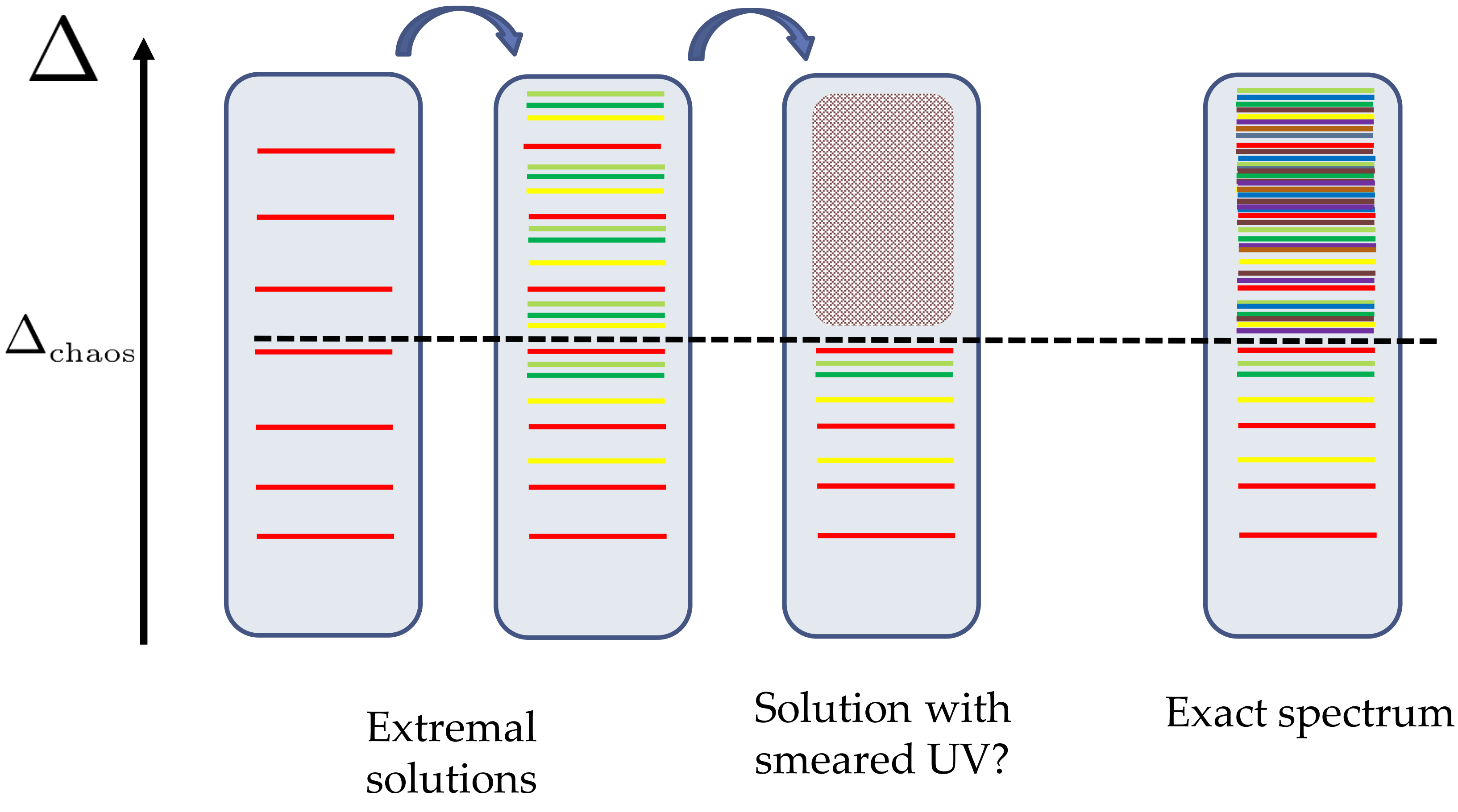
- The extremal functional picture provides us with a simple picture of expected bootstrap spectra:
- Including constraints from an ensemble of four point functions, one first imagines treating each external operator as a separate free field.
- One then constructs all possible double traces of such fields
- A generic extremal solution to the crossing constraints of all such correlators will have all such operators appearing in all OPEs (modulo symmetry constraints).
- This immediately tells us that the OPE density of states will grow linearly for any extremal solution to crossing – very far from the expected exponential growth.

This implies that bootstrapping more correlation functions has rapidly diminishing returns… (good and bad news)

Systematics



Bootstrapping chaos?

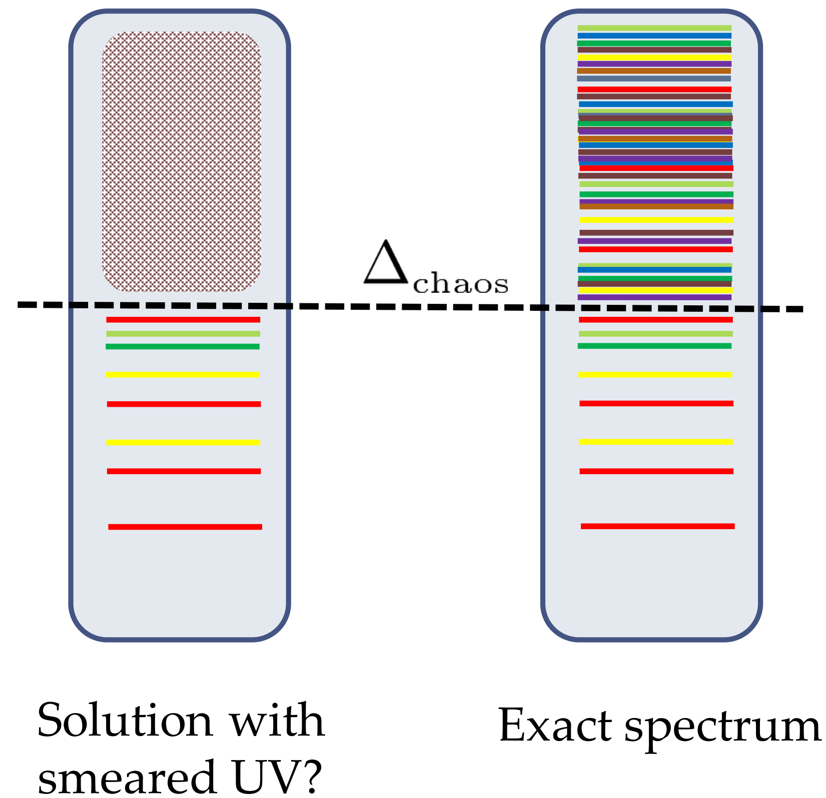


Bootstrapping chaos?

- How can we characterize the scale at which the spectrum is ``sufficiently'' complicated?
- This scale can be set parametrically high. E.g. take an integrable model with small integrability breaking deformation,

$$H_0^{\text{int}} + \epsilon H_1$$

Then for which *E* does chaotic statistics kick in as opposed to Poisson?

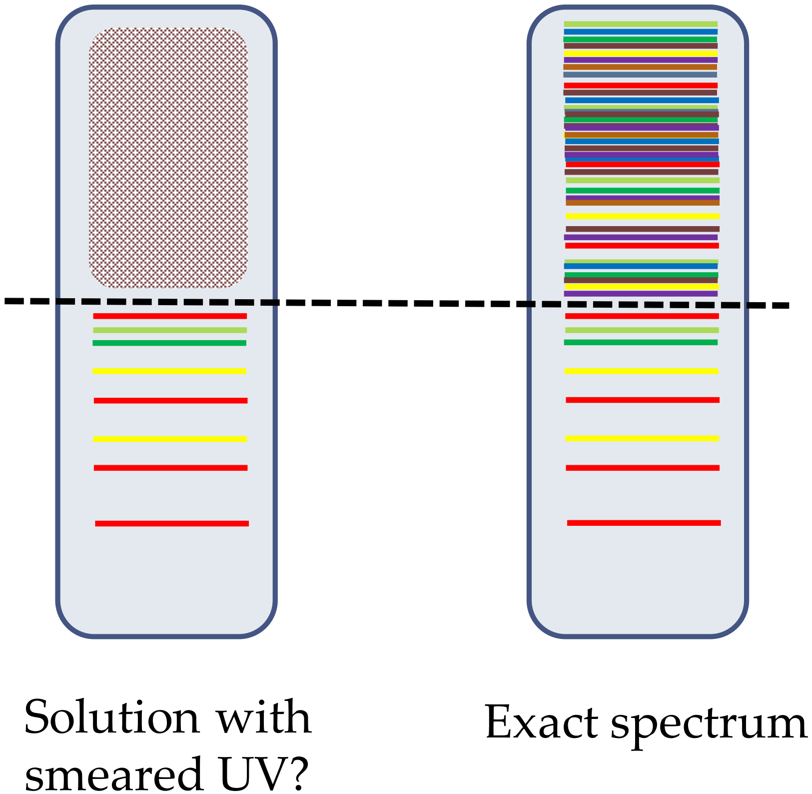


Bootstrapping chaos?

- For QFTs in AdS, can take free theory with relevant operator with small coupling. Perturbation theory breaks down for large operators:

$$\gamma_{\phi^n} \sim g n^2 \,,$$

- OPEs can't save you: similar breakdown there + exponential number of states…
- But it should still be possible to say *something*: new saddles for the action as for large charge story…?
- Can this helps us understand how to perform the correct smearing?



Outlook

- An extremal picture of the bootstrap program:

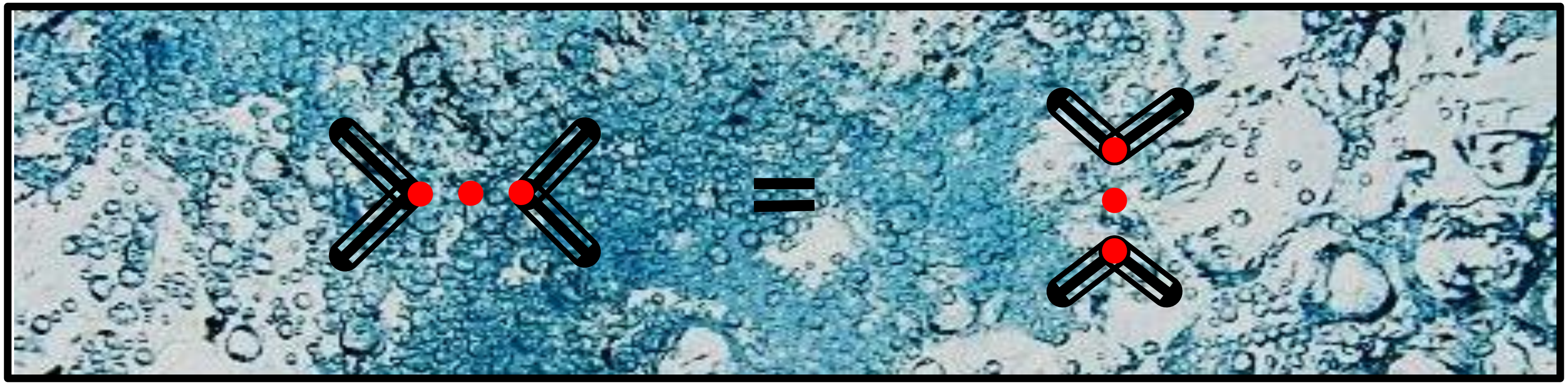
  - Extremal solutions naturally lead to bounds which universally hold *for any CFT*. Conversely, all bounds are saturated by extremal solutions. But notion independent of unitarity.

  - Extremal basis manifest UV/IR decoupling – double zeros at expected UV spectrum – numerics ``converges''

  - The same equations can be used for analytic bootstrap *and* numeric bootstrap – clean CFT identification of numerical results, possibility of rigorous hybrid bootstrap, etc.

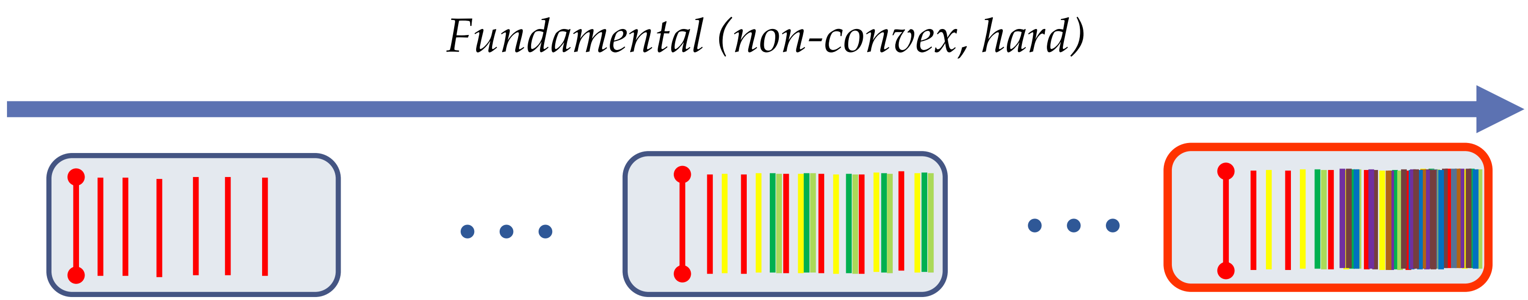
Outlook

- Lots to do:
  - **Beat the technical to tackle fundamental**: efficiently exploring the extremality landscape
  - Can we reach the chaos scale? How to ``UV complete''? Can we exploit multipoint crossing?
  - Analytic understanding of extremality via Basicity? Classify solutions? Integrable 1d CFTs?
  - Extremality in higher D?
  - Many applications…



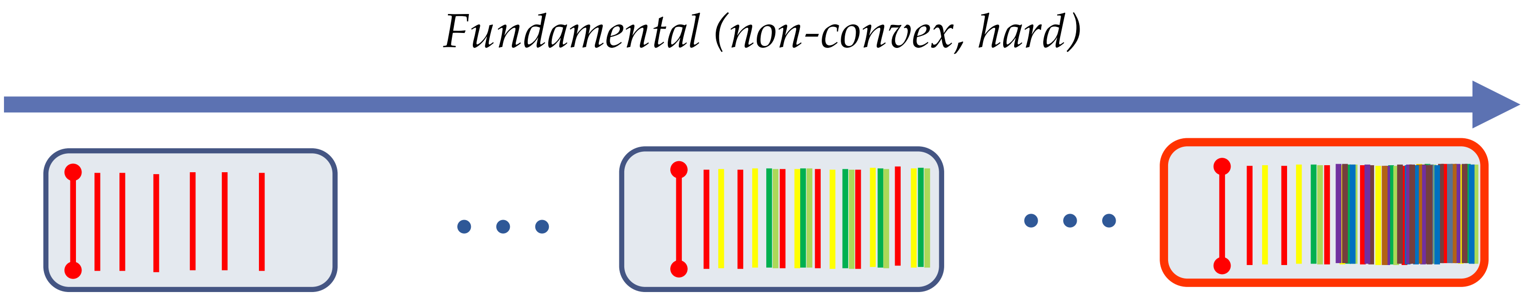
Thank you!

Difficulties and issues



- As we add extra correlators, must perform searches in high dimensional spaces.
- Technologies such as navigator and skydiving in principle settle this (?).
- Bottleneck: computation of functional actions. But efficient algorithms exist already. Computations much less demanding in precision.
- Unclear as of yet how far in number of correlators we will be able to push. Preliminary goal: $\phi, \phi^2, \phi^3, \phi^4$

Difficulties and issues



- As we add extra correlators, there are extra finite degrees of freedom.
- Equivalently, extremal solutions require a finite but growing set of assumptions.
- Origin: huge space of solutions coming from QFTs in AdS
- Must input information about desired set of theories. It seems this can be achieved by demanding existence of dual AdS local operators.
- Numerically: larger semidefinite programs combining OPE and BOE information

Levine, MP '23
Meineri, Penedones, Spirig '23

Application in D=2

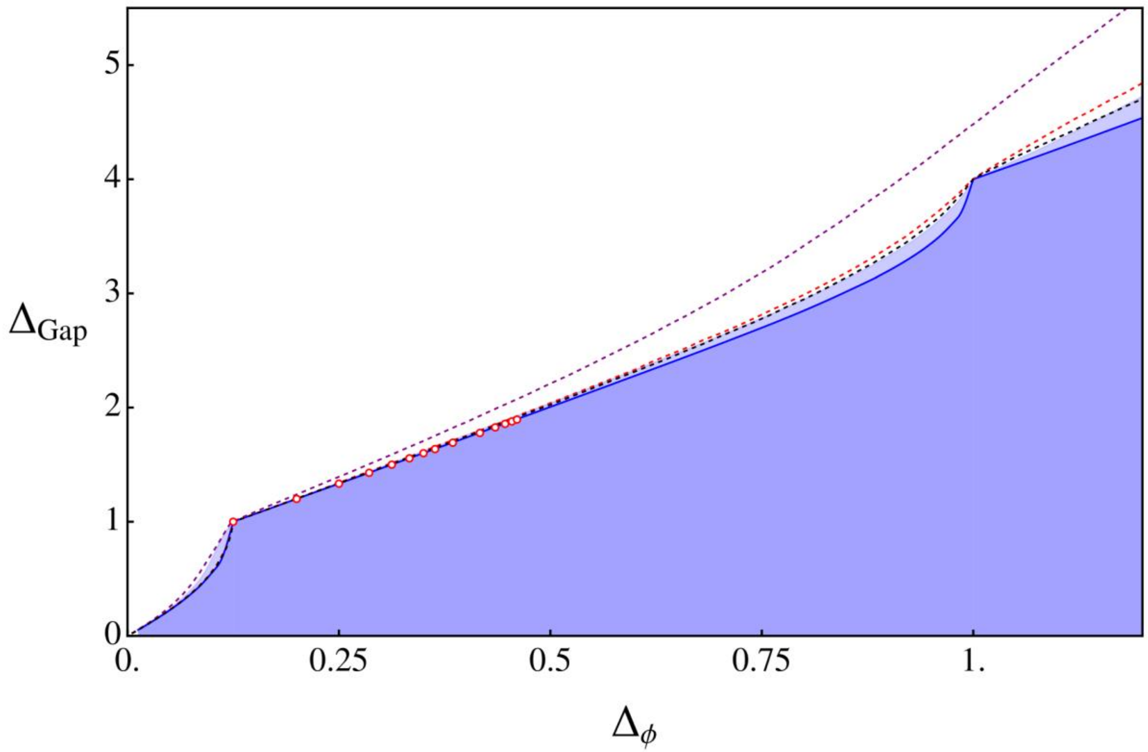
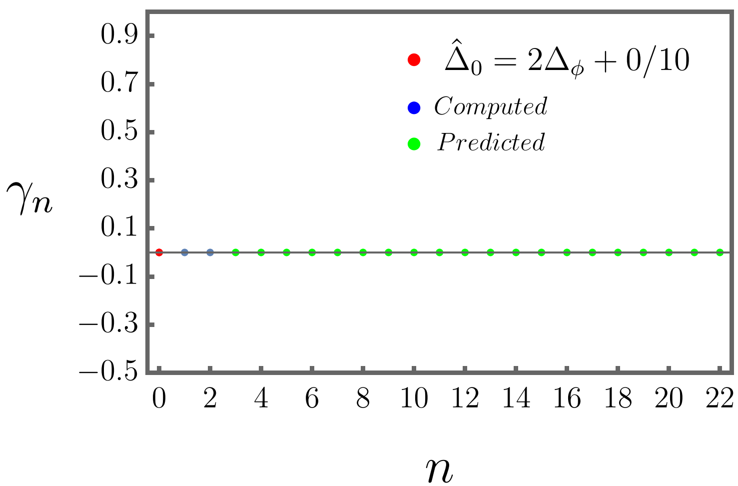
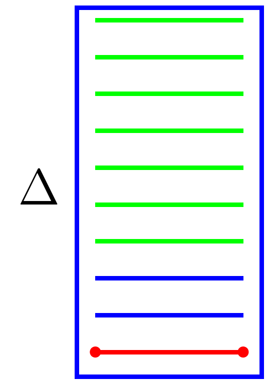
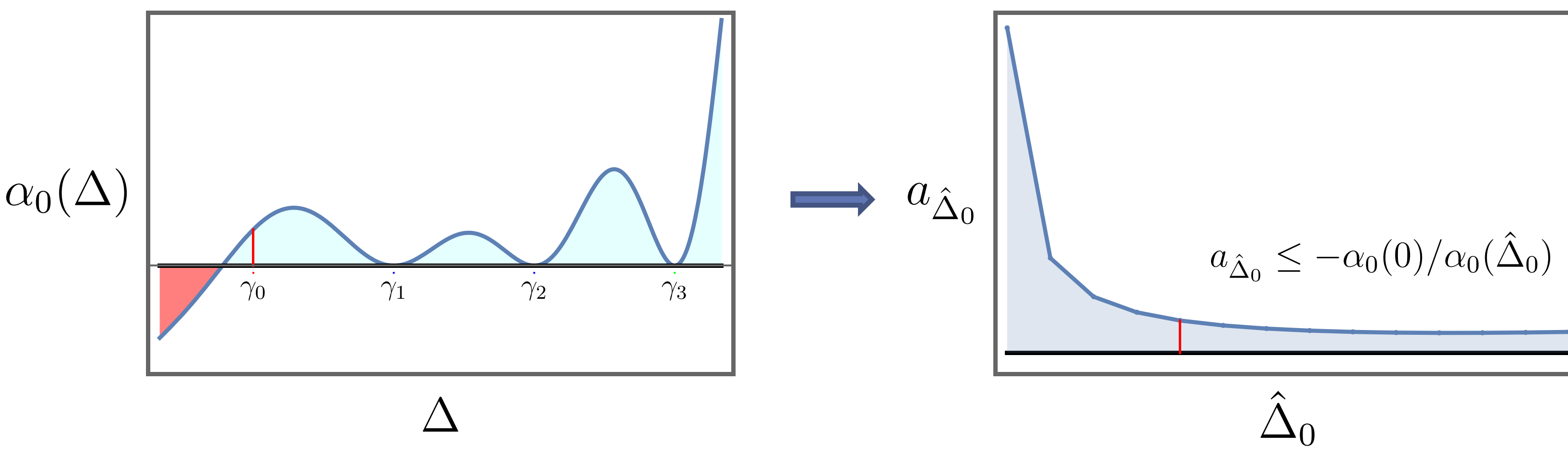


Figure 7: Spin $\ell = 0$ gap maximization. The gap for 60 F^+F^- functionals is depicted by the darker blue region, while the gap for 12 F^+F^- functionals is depicted by the lighter blue region. The purple, red, and black dashed lines represent the gap maximization with 15, 91, and 171 derivatives, respectively. The dotted circles indicate the positions of several selected minimal models.

Hybrid Bootstrap

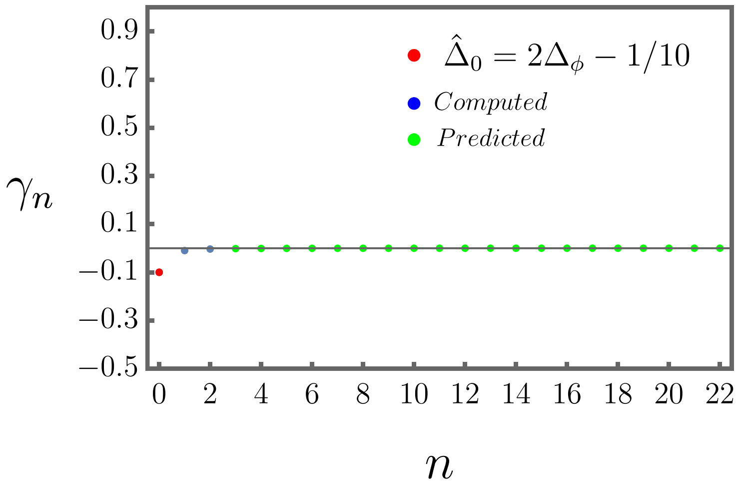


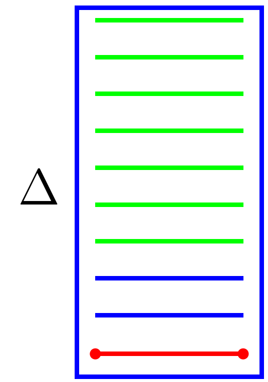


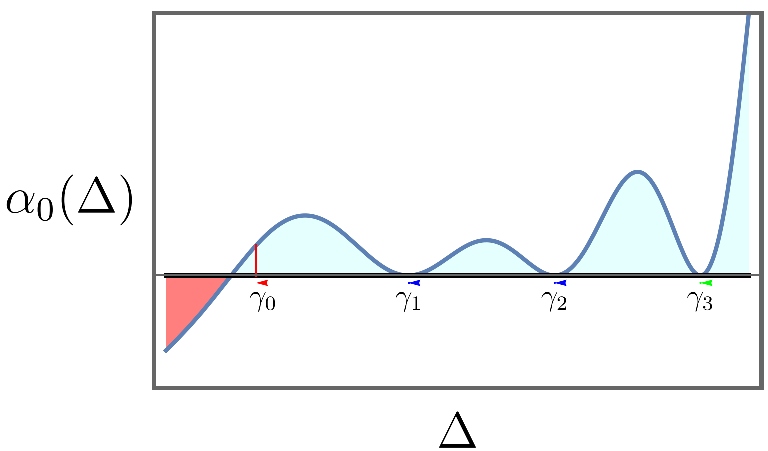


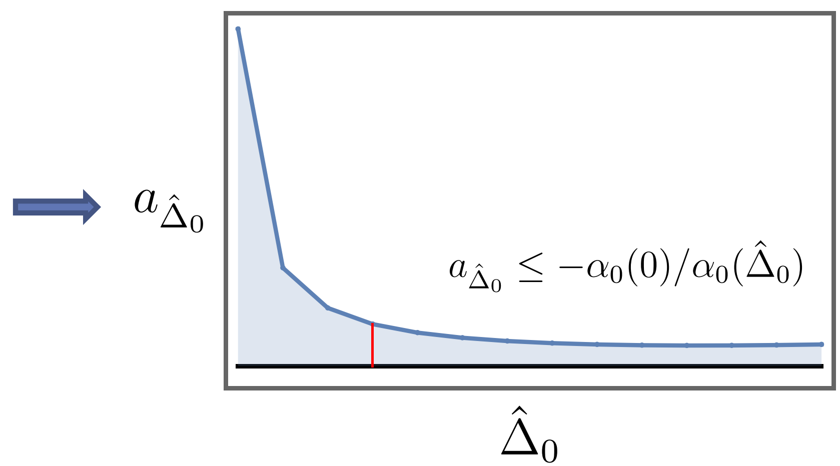
$$\alpha_0 = \sum_{n=0}^{N} [a_n \alpha_n^b + b_n \beta_n^b] + \sum_{n=N+1}^{\infty} [c_n \alpha_n^b + d_n \beta_n^b]$$

Hybrid Bootstrap









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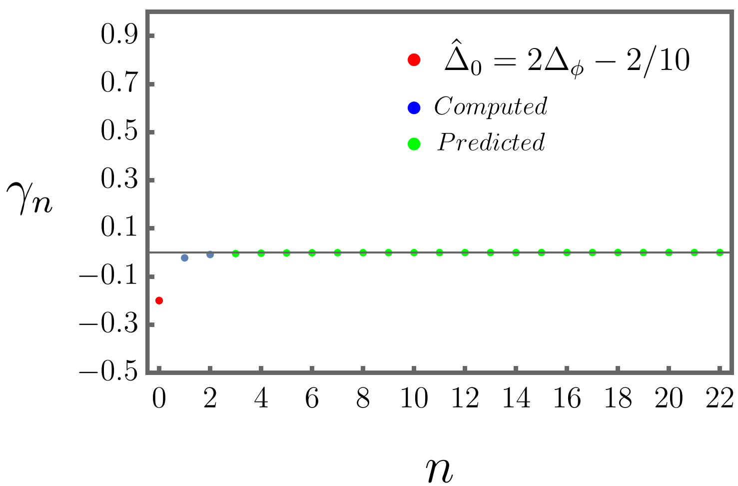

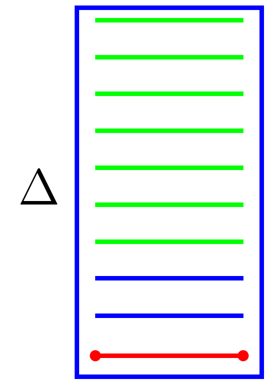

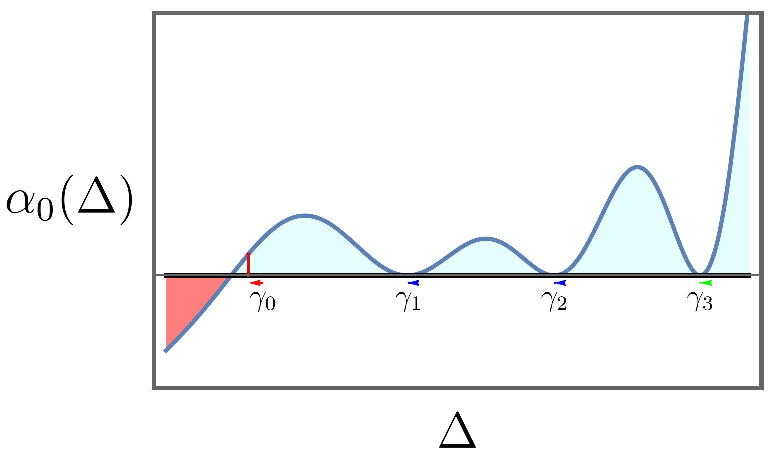

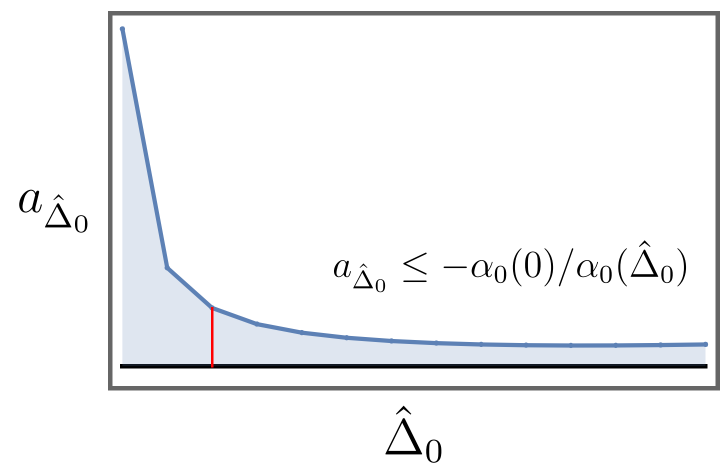

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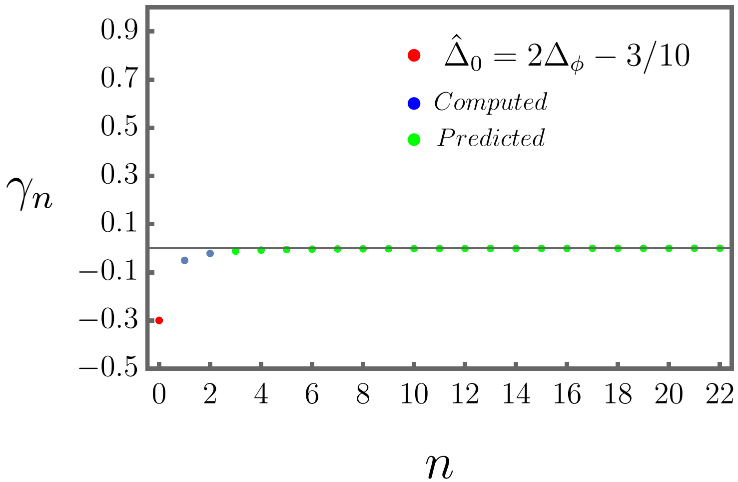

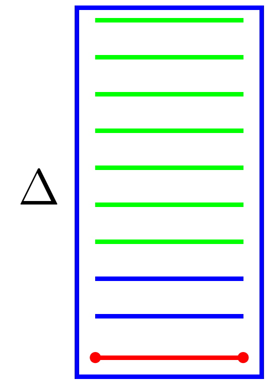

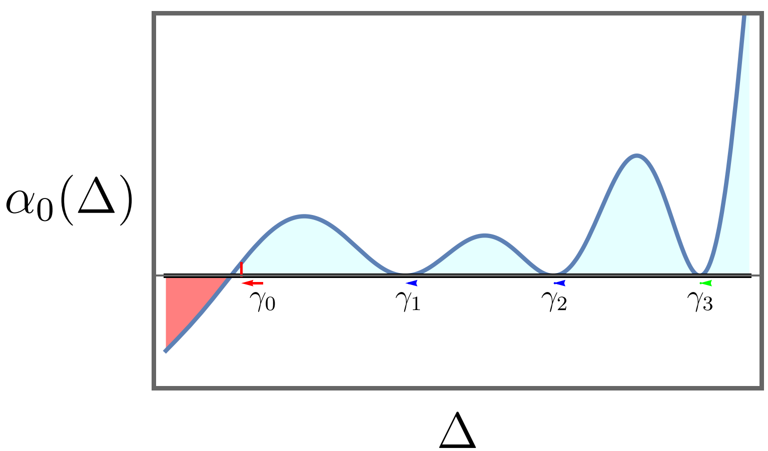

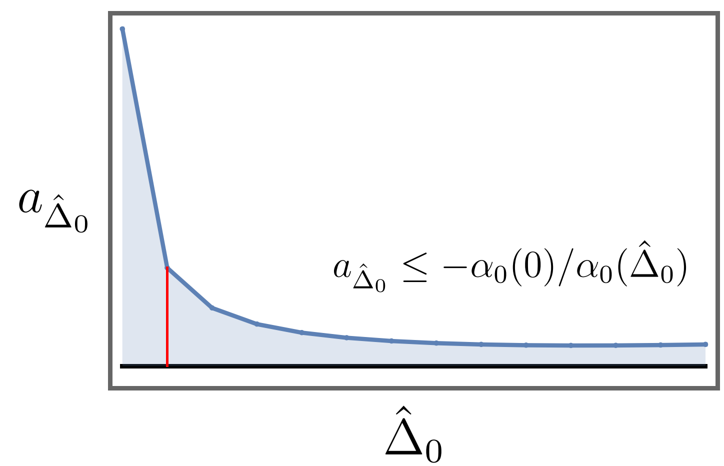

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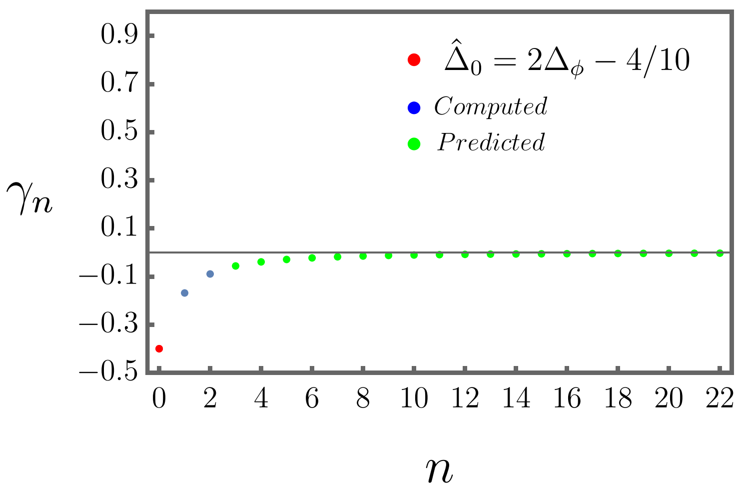

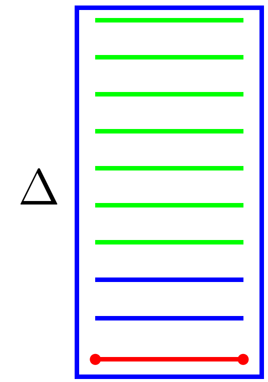

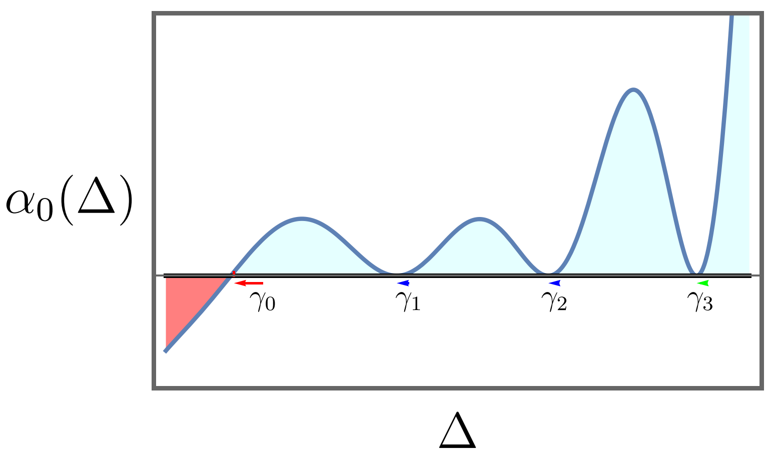

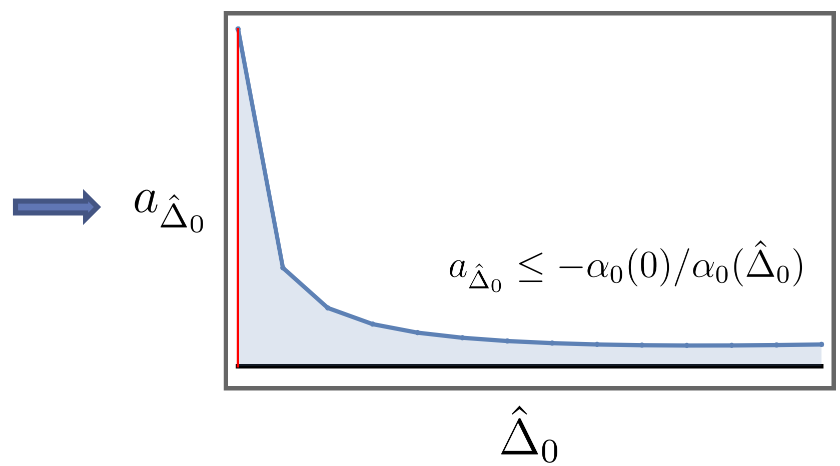

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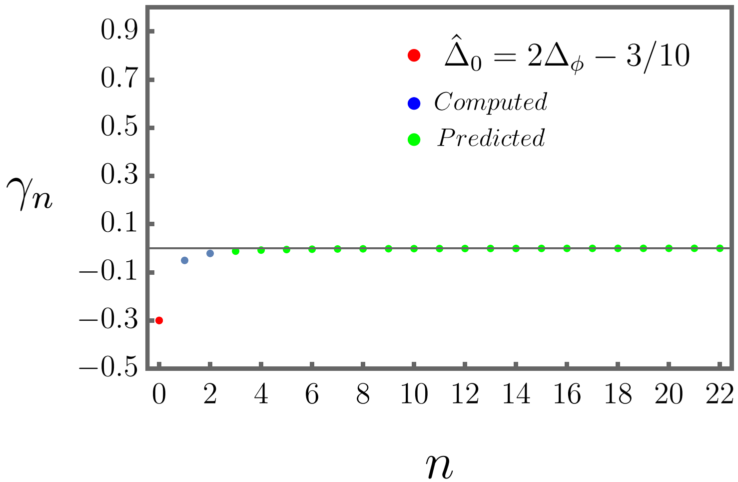

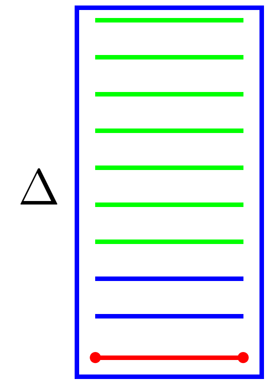

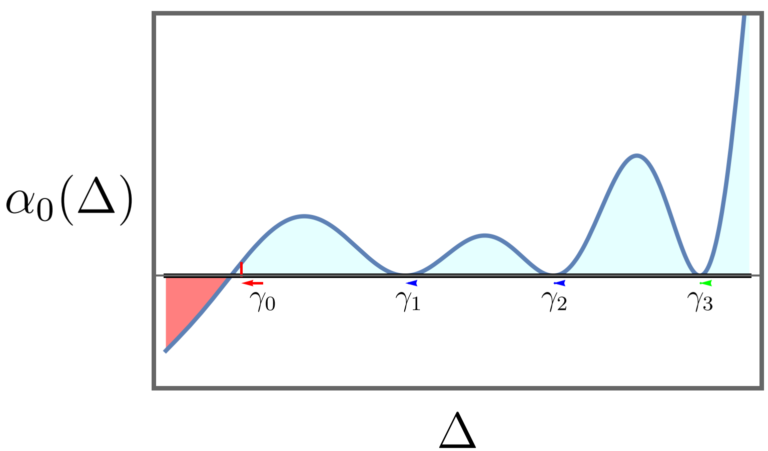

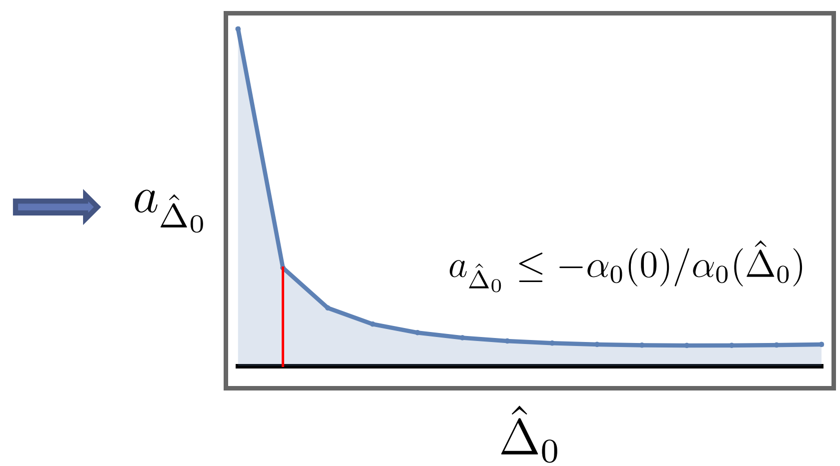

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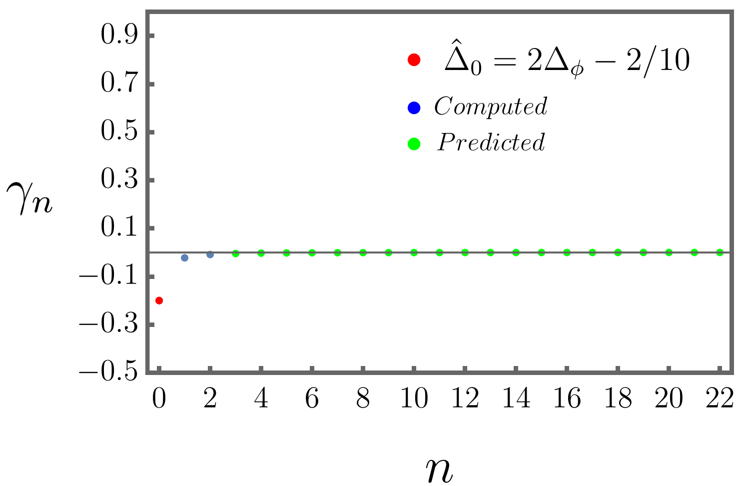

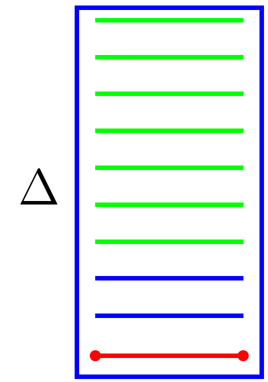

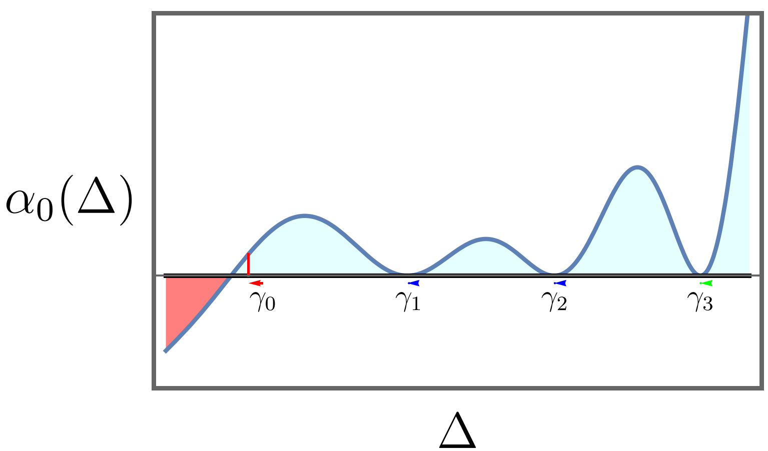

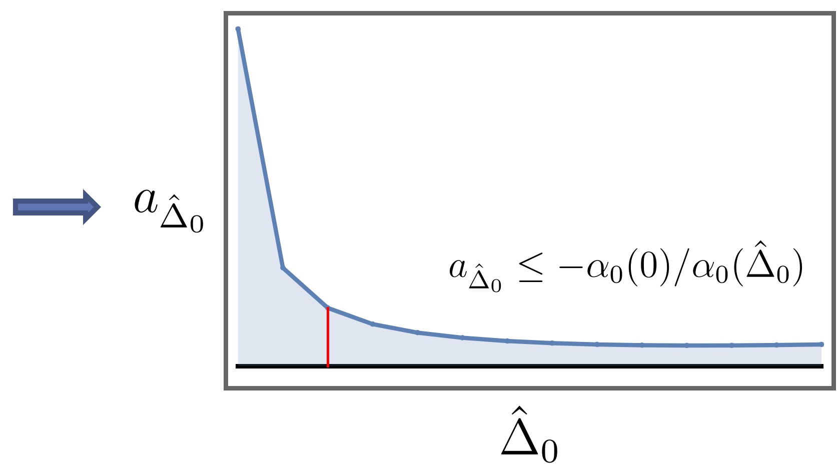

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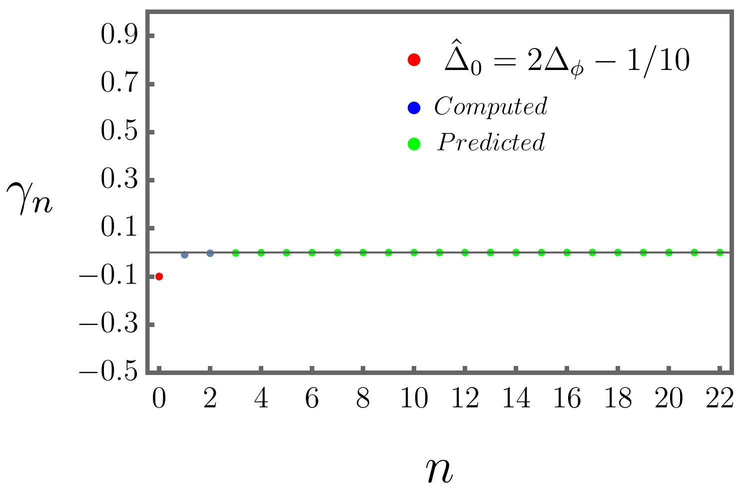

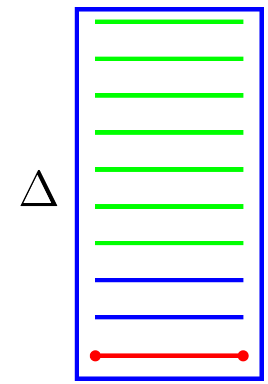

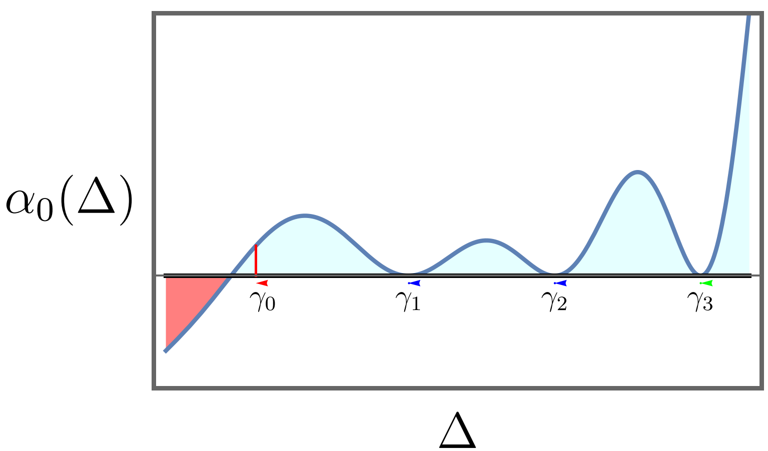

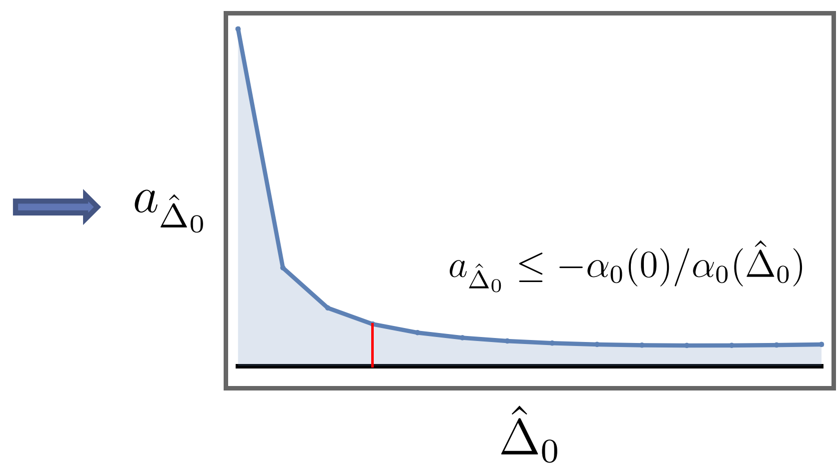

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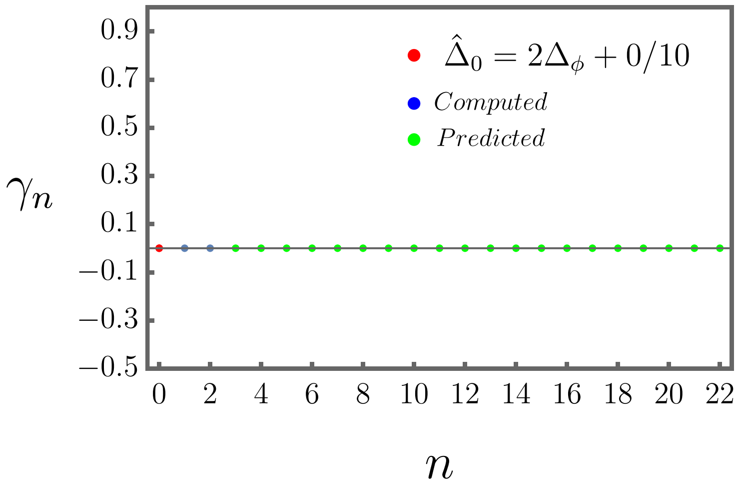
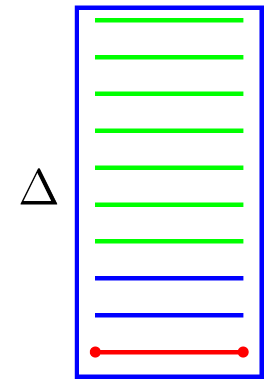
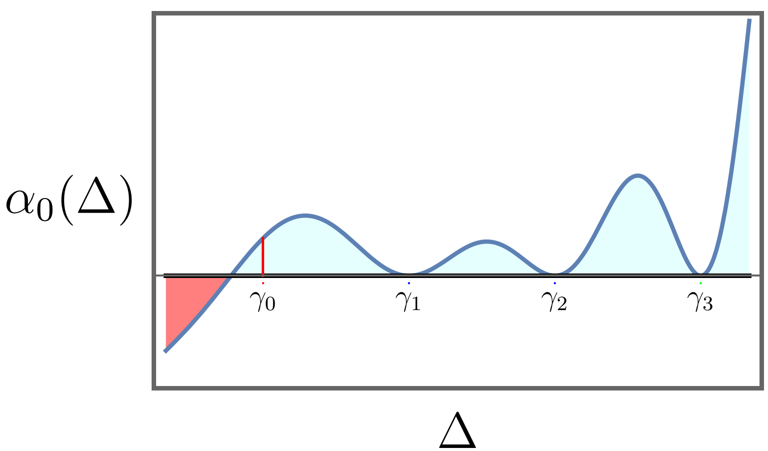
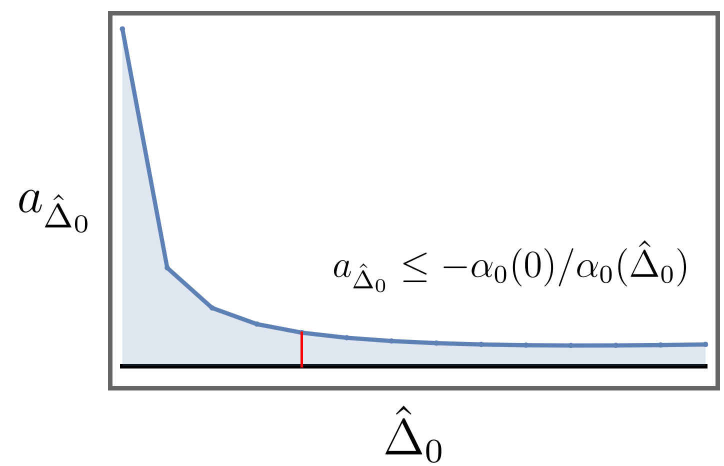
Hybrid Bootstrap









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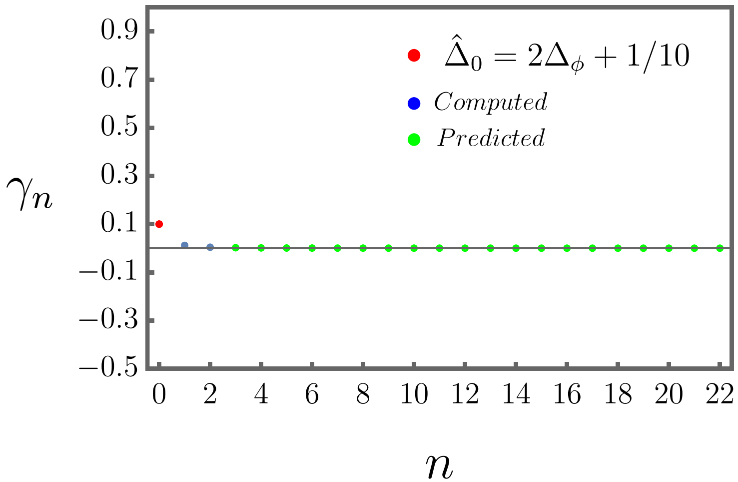

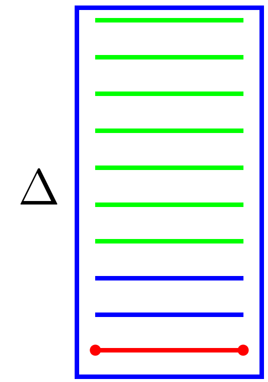

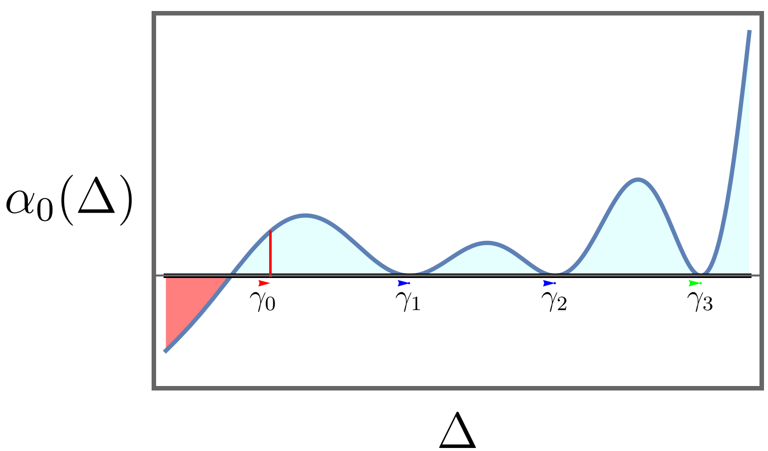

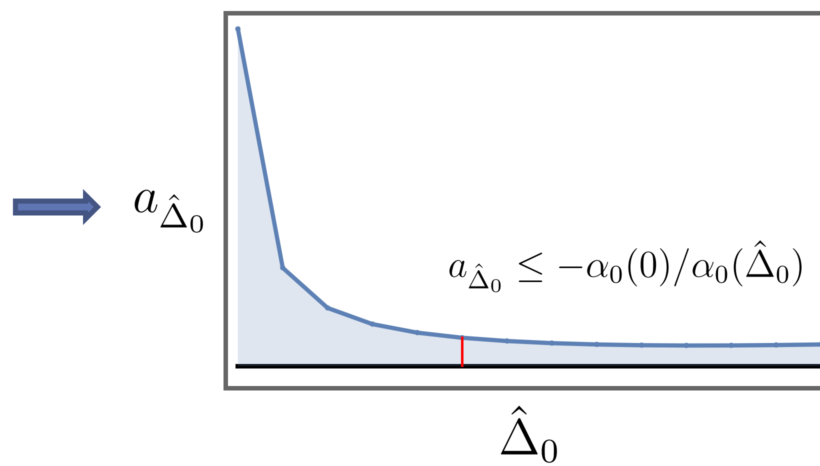

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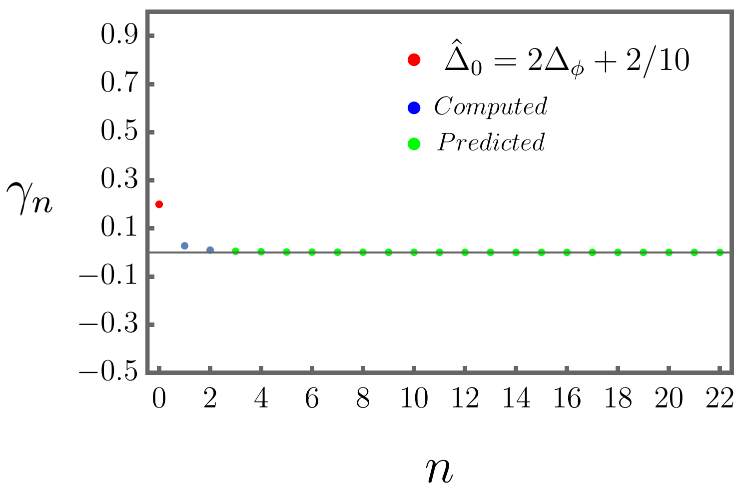
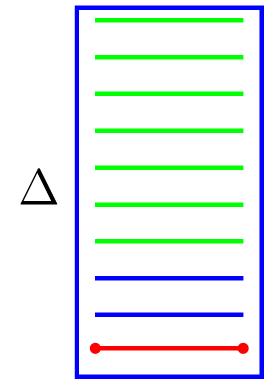
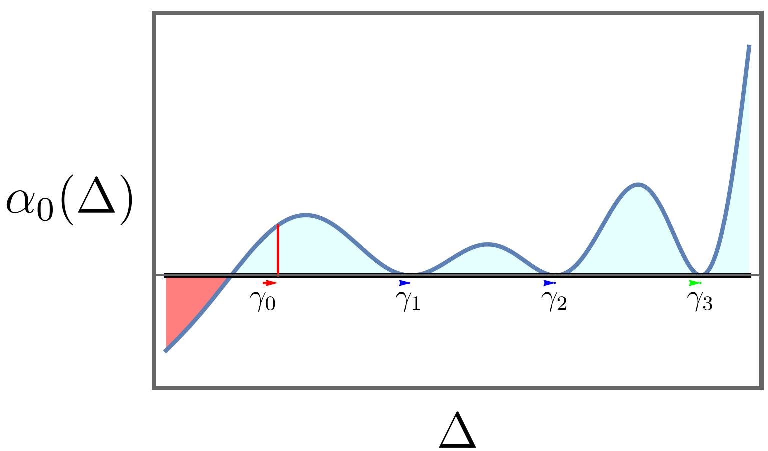
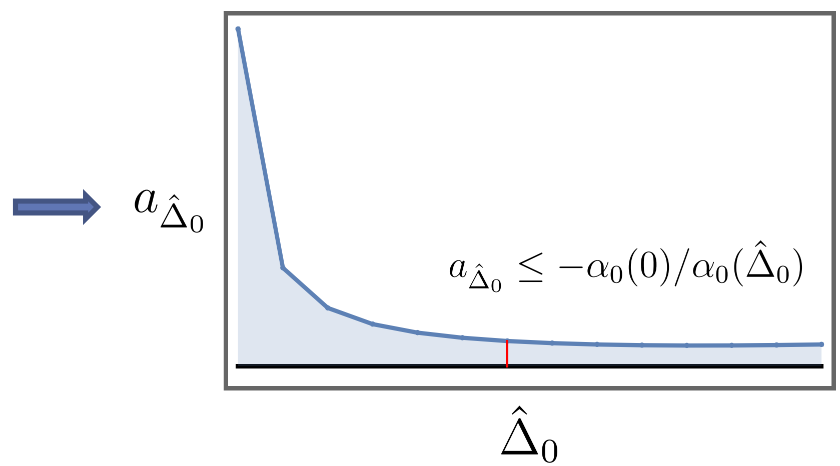


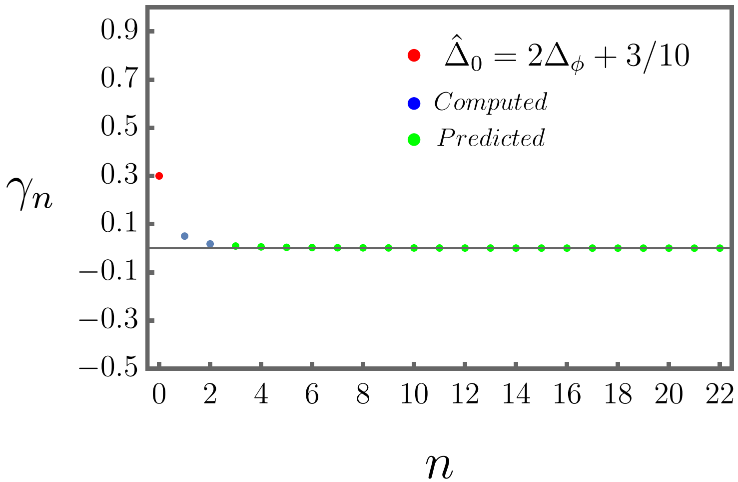


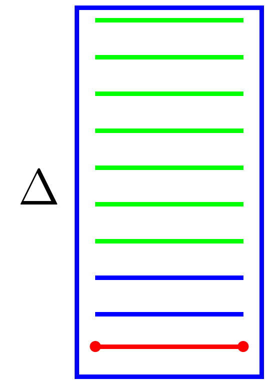


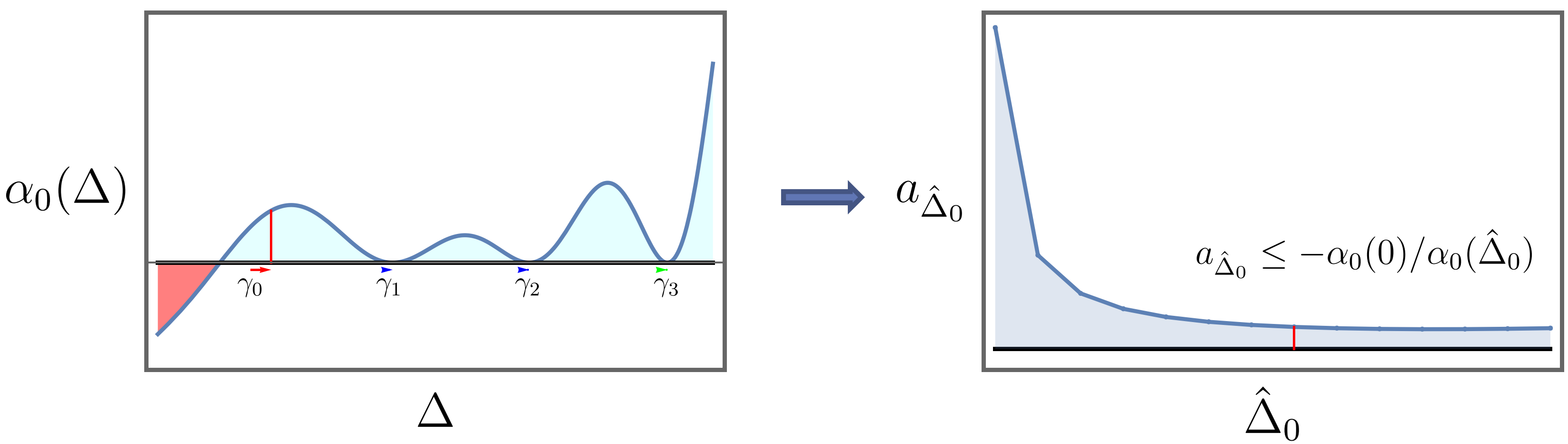


$$\alpha_0 = \sum_{n=0}^{N} [a_n \alpha_n^b + b_n \beta_n^b] + \sum_{n=N+1}^{\infty} [c_n \alpha_n^b + d_n \beta_n^b]$$

Hybrid Bootstrap







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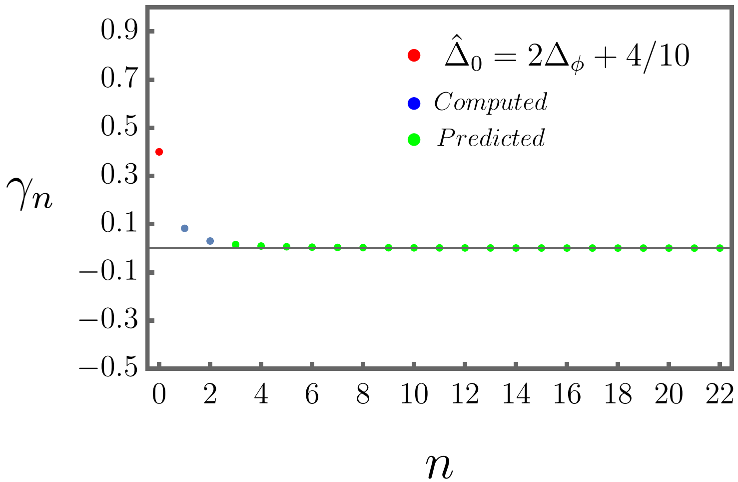
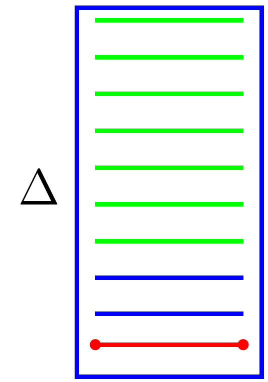
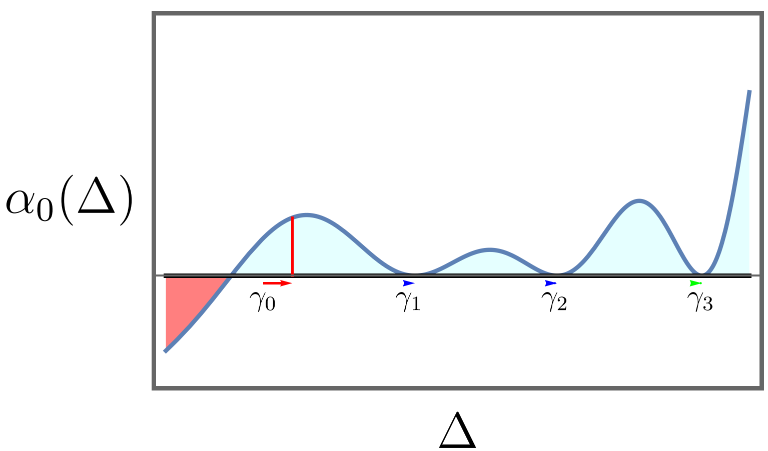
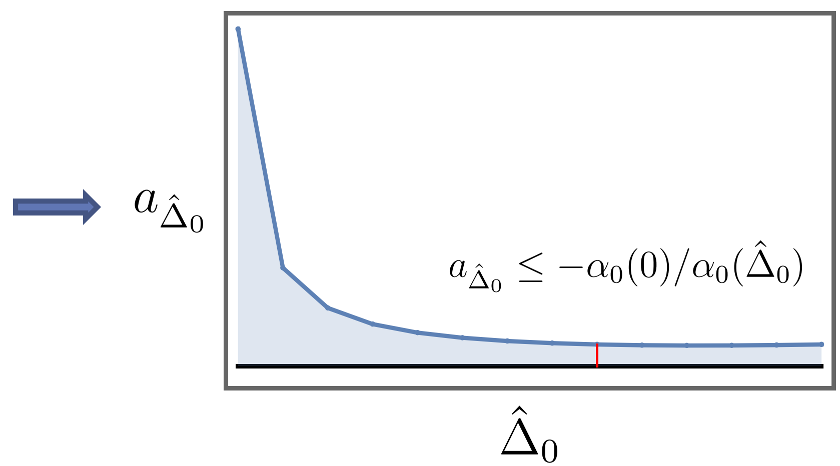


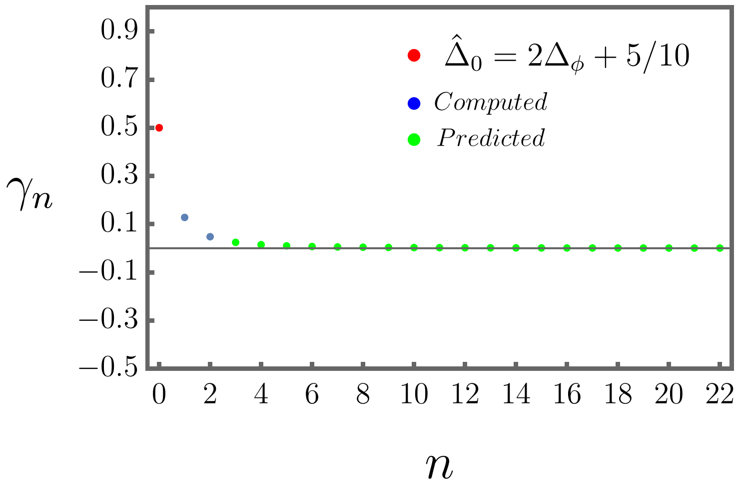


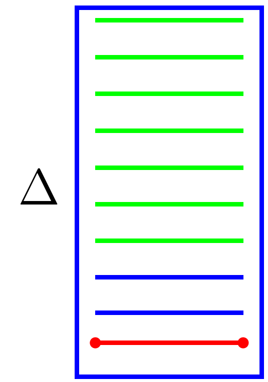


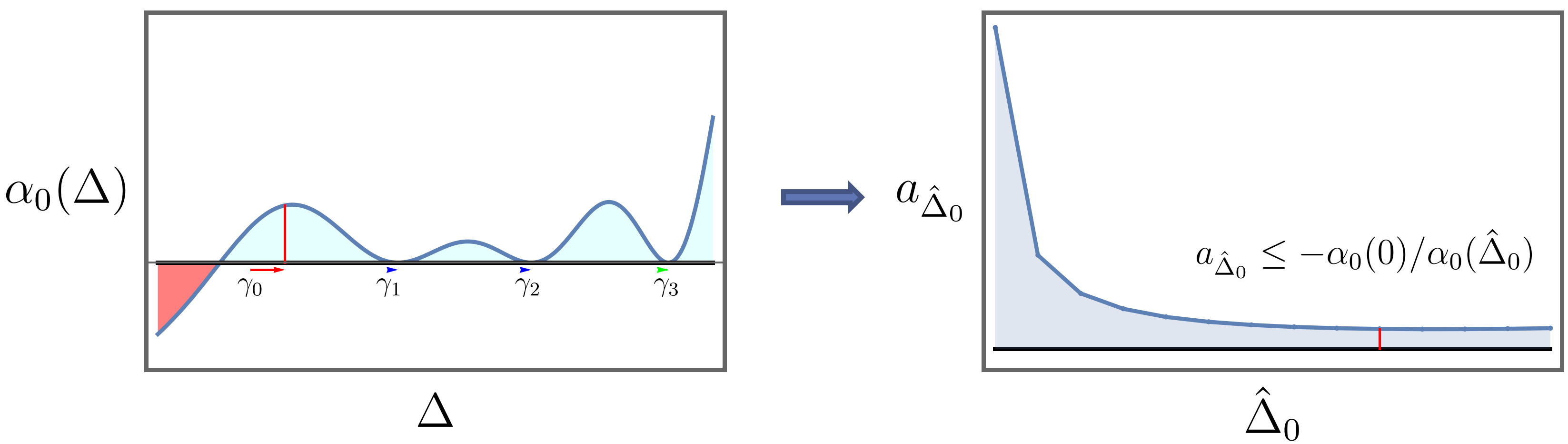


$$\alpha_0 = \sum_{n=0}^{N} [{\color{blue}a_n}\alpha_n^b + {\color{blue}b_n}\beta_n^b] + \sum_{n=N+1}^{\infty} [{\color{green}c_n}\alpha_n^b + {\color{green}d_n}\beta_n^b]$$

Hybrid Bootstrap







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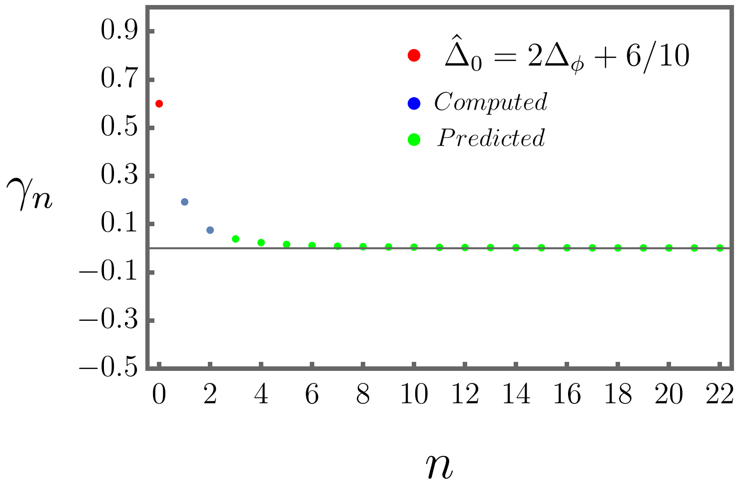
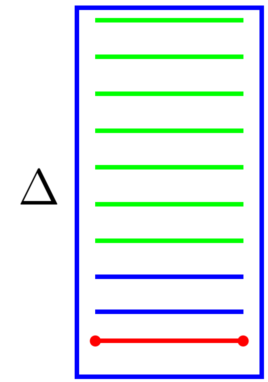
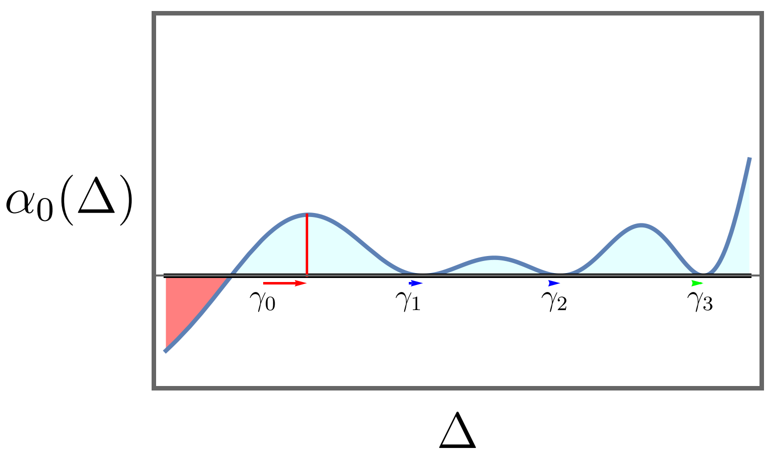
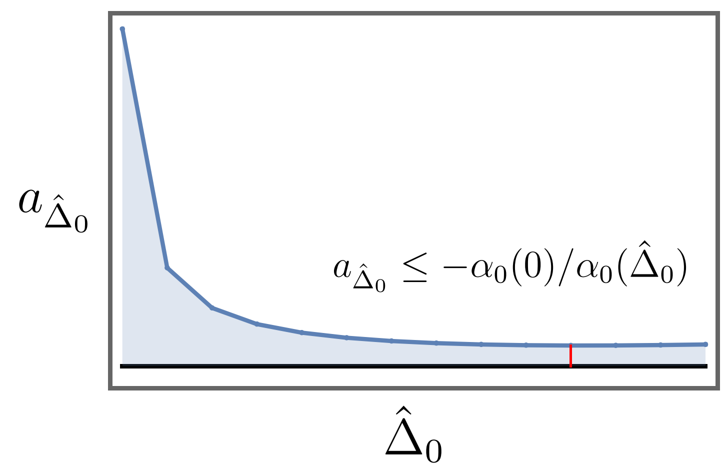
Hybrid Bootstrap









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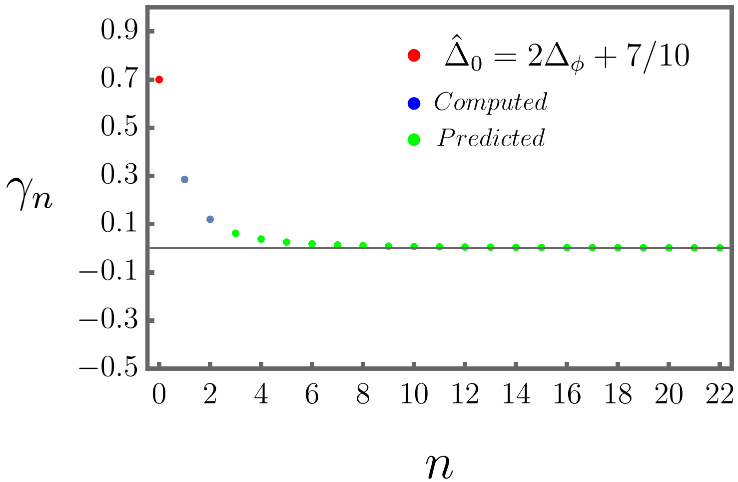
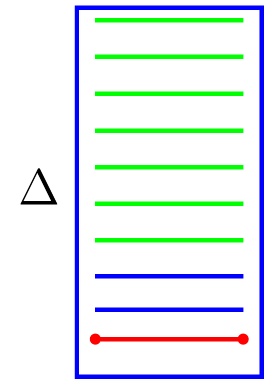
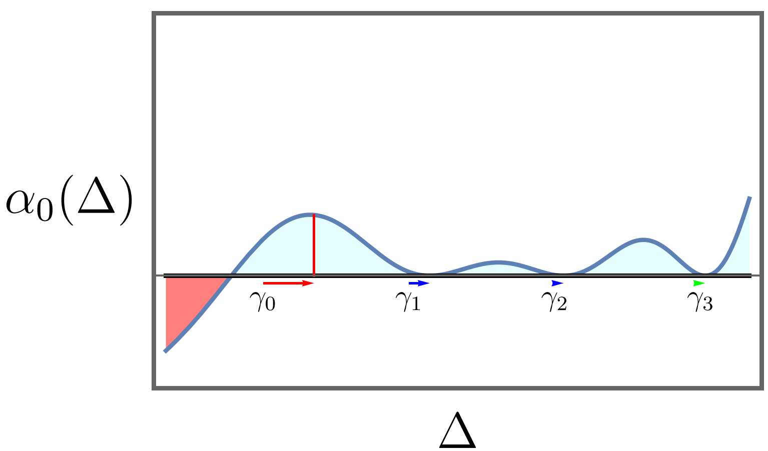
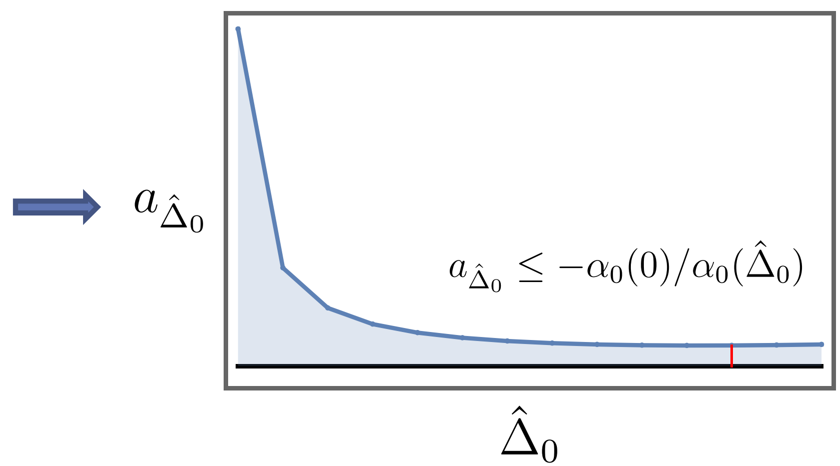
Hybrid Bootstrap









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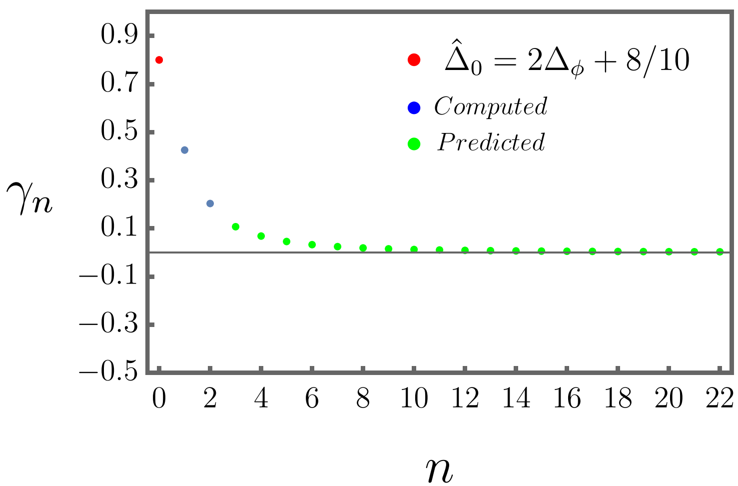
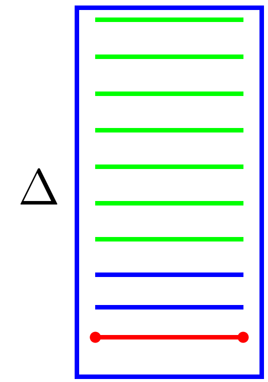
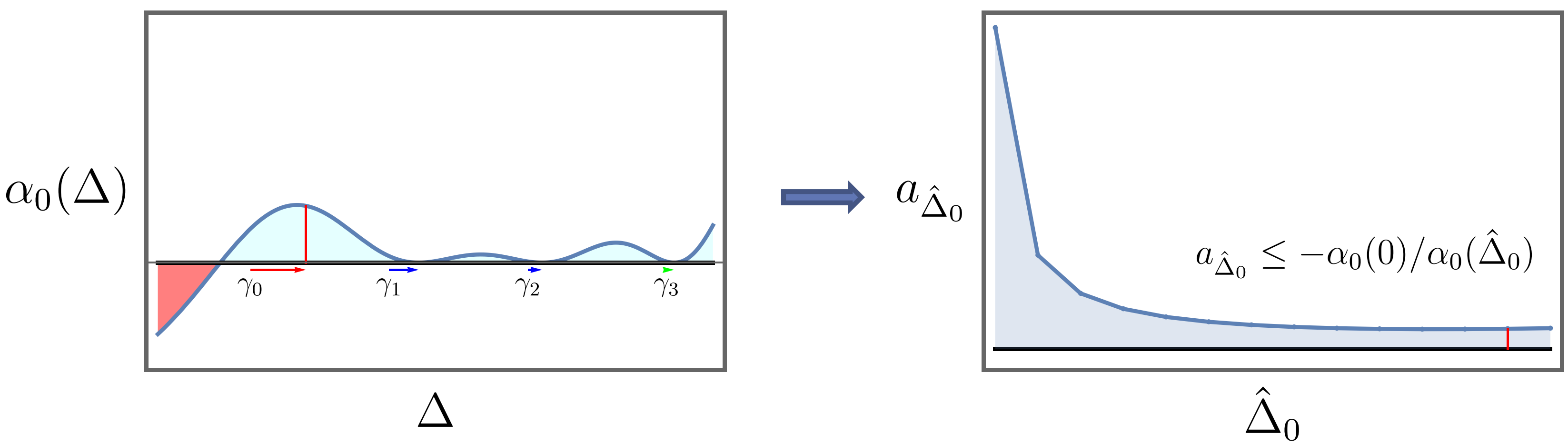
Hybrid Bootstrap







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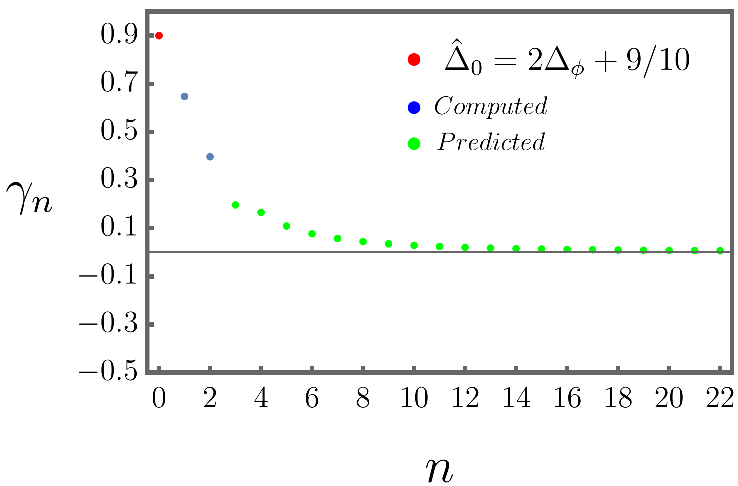
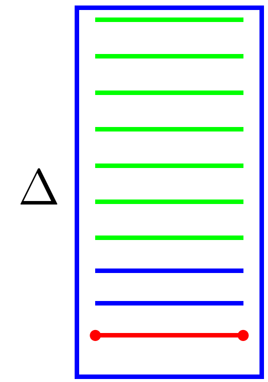
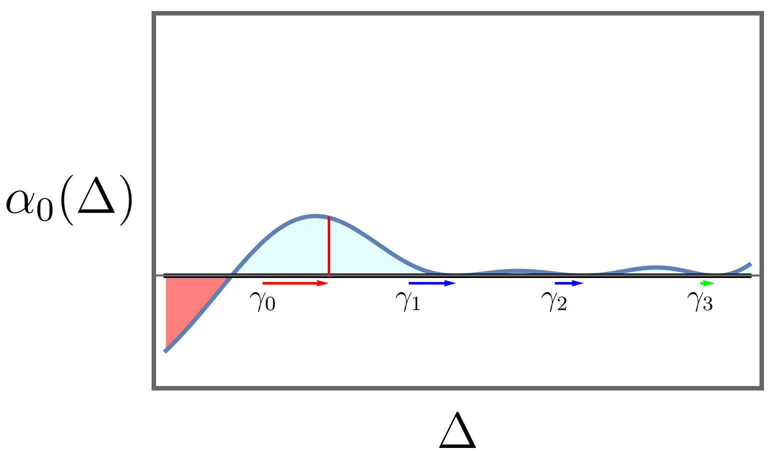
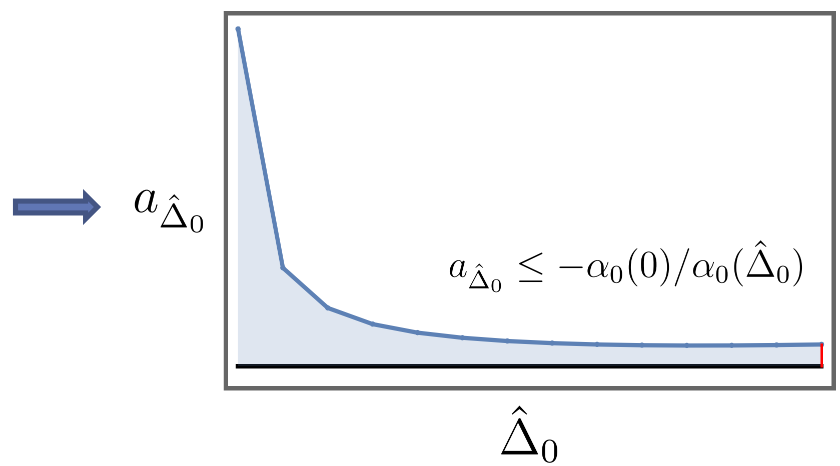
Hybrid Bootstrap









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Hybrid Bootstrap

- Schematically we run a loop:

    1. Solve extremality conditions *numerically* to get an approximation to the first N operators in the extremal solution.

    2. Use the results to *analytically* predict the data for remaining operators.

    3. Use this information to find an improved functional basis* and re-solve the numerical problem for the first N operators.

- Depending on accuracy with which how 2 and 3 are implemented one can improve power law numerical convergence in N by any desired amount.

**Key step where our approach differs from N. Su '22*

Hybrid Bootstrap

- Output is:
  - An approximation for *all* CFT data in the extremal solution.
  - A set of extremal functionals which give rigorous bounds *on any CFT* consistent with the initial fixed set assumptions (saturated by associated extremal solution)

- **Example**: bootstrap the extremal solution with fixed data:
  - Identity
  - Operator with dimension $\hat{\Delta}_0 = 2\Delta_\phi + \gamma_0$

Extremal bases

$$F_0 + \sum_{i=1}^{K} \hat{a}_i F_{\hat{\Delta}_i}$$

Plug in basis decomposition

$$F_\Delta(z) = \sum_{n=0}^{\infty} [\alpha_n(\Delta) F_{\Delta_n}(z) + \beta_n(\Delta) \partial_\Delta F_{\Delta_n}(z)]$$

Extremal bases

$$F_0 + \sum_{i=1}^{K} \hat{a}_i F_{\hat{\Delta}_i} = -\sum_{n=0}^{\infty} \left[a_n F_{\Delta_n}(z) + b_n \partial_\Delta F_{\Delta_n}(z) \right]$$

Allowed (OPE coefficients)

Disallowed, must cancel

$$F_\Delta(z) = \sum_{n=0}^{\infty} \left[\alpha_n(\Delta) F_{\Delta_n}(z) + \beta_n(\Delta) \partial_\Delta F_{\Delta_n}(z) \right]$$

Extremal bases

$$F_0 + \sum_{i=1}^{K} \hat{a}_i F_{\hat{\Delta}_i} + \sum_{n=0}^{\infty} \left[a_n F_{\Delta_n}(z) \right] = 0$$

$$\Updownarrow$$

$$a_n := -\left[\alpha_n(0) + \sum_{i=1}^{K} \hat{a}_i \alpha_n(\hat{\Delta}_i) \right] \qquad n = 0, 1, \ldots$$

$$b_n := \beta_n(0) + \sum_{i=1}^{K} \hat{a}_i \beta_n(\hat{\Delta}_i) \stackrel{!}{=} 0$$

Extremal bases

$$\beta_n(0) + \sum_{i=1}^{K} \hat{a}_i \beta_n(\hat{\Delta}_i) = 0$$

with $\left\{\hat{\Delta}_i, \hat{a}_i\right\}$ *freely chosen* (label extremal solution)

- These **Extremality Conditions** are now *implicit* equations for $\{\Delta_n\}$
- Functional basis *and* extremal solution can be constructed simultaneously by:
  - extremality conditions
  - duality conditions