The perturbative gradient flow at higher orders

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12 February 2025

Gradient Flow Workshop
Zürich (12-14 Feb 2025)

supported by: Deutsche Forschungsgemeinschaft

The gradient flow

flowed gauge field:

$$\frac{\partial}{\partial t} B_\mu(t,x) = \mathscr{D}_\nu G_{\nu\mu}(t,x)$$
$$B_\mu(t=0,x) = A_\mu(x)$$

flowed quark field:

$$\frac{\partial}{\partial t} \chi(t,x) = \mathscr{D}^2 \chi(t,x)$$
$$\chi(t=0,x) = \psi(x)$$

Narayanan, Neuberger 2006
Lüscher 2009
Lüscher 2010
Lüscher, Weisz 2011
Lüscher 2013

Schematically…

$$\frac{\partial}{\partial t} B_\mu(t) = \mathscr{D}_\nu G_{\nu\mu}(t)$$

$$\mathscr{D}_\mu = \partial_\mu - iT^a g_0 B^a_\mu(t)$$

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$$\frac{\partial}{\partial t} B = \mathscr{D}\, G$$

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flow equation: $$\partial_t B \sim \partial^2 B + g_0 \partial B^2 + g_0^2 B^3$$

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momentum space: $\tilde{B}_1(t) = e^{-tp^2} \tilde{A}(p)$

$$\tilde{B}_2(t,p) = \int_0^t ds \int d^4q\, K(t,s,p,q)\, A(p)\, A(p-q)$$

$$K(t,s,p,q) \sim \exp\big[-tp^2 - 2sq(q-p)\big]$$

etc.

Exponential damping in momentum integrals!

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$$\mathscr{L} = \mathscr{L}_{\text{QCD}} + \mathscr{L}_B + \mathscr{L}_\chi$$

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right)$$

$$\mathscr{L}_\chi \sim \int_0^\infty dt\, \bar{\lambda} \left(\partial_t - \mathscr{D}^2\right) \chi + \text{h.c.}$$

Lüscher, Weisz 2011

Lüscher 2013

Perturbative approach

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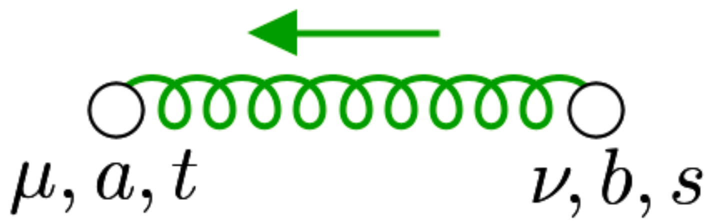
$$\frac{\delta^{ab}}{p^2}\left(\delta_{\mu\nu} - \xi \frac{p_\mu p_\nu}{p^2}\right) e^{-(t+s)p^2}$$

$$\sim \langle 0 | T\, B_\mu^a(t,x)\, B_\nu^b(s,0) | 0 \rangle$$



$$\delta_{ab}\delta_{\mu\nu}\,\theta(t-s)\, e^{-(t-s)p^2}$$

"gluon flow line"

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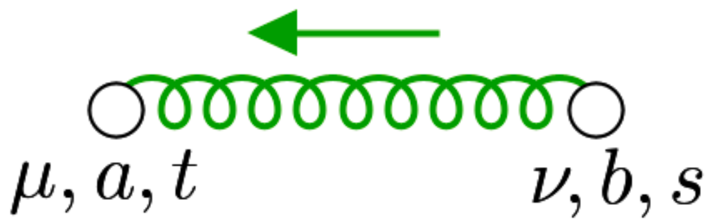

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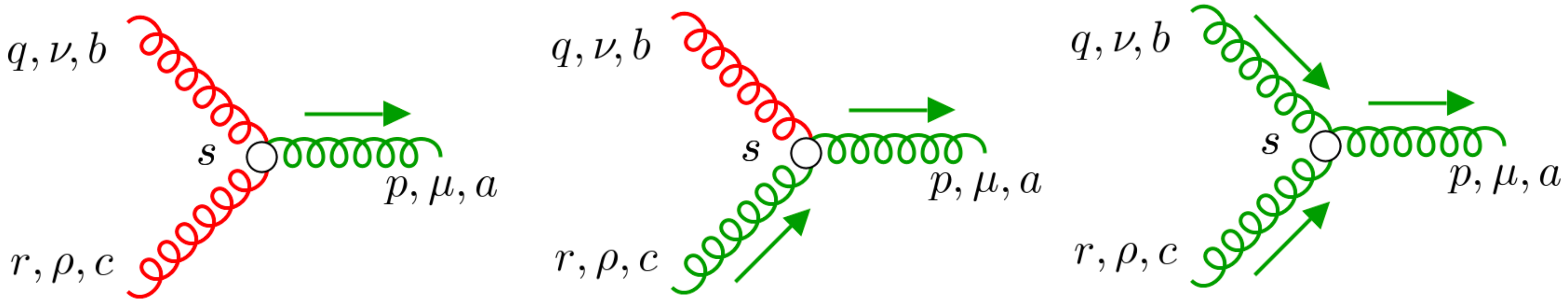
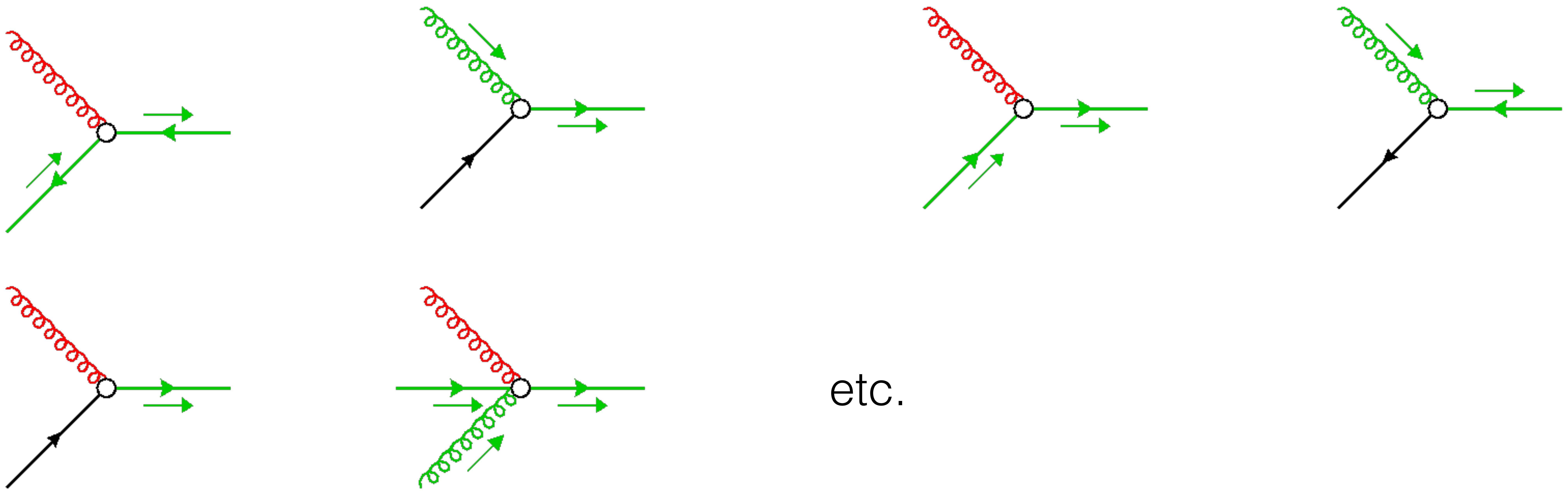


$$-\mathrm{i}g f^{abc}\int_0^\infty \mathrm{d}s\,\big(\delta_{\nu\rho}(r-q)_\mu + 2\delta_{\mu\nu}q_\rho - 2\delta_{\mu\rho}r_\nu$$
$$+ (\kappa-1)(\delta_{\mu\rho}q_\nu - \delta_{\mu\nu}r_\rho)\big)$$

+ 4-gluon vertex

Perturbative approach

$$\mathscr{L}_\chi \sim \int_0^\infty dt\, \bar{\lambda}\left(\partial_t - \mathscr{D}^2\right)\chi + \mathrm{h.c.}$$



etc.

Renormalization

bulk ($t > 0$) is UV regulated ⇒

renormalization of QCD parameters unaffected

$$\mathscr{L} = \mathscr{L}_{\text{QCD}} + \mathscr{L}_B + \mathscr{L}_\chi$$

$$\mathscr{L}_B \sim \int_0^\infty dt\, L_\mu \left(\partial_t B_\mu - \mathscr{D}_\nu G_{\nu\mu}\right)$$

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renormalization of flowed fields:

$$B_\mu^{\text{R}}(t) = Z_B^{1/2} B_\mu(t) \qquad Z_B = 1$$

Lüscher 2010

Lüscher, Weisz 2011

$$\chi^{\text{R}}(t) = Z_\chi^{1/2} \chi(t)$$

→ see Janosch Borgulat's talk (up next!)

Let’s calculate

$$E(t) \equiv \frac{1}{4}\langle G^a_{\mu\nu}(t) G^{a,\mu\nu}(t)\rangle$$

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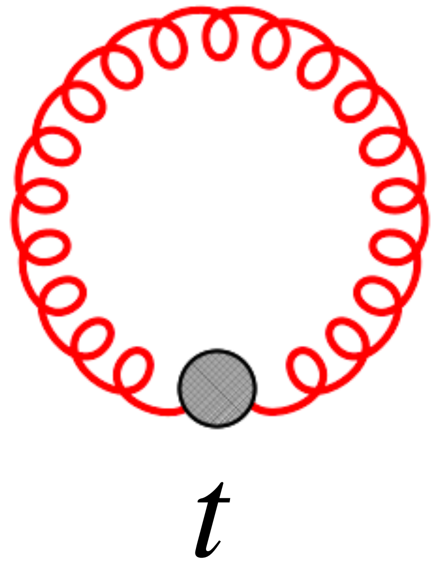

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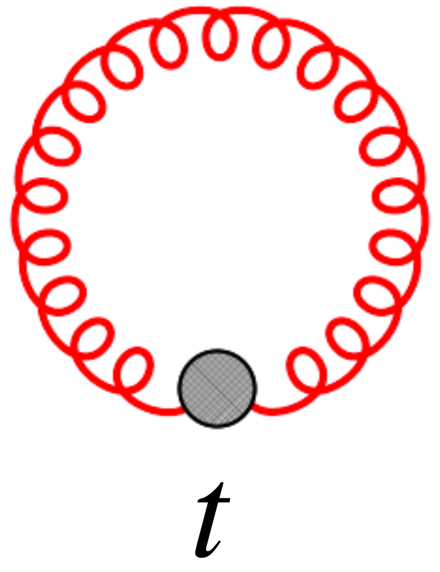

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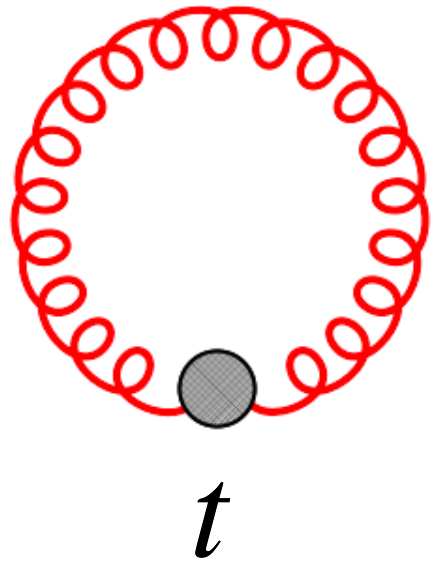

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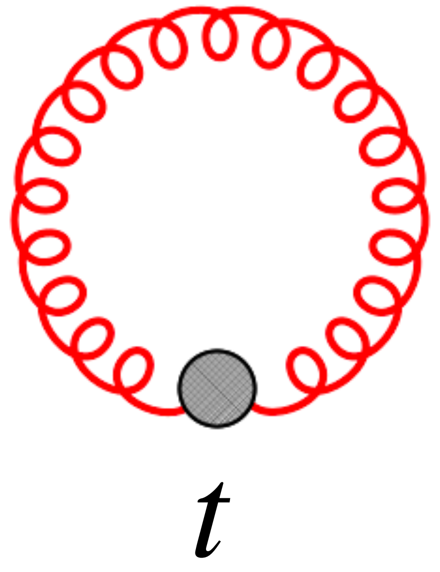

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explicitly: $$E(t) = \frac{3\alpha_s}{4\pi t^2} + \mathcal{O}(\alpha_s^2)$$

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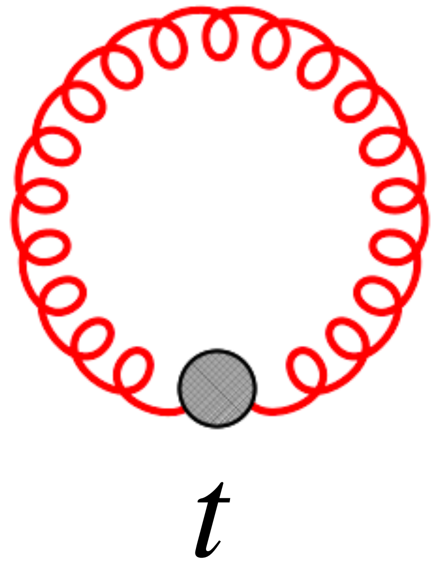

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→ measure α_s on the lattice?

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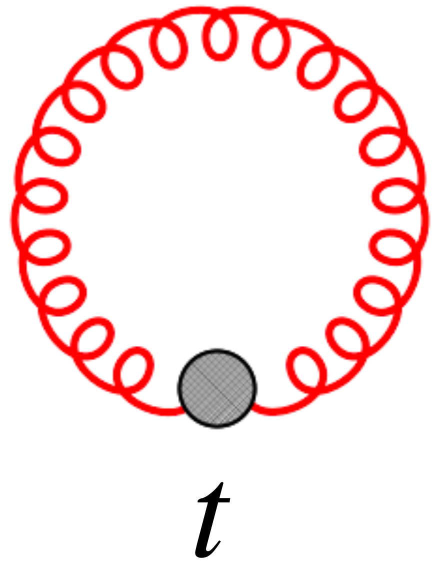

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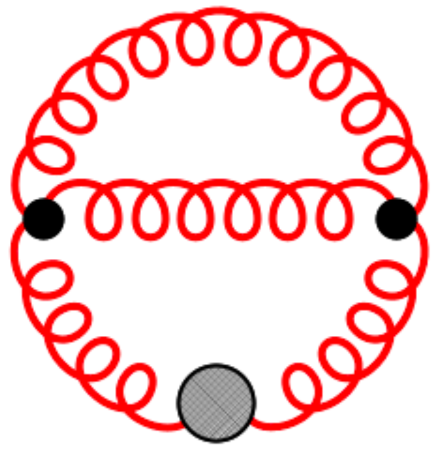
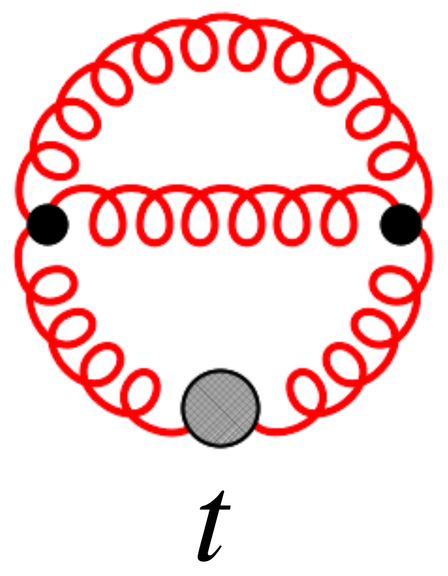
$$\alpha_s = \alpha_s(\mu)$$

Higher orders



$$\sim \int_p \int_k \frac{e^{-2\,t\,p^2}}{p^4\,k^2\,(p-k)^2}$$

Higher orders



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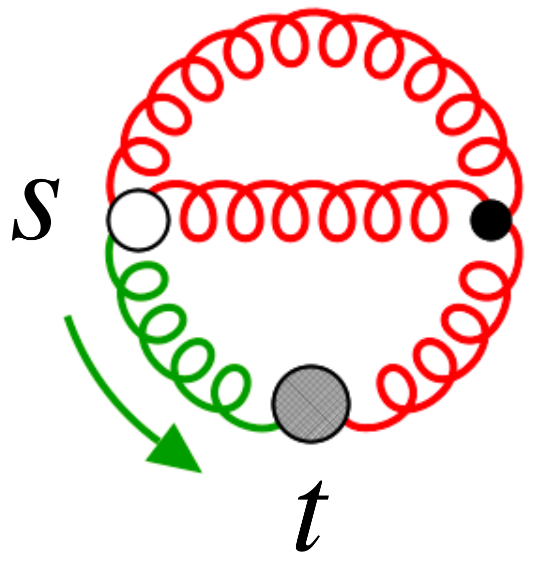
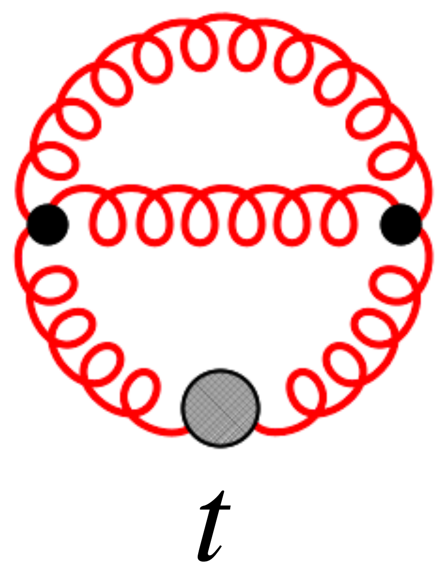


$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2 k^2 (p-k)^2}$$

Higher orders



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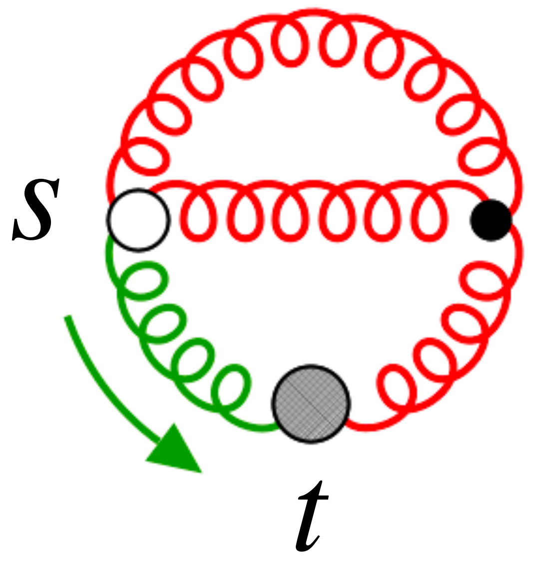

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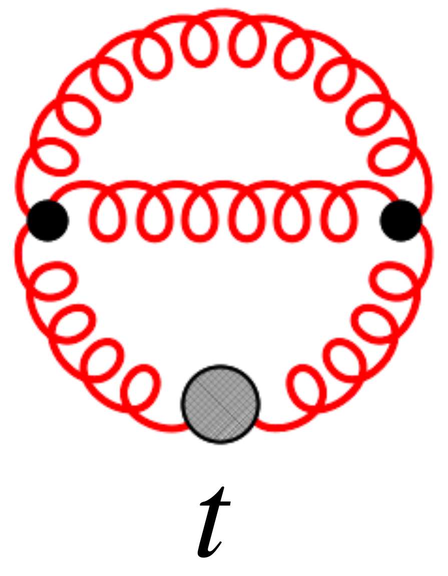
- generalized loop integrals

Higher orders



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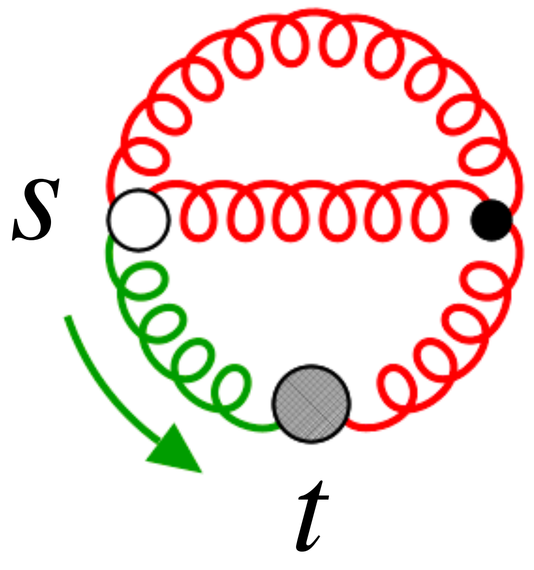
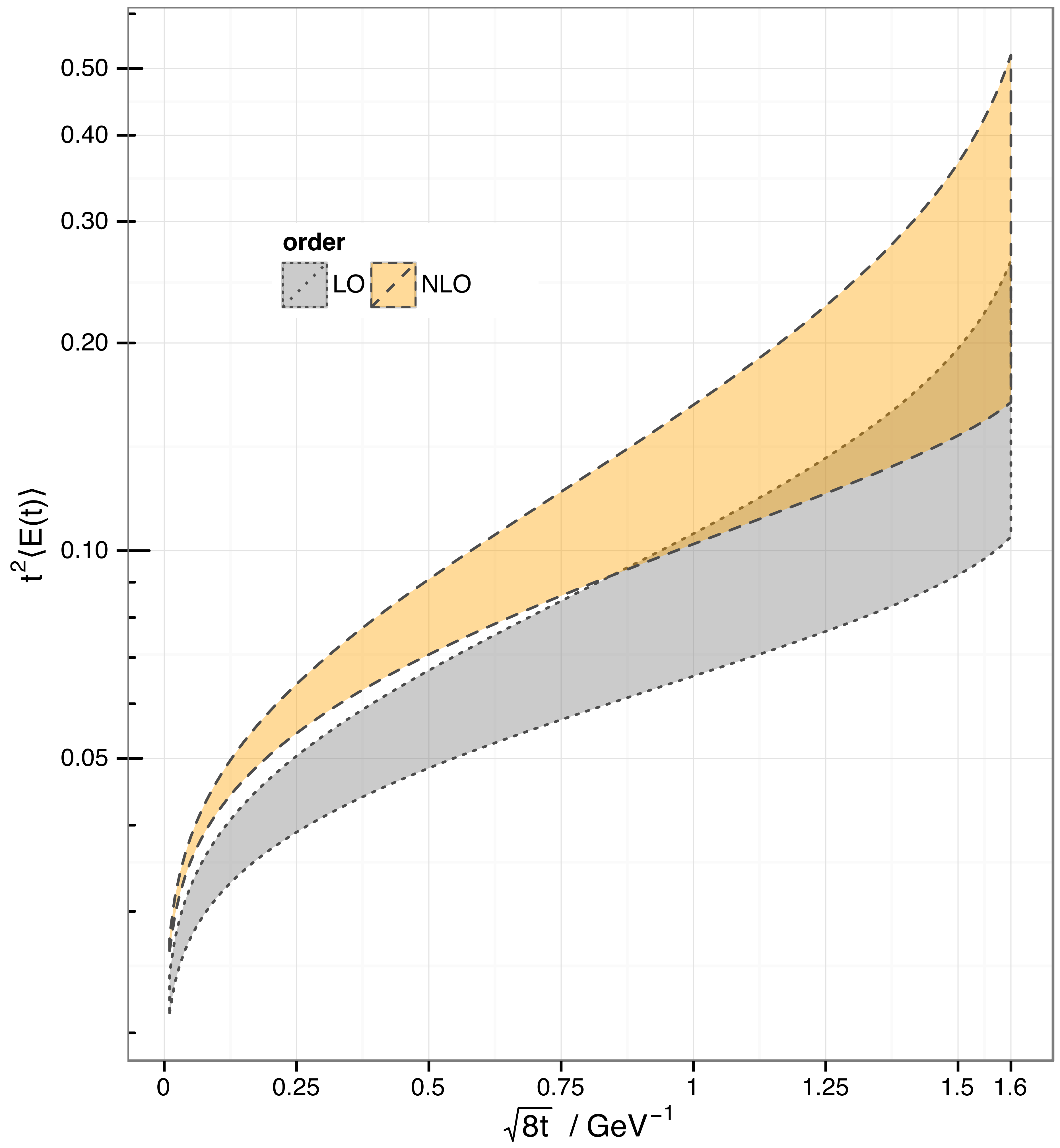


$$\int_0^t ds \int_p \int_k \frac{e^{-(2t-s)p^2}}{p^2\, k^2\, (p-k)^2}$$

- generalized loop integrals
- integration over flow-time parameters

$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu)\right]$$

Lüscher 2010



$$k_1 = \left(\frac{52}{9} + \frac{22}{3}\ln 2 - 3\ln 3\right) C_A - \frac{8}{9} n_f T_R + \beta_0 L_{t\mu}$$

$$L_{t\mu} = \ln 2\mu^2 t + \gamma_E$$

$$\mu_0 = \frac{1}{\sqrt{8t}}$$

resulting perturbative accuracy on α_s: ± 3-5%

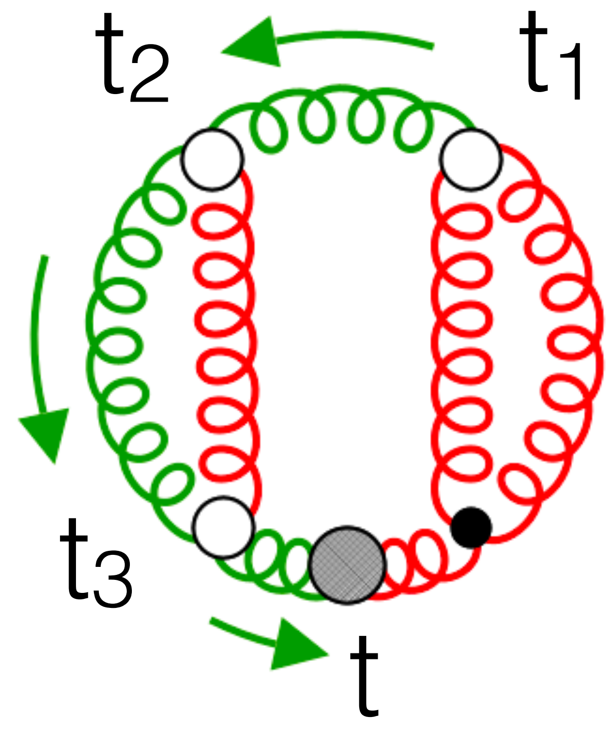
PDG: ± 1%

Three-loop calculation

Three-loop calculation

The usual problems:

- many diagrams (NLO: 20; NNLO: 3651)
- many integrals
- complicated integrals



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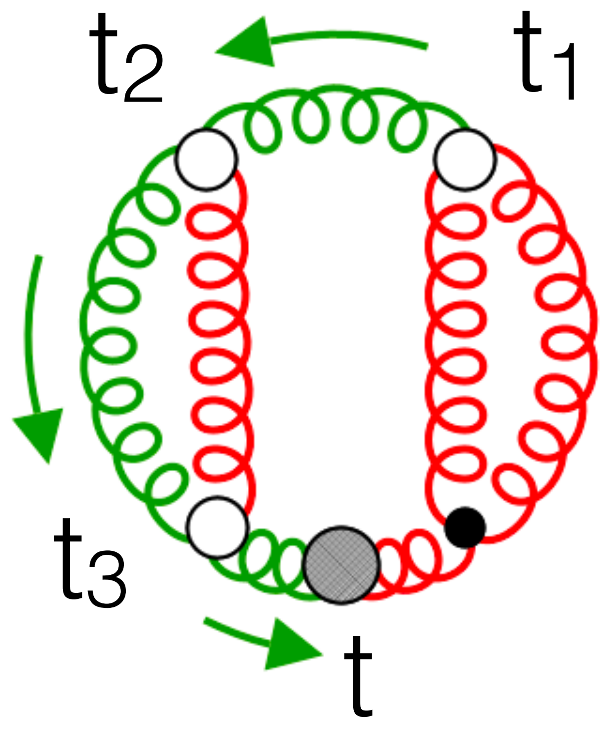


The usual solutions:

- automatic diagram generation
- reduce to master integrals
- evaluate master integrals

Artz, RH, Lange, Neumann, Prausa ’19

The perturbative toolbox

[For details, see: Artz, RH, Lange, Neumann, Prausa 2019]

diagram generation: qgraf — Nogueira 1993

diagram analyzation: q2e/exp — RH, Seidensticker, Steinhauser 1997

$\longrightarrow$ tapir/exp — Gerlach, Herren, Lang 2022

algebraic manipulations: FORM — Vermaseren 2000, …

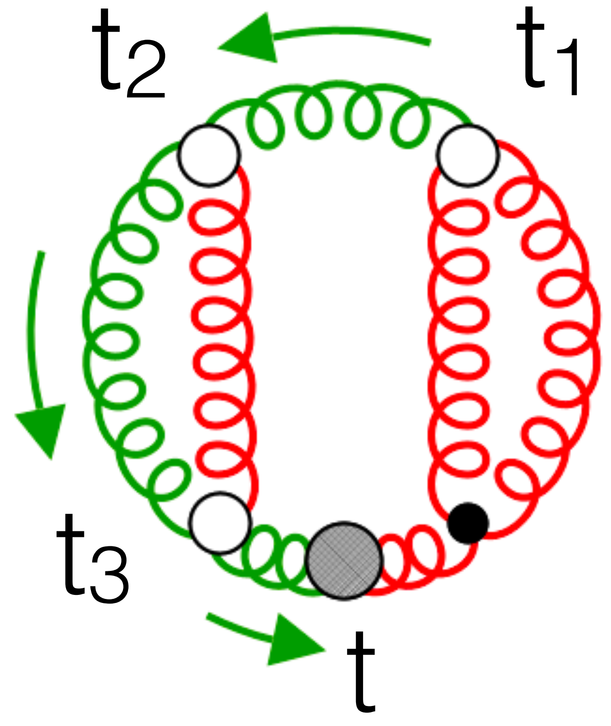
reduction to masters: Kira $\otimes$ FireFly — Usovitsch, Uwer, Maierhöfer 2017 $\otimes$ Klappert, Klein, Lange 2019

Chetyrkin, Tkachov 1981
Laporta 2000

Three-loop calculation

$$I(\{c_1, \cdots, c_f\}, \{a_1(u), \cdots, a_6(u)\}, \{b_1, \cdots, b_6\}) =$$

$$= \left(\prod_{i=1}^{f} \int_0^1 du_i \, u_i^{c_i} \right) \int d^D p_1 \, d^D p_2 \, d^D p_3 \, \frac{\exp\left[-t \left(a_1(u) p_1^2 + \cdots + a_6(u) p_6^2 \right) \right]}{(p_1^2)^{b_1} \, \cdots \, (p_6^2)^{b_6}}$$



$$u_i = t_i / t$$

Three-loop calculation

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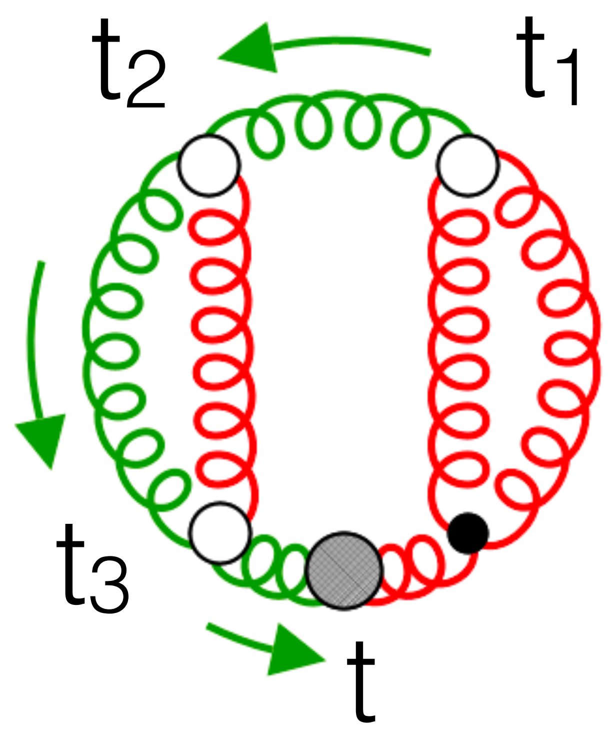

$u_i = t_i/t$

IbP identities:

$$\frac{\partial}{\partial p_i}\cdot p_j\, I(c,a,b) = D\,\delta_{ij}\, I(c,a,b) + \sum I(c',a,b')$$

$$\frac{\partial}{\partial u_i} I(c,a,b) = I(c',a(u=1),b') - I(c',a(u=0),b')$$

Artz, RH, Lange, Neumann, Prausa 2019

Three-loop calculation

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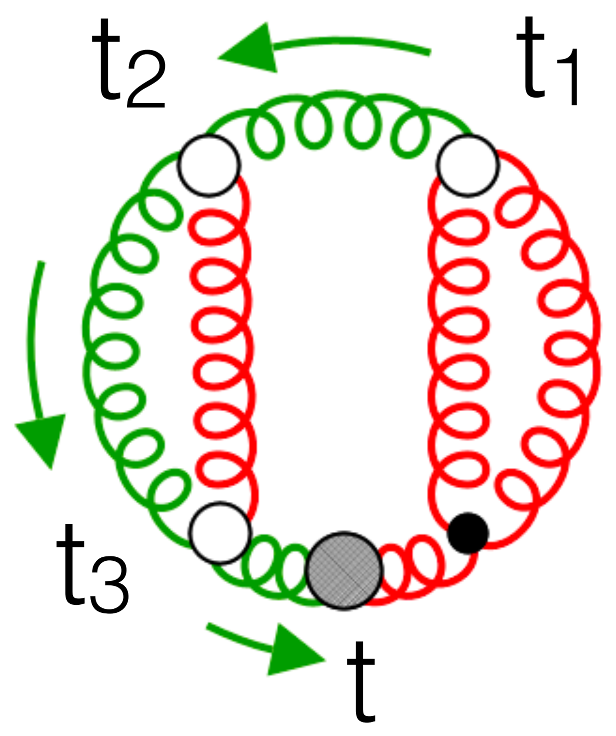

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Artz, RH, Lange, Neumann, Prausa 2019

Huge systems of linear equations, solved by “master integrals”.

The perturbative toolbox

[For details, see: Artz, RH, Lange, Neumann, Prausa 2019]

diagram generation: **qgraf** Nogueira 1993

diagram analyzation: **q2e/exp** RH, Seidensticker, Steinhauser 1997

$\longrightarrow$ **tapir/exp** Gerlach, Herren, Lang 2022

algebraic manipulations: **FORM** Vermaseren 2000, …

reduction to masters: **Kira** $\otimes$ **FireFly** Usovitsch, Uwer, Maierhöfer 2017 $\otimes$ Klappert, Klein, Lange 2019

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sector decomposition:

Binoth, Heinrich 2002

$$\int d^Dk \int d^Dp \int_0^t ds \frac{e^{-tp^2 - s(k-p)^2}}{k^2 p^2 (k-p^2)} = \frac{A}{\epsilon^2} + \frac{B}{\epsilon} + C + \ldots$$

$$\left(\prod_{i=1}^{f}\int_0^1 du_i\, u_i^{c_i}\right)\int d^Dp_1\, d^Dp_2\, d^Dp_3\, \frac{\exp\left[-t\left(a_1(u)p_1^2+\cdots+a_6(u)p_6^2\right)\right]}{(p_1^2+m_1^2)^{b_1}\,\cdots\,(p_6^2+m_6^2)^{b_6}}$$

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$$c_1 = c_2 = 0$$

$$a_1 = u_1u_2\,,\quad a_2 = u_2\,,\quad a_3 = u_2 - u_1u_2$$

$$a_4 = 1\,,\quad a_5 = 1 + u_1u_2\,,\quad a_6 = 1 - u_2$$

$$b_1 = b_4 = 1$$

$$b_2 = b_3 = b_5 = b_6 = 0$$

$$m_1 = \cdots = m_6 = 0$$

$$\left(\prod_{i=1}^{f}\int_0^1 du_i\, u_i^{c_i}\right)\int d^Dp_1\, d^Dp_2\, d^Dp_3\, \frac{\exp\left[-t\left(a_1(u)p_1^2+\cdots+a_6(u)p_6^2\right)\right]}{(p_1^2+m_1^2)^{b_1}\,\cdots\,(p_6^2+m_6^2)^{b_6}}$$

$$c_1 = c_2 = 0$$

$$a_1 = u_1u_2\,,\quad a_2 = u_2\,,\quad a_3 = u_2 - u_1u_2$$

$$a_4 = 1\,,\quad a_5 = 1 + u_1u_2\,,\quad a_6 = 1 - u_2$$

$$b_1 = b_4 = 1$$

$$b_2 = b_3 = b_5 = b_6 = 0$$

$$m_1 = \cdots = m_6 = 0$$

ftint RH, Nellopoulos, Olsson, Wesle ‘24

(based on pySecDec)

Heinrich, Magerya, Kerner, Jones, …

$$\left(\prod_{i=1}^{f}\int_0^1 du_i\, u_i^{c_i}\right)\int d^Dp_1\, d^Dp_2\, d^Dp_3\, \frac{\exp\left[-t\left(a_1(u)p_1^2+\cdots+a_6(u)p_6^2\right)\right]}{(p_1^2+m_1^2)^{b_1}\cdots(p_6^2+m_6^2)^{b_6}}$$

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ftint RH, Nellopoulos, Olsson, Wesle '24

(based on pySecDec)

Heinrich, Magerya, Kerner, Jones, …

```
f[{{0,0},{u1*u2,u2,u2-u1*u2,1,1+u1*u2,1-u2}},{1,0,0,1,0,0}] -> (
    +eps^-1*(+8.333333333333343*10^-02+0.0000000000000000*10^+00*I)
    +eps^-1*(+1.443389544408614 5*10^-15+0.0000000000000000*10^+00*I)*plusminus
    +eps^0*(+3.023827028456266 3*10^-01+0.0000000000000000*10^+00*I)
    +eps^0*(+1.691836274649922 8*10^-08+0.0000000000000000*10^+00*I)*plusminus
    +eps^1*(+6.553101045801212 9*10^-01+0.0000000000000000*10^+00*I)
    +eps^1*(+3.785726080291666 2*10^-08+0.0000000000000000*10^+00*I)*plusminus
  ),
```

$$\left(\prod_{i=1}^{f}\int_0^1 du_i\, u_i^{c_i}\right)\int d^Dp_1\, d^Dp_2\, d^Dp_3\, \frac{\exp\left[-t\left(a_1(u)p_1^2+\cdots+a_6(u)p_6^2\right)\right]}{(p_1^2+m_1^2)^{b_1}\,\cdots\,(p_6^2+m_6^2)^{b_6}}$$

$$c_1 = c_2 = 0$$

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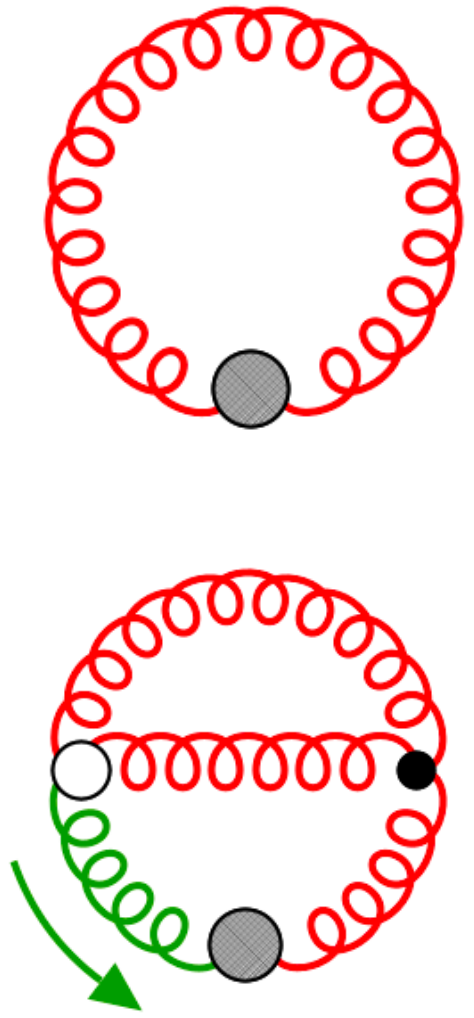
ftint RH, Nellopoulos, Olsson, Wesle '24
(based on pySecDec)
Heinrich, Magerya, Kerner, Jones, …

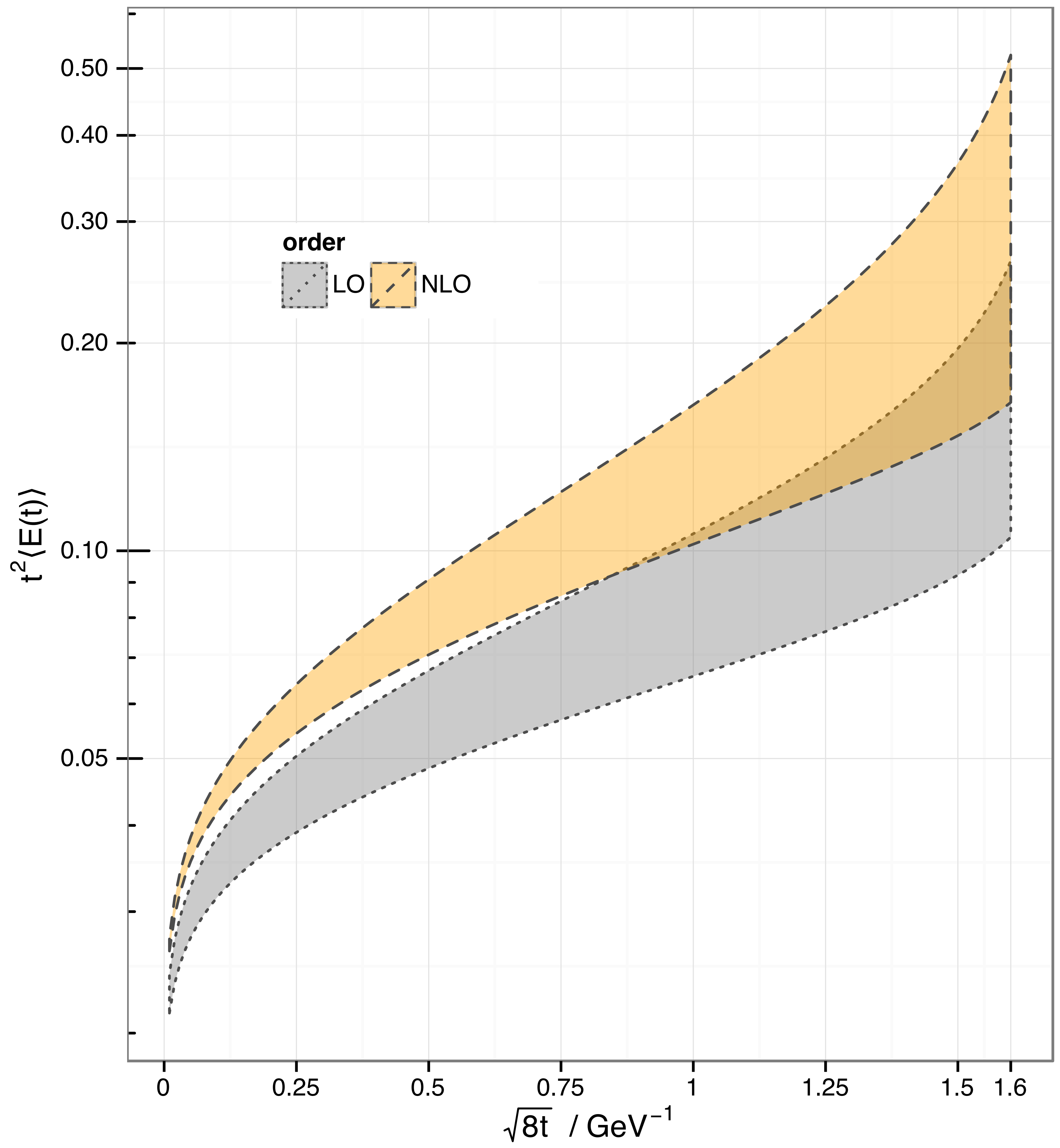
→ see Robert Mason's talk (today, 11:30am)

```
f[{{0,0},{u1*u2,u2,u2-u1*u2,1,1+u1*u2,1-u2}},{1,0,0,1,0,0}] -> (
    +eps^-1*(+8.3333333333333343*10^-02+0.0000000000000000*10^+00*I)
    +eps^-1*(+1.4433895444086145*10^-15+0.0000000000000000*10^+00*I)*plusminus
    +eps^0*(+3.0238270284562663*10^-01+0.0000000000000000*10^+00*I)
    +eps^0*(+1.6918362746499228*10^-08+0.0000000000000000*10^+00*I)*plusminus
    +eps^1*(+6.5531010458012129*10^-01+0.0000000000000000*10^+00*I)
    +eps^1*(+3.7857260802916662*10^-08+0.0000000000000000*10^+00*I)*plusminus
  ),
```

$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu)\right]$$

Lüscher 2010





$$k_1 = \left(\frac{52}{9} + \frac{22}{3}\ln 2 - 3\ln 3\right) C_A - \frac{8}{9} n_f T_R + \beta_0 L_{t\mu}$$

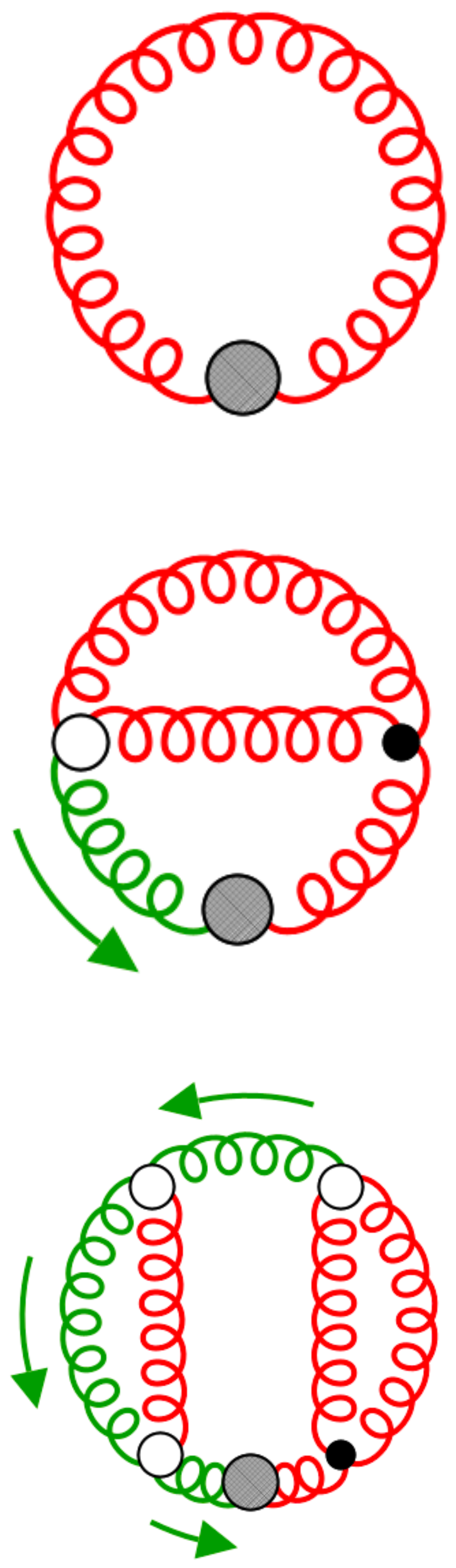
$$L_{t\mu} = \ln 2\mu^2 t + \gamma_E$$

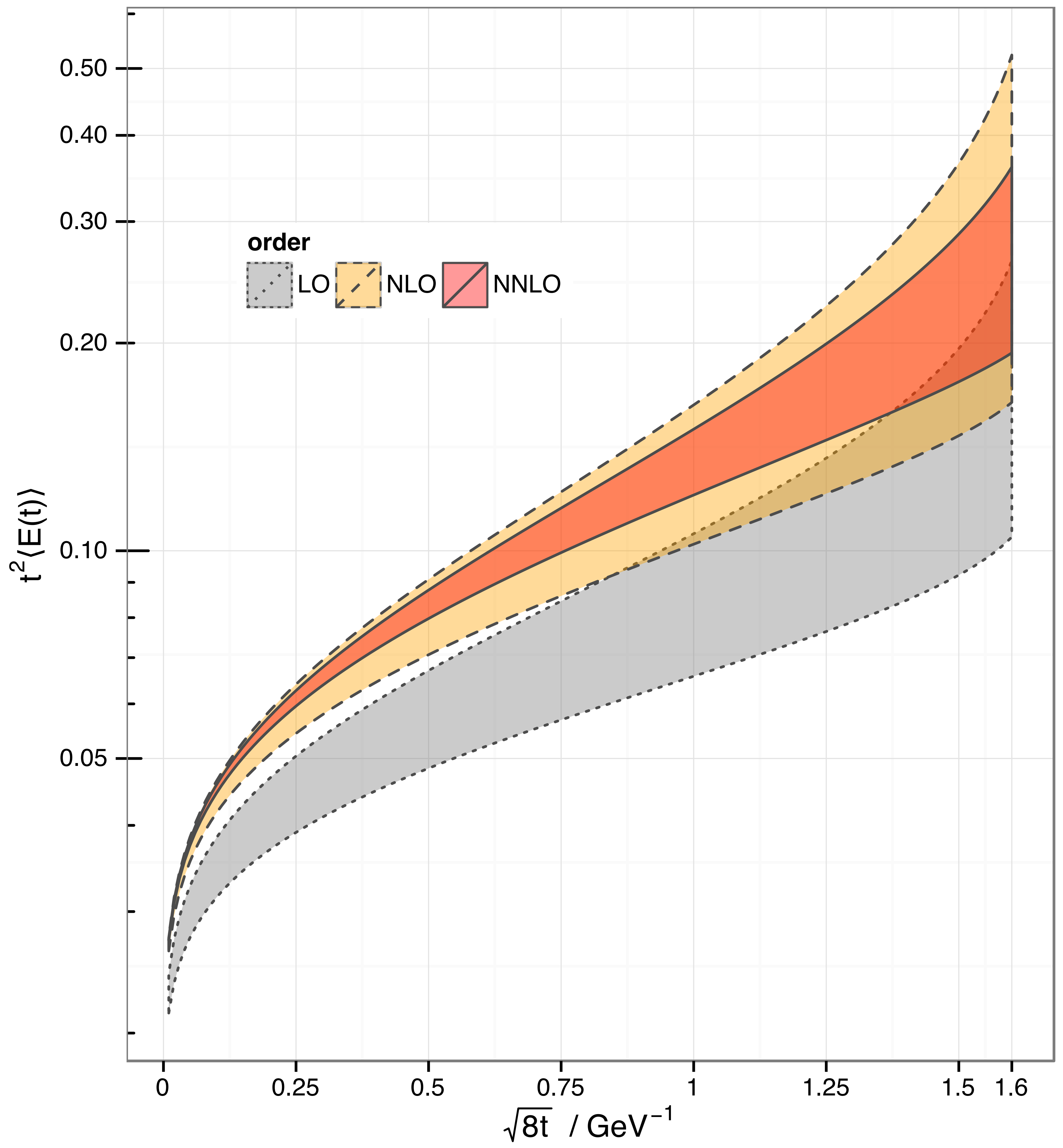
resulting perturbative accuracy on α_s: ± 3-5%

PDG: ± 1%

$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right]$$

RH, Neumann 2016





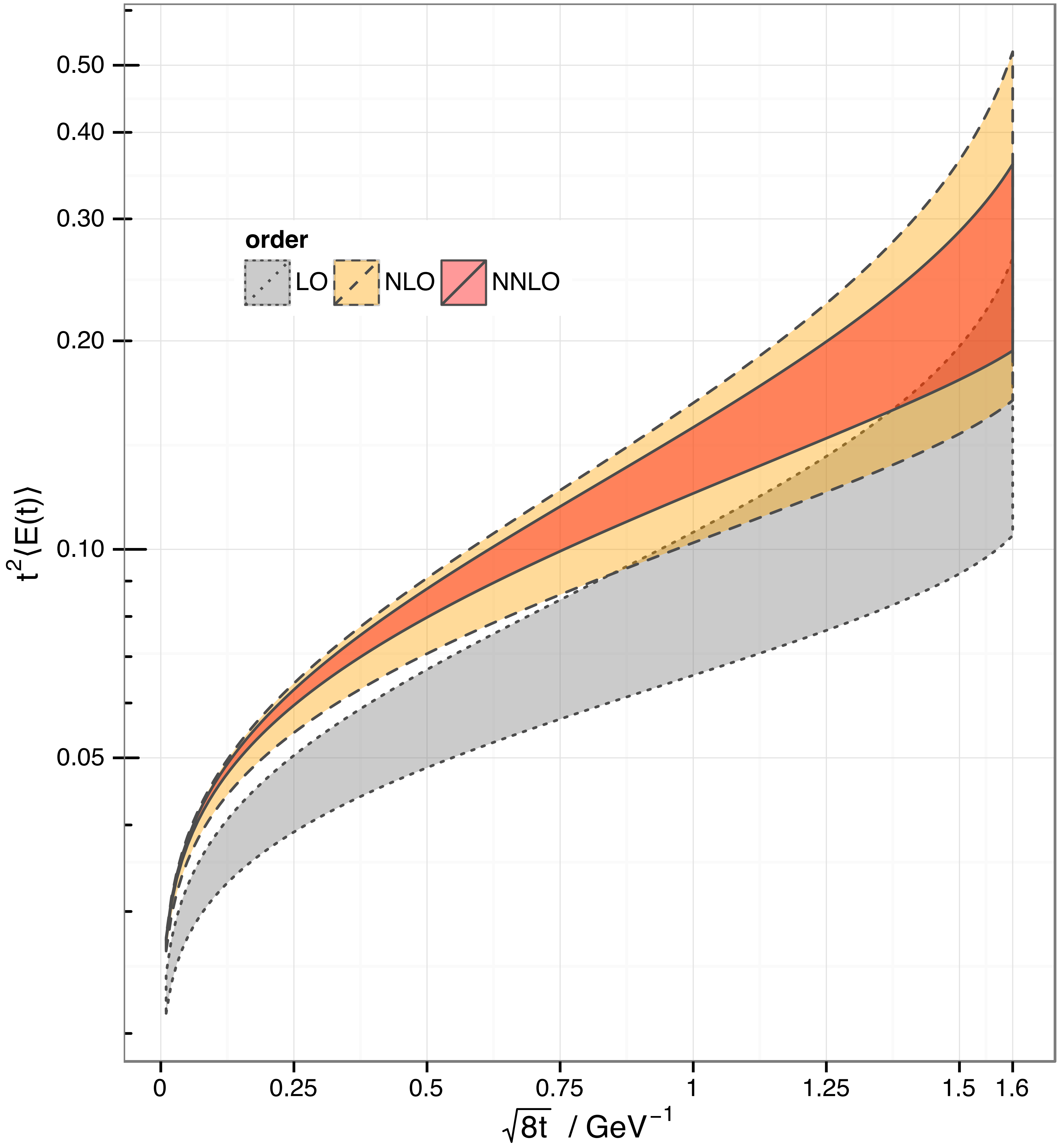
resulting perturbative accuracy on α_s: O(1%)

PDG: ± 1%

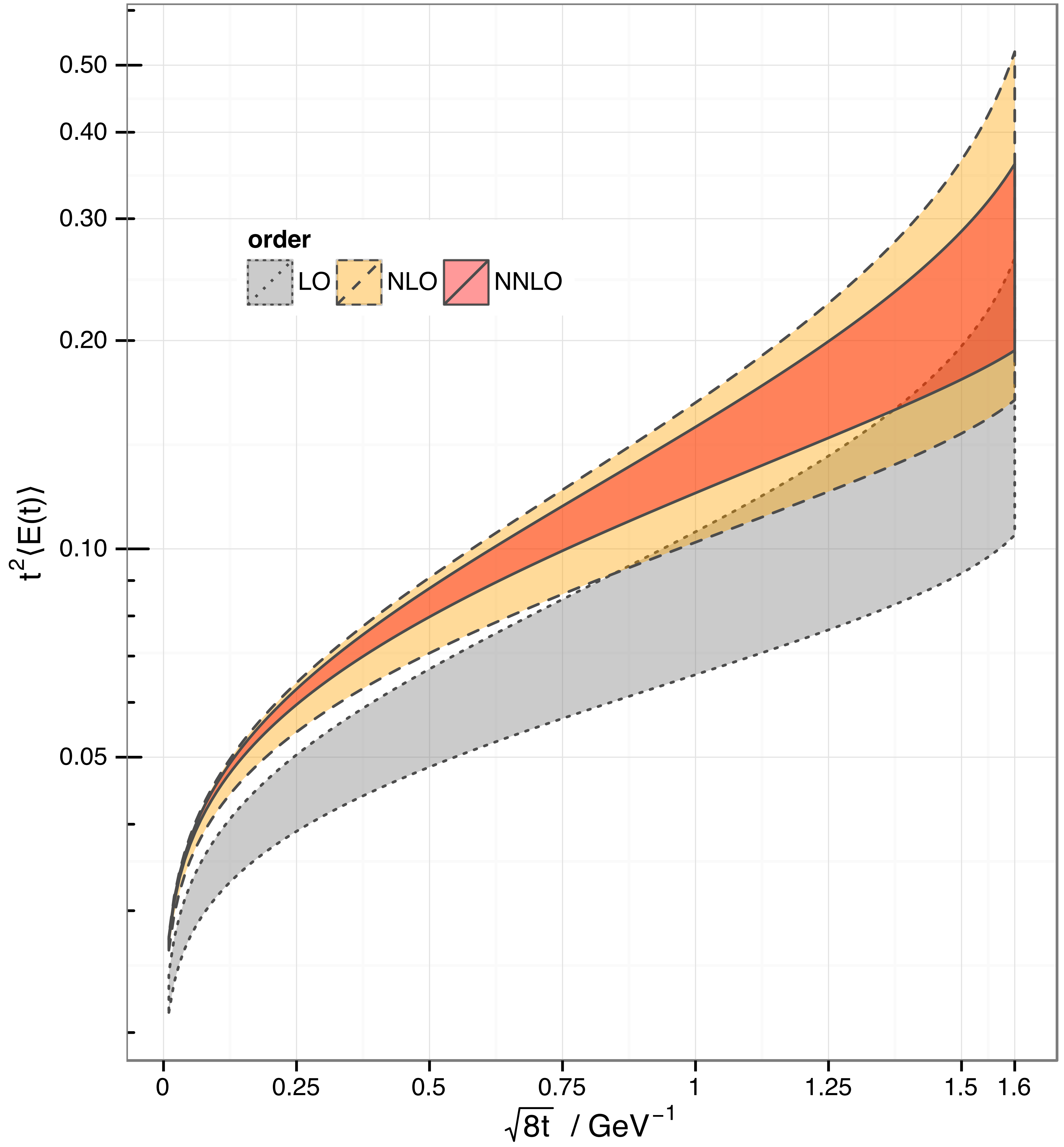
Derive $\alpha_s(m_Z)$

	$t^2\langle E(t)\rangle \cdot 10^4$							
q_8	2 GeV		10 GeV			m_Z		
$\alpha_s(m_Z)$	$n_f = 3$	$n_f = 4$	$n_f = 3$	$n_f = 4$	$n_f = 5$	$n_f = 3$	$n_f = 4$	$n_f = 5$
0.113	744	755	424	446	456	267	285	299
0.1135	753	764	426	449	459	268	286	301
0.114	762	773	429	452	462	269	287	302
0.1145	771	782	432	455	466	270	289	303
0.115	780	792	435	458	469	272	290	305
0.1155	789	802	438	461	472	273	291	306
0.116	798	811	440	465	476	274	292	308
0.1165	808	821	443	468	479	275	294	309
0.117	818	832	446	471	483	276	295	311
0.1175	827	842	449	474	486	277	296	312
0.118	837	852	452	478	490	278	298	314
0.1185	847	863	455	481	493	279	299	315
0.119	858	874	457	484	497	280	300	316
0.1195	868	885	460	488	500	281	301	318
0.12	879	896	463	491	504	282	303	319

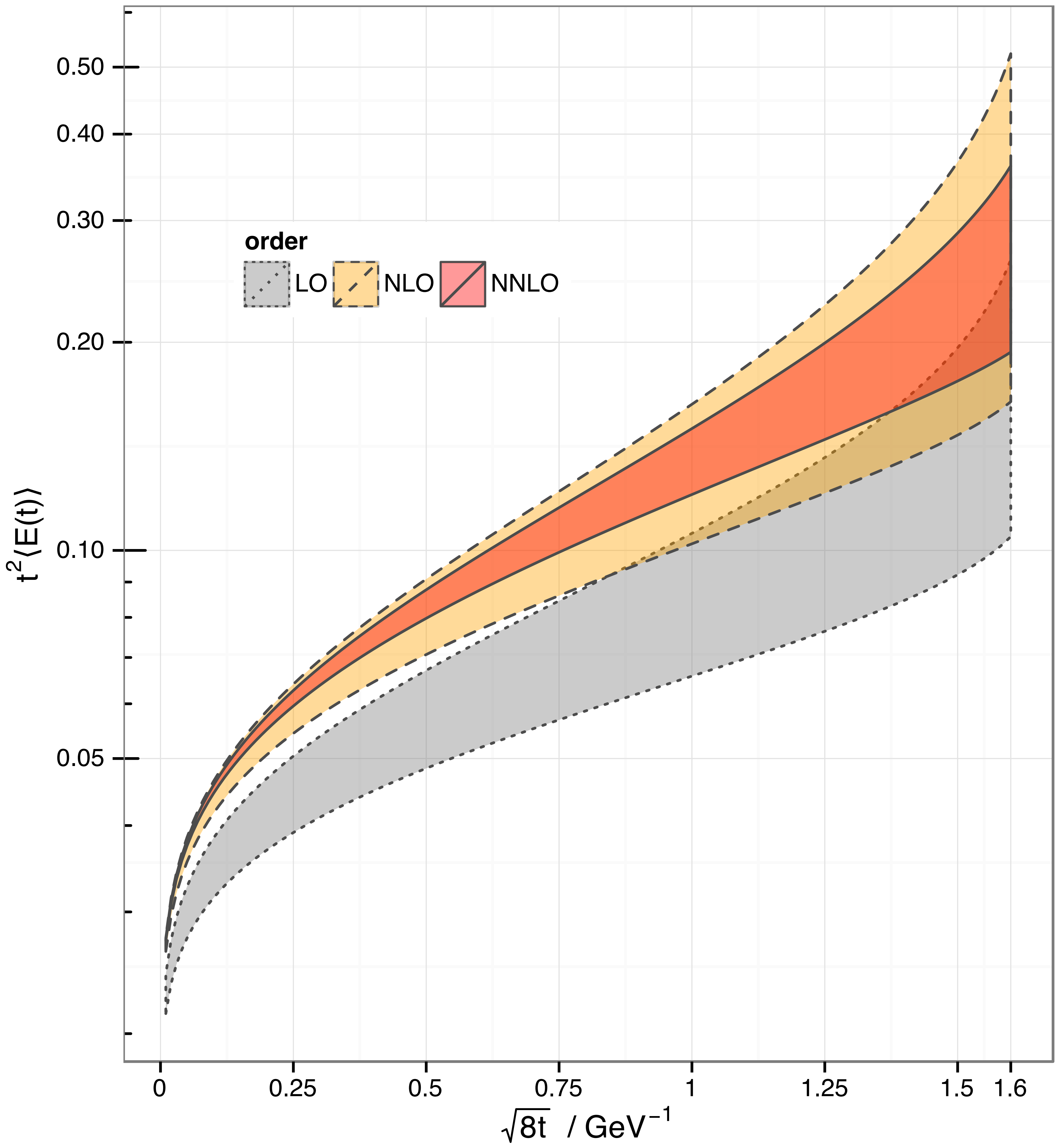
$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right]$$



$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\hat{\alpha}_s(t) = \frac{3}{4}\hat{a}_s(t)$$



$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \qquad \equiv \frac{3}{4\pi}\hat{\alpha}_s(t) = \frac{3}{4}\hat{a}_s(t)$$



$$t\frac{d}{dt}\hat{a}_s(t) = \hat{\beta}(\hat{a}_s)$$

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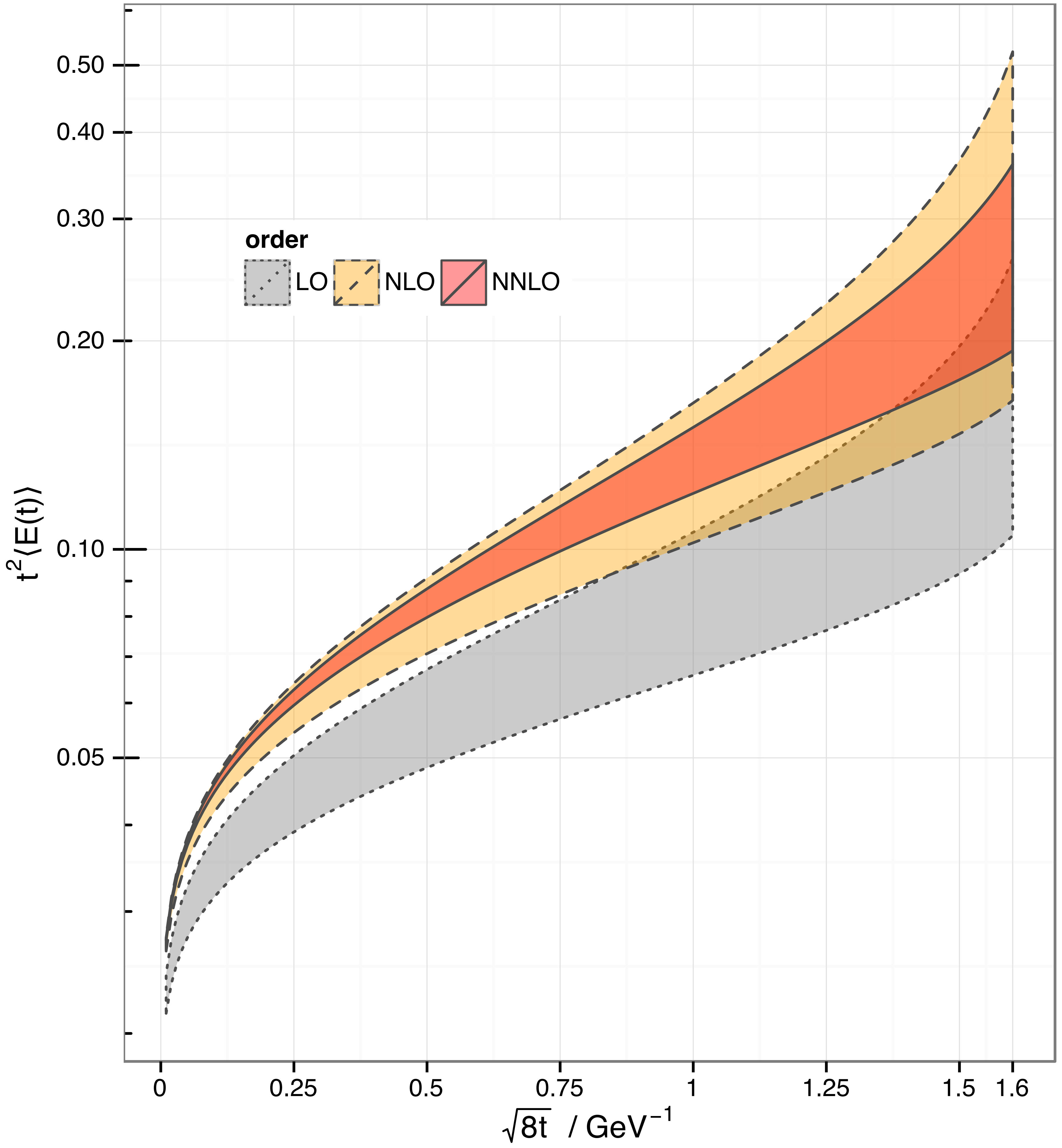

$$t\frac{d}{dt}\hat{a}_s(t) = \hat{\beta}(\hat{a}_s)$$

$$= \hat{a}_s^2\left[\hat{\beta}_0 + \hat{a}_s\hat{\beta}_1 + \hat{a}_s^2\hat{\beta}_2 + \ldots\right]$$

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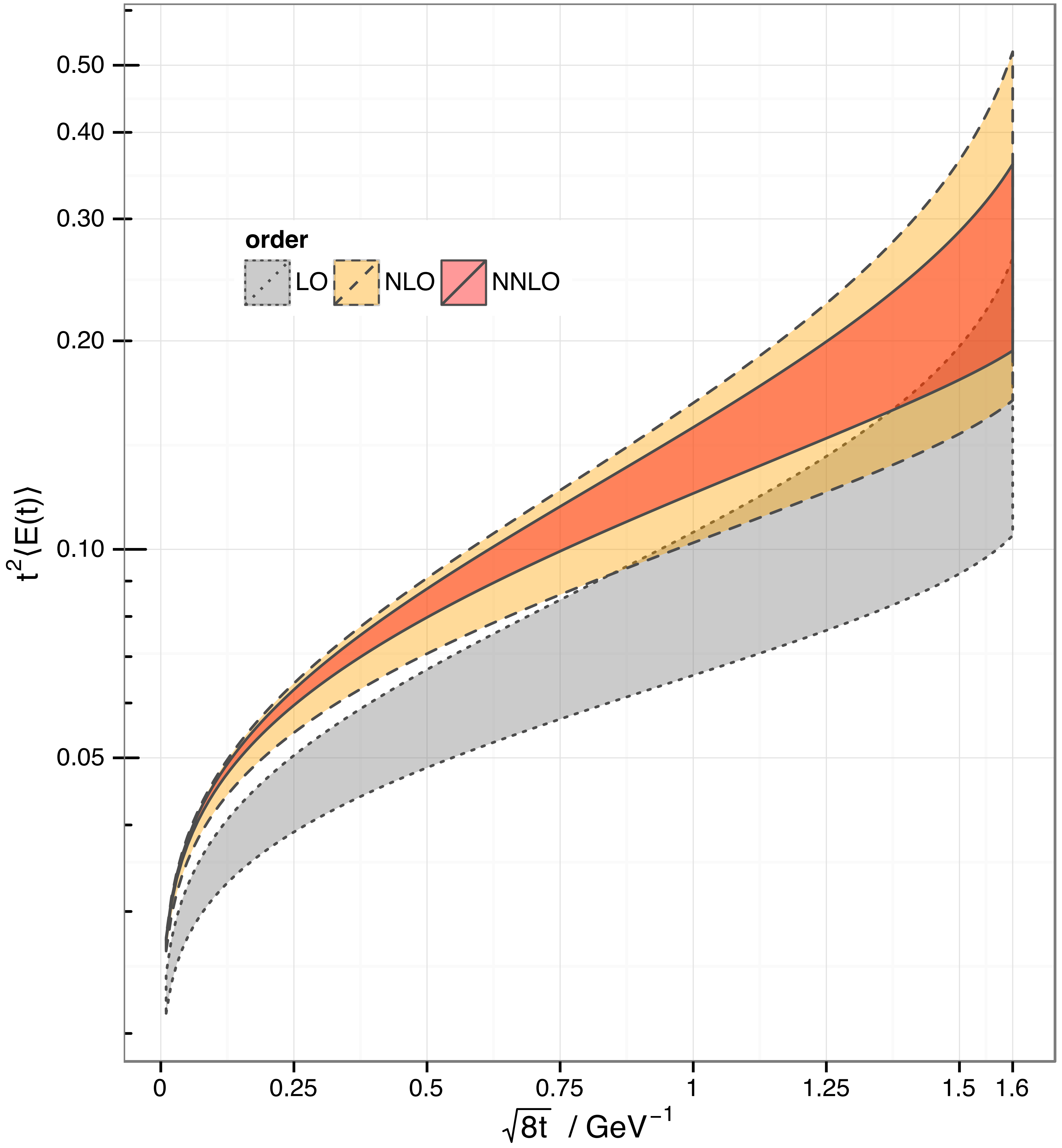

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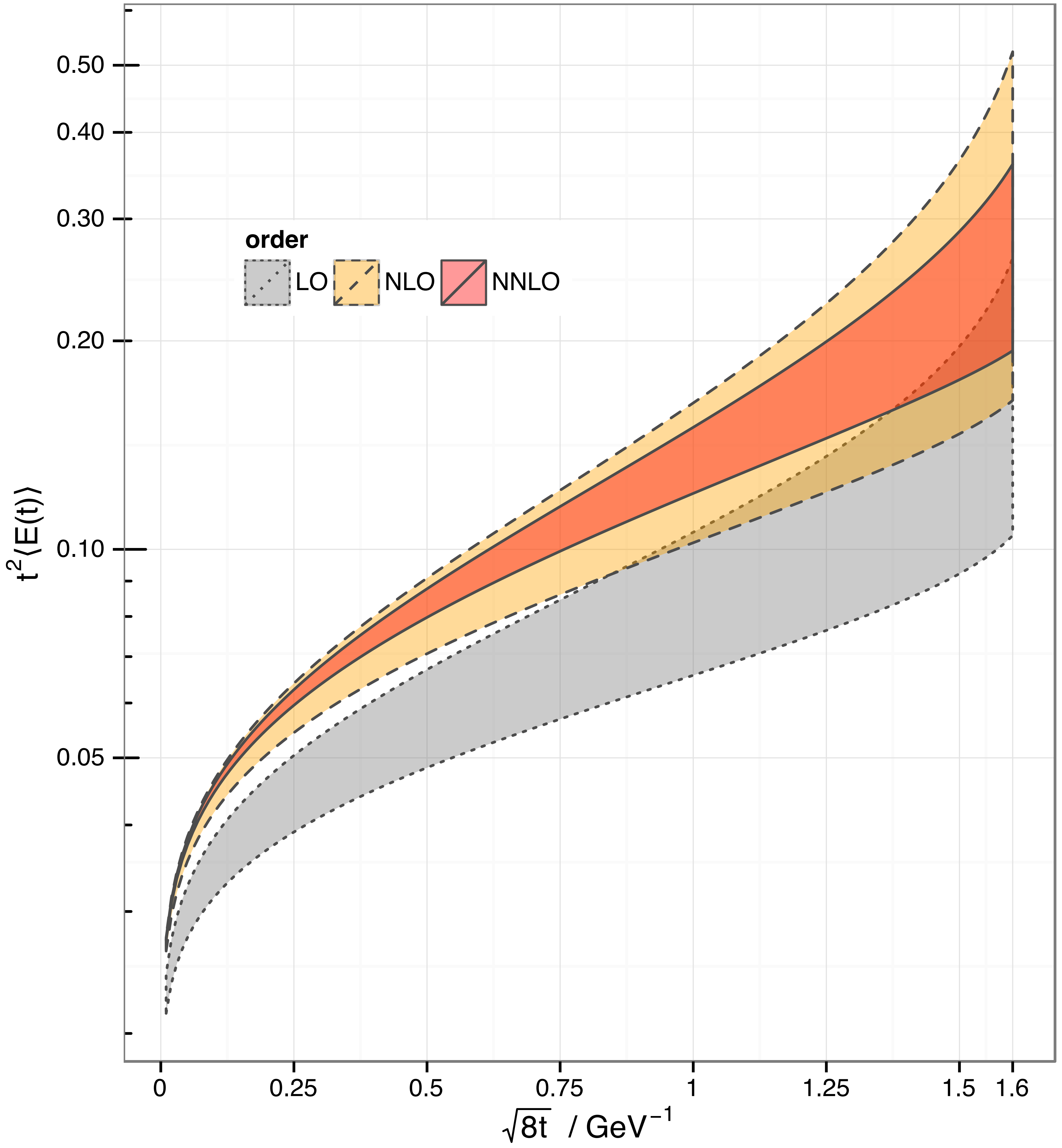
universal

$$t^2 E(t) = \frac{3\alpha_s(\mu)}{4\pi}\left[1 + k_1(t,\mu)\,\alpha_s(\mu) + k_2(t,\mu)\,\alpha_s^2(\mu)\right] \equiv \frac{3}{4\pi}\hat{\alpha}_s(t) = \frac{3}{4}\hat{a}_s(t)$$



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GF specific depends on k_2

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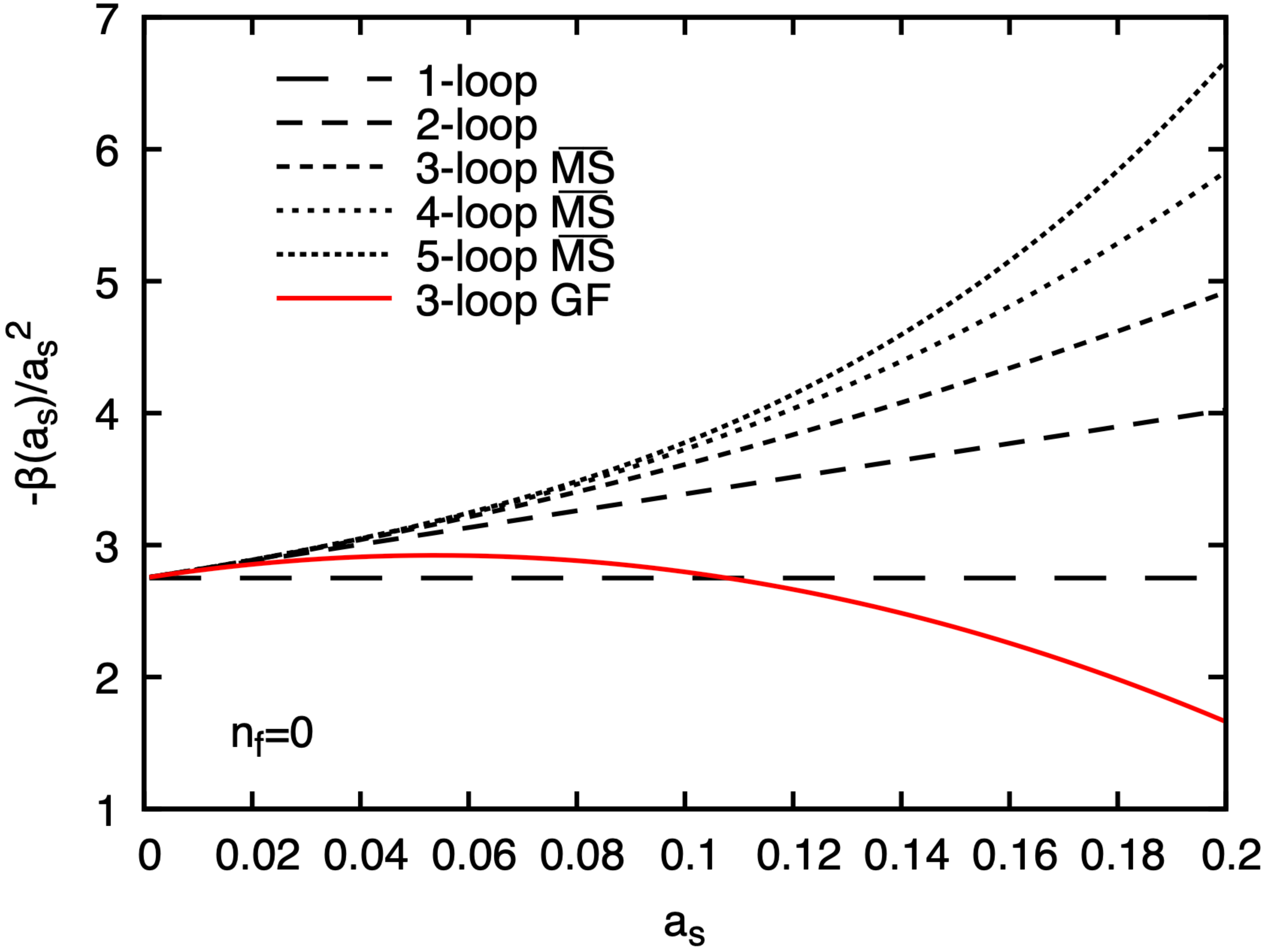

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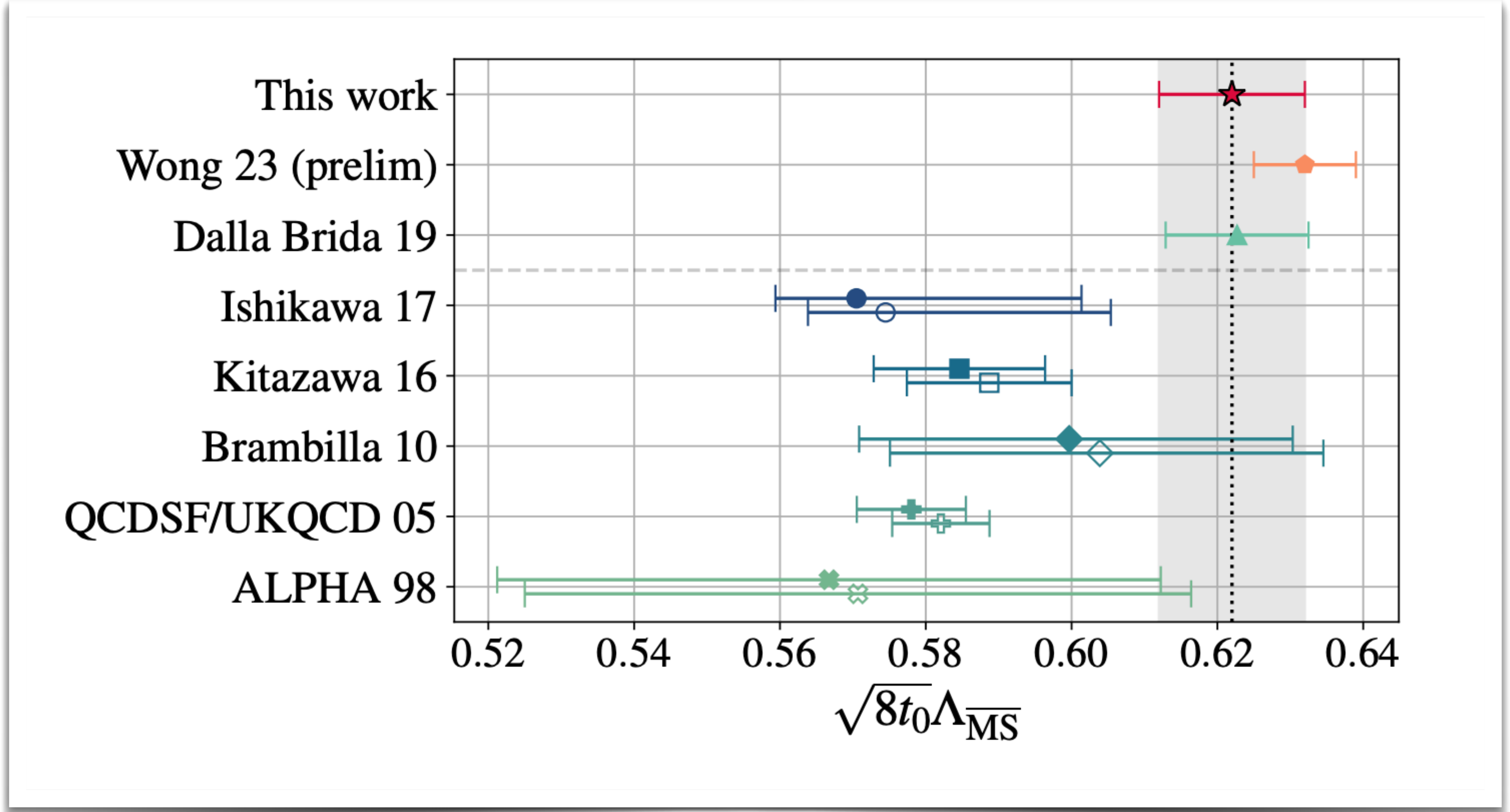
universal

GF specific depends on k_2

Determine Λ_{QCD}



Hasenfratz, Peterson, van Sickle, Witzel (2023)

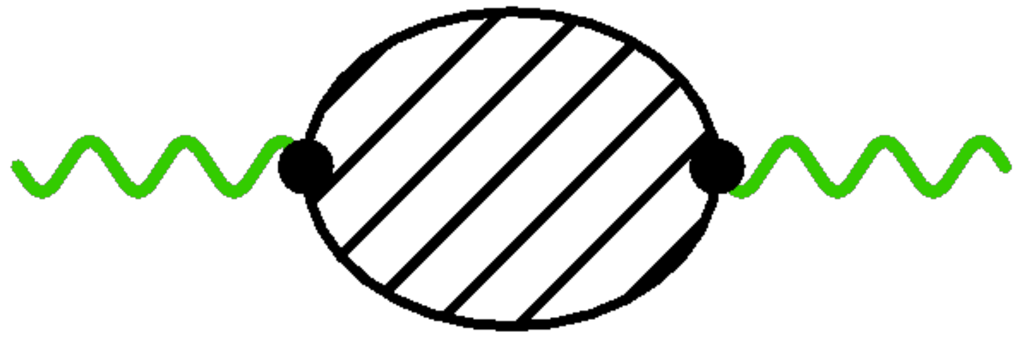
see also Wong, Borsanyi, Fodor, Holland, Kuti (2023)

→ see Anna Hasenfratz's talk (Friday, 9am)

Renormalon subtraction

Beneke, Takaura '23

Adler function: $$D(s) = -s\frac{d\Pi(s)}{ds}$$



Renormalon subtraction

Beneke, Takaura '23

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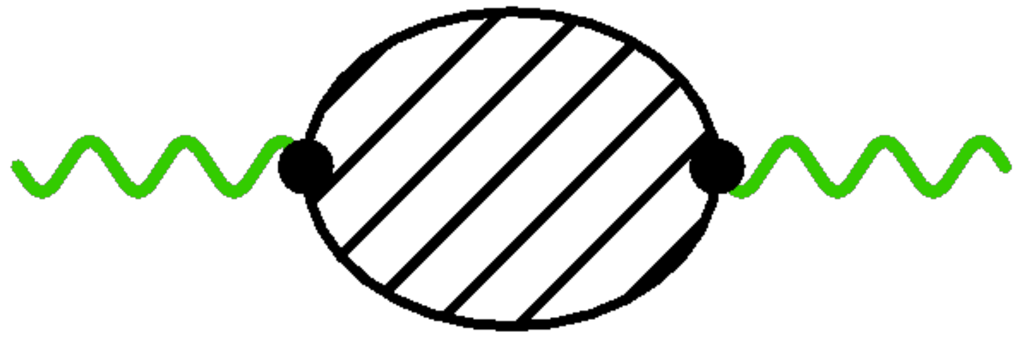

$$D(Q^2) = \frac{N_c}{12\pi^2}\left(C_1(Q^2) + \frac{C_{G^2}(Q^2)}{Q^4}\langle\frac{\alpha_s}{\pi}G^2\rangle + \mathcal{O}(1/Q^6)\right)$$

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Beneke, Takaura '23

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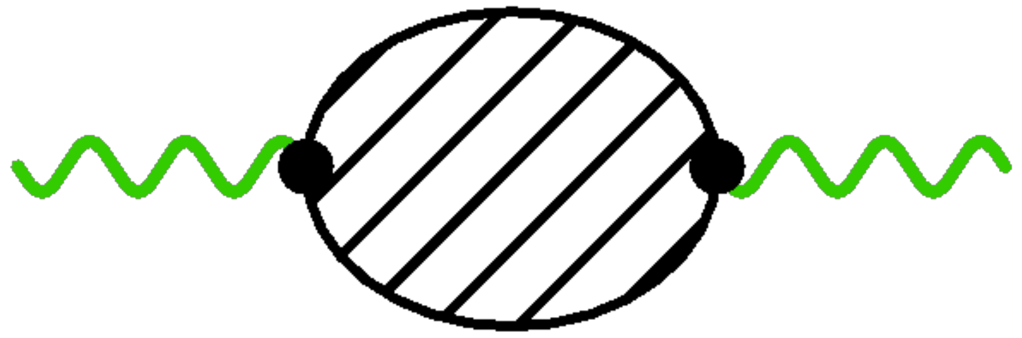

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perturbative $E(t)$

$$= \frac{N_c}{12\pi^2}\left(\underbrace{\left[C_1(Q^2) - \frac{r}{t^2Q^4}\widetilde{C}_1(t)\right]}_{\text{renormalon cancels}} + \underbrace{\frac{r}{Q^4}\frac{E(t)}{\pi^2}}_{\text{nonpert. defined}} + \mathcal{O}(1/Q^6)\right)$$

with

$$r = \frac{C_{G^2}(Q^2)}{\widetilde{C}_{G^2}(t)} = \frac{2\pi^2}{3}\left(\frac{1}{6} - \frac{35}{24}\frac{\alpha_s(\mu)}{\pi} + \mathcal{O}(\alpha_s^2)\right).$$

Renormalon subtraction

Beneke, Takaura '23

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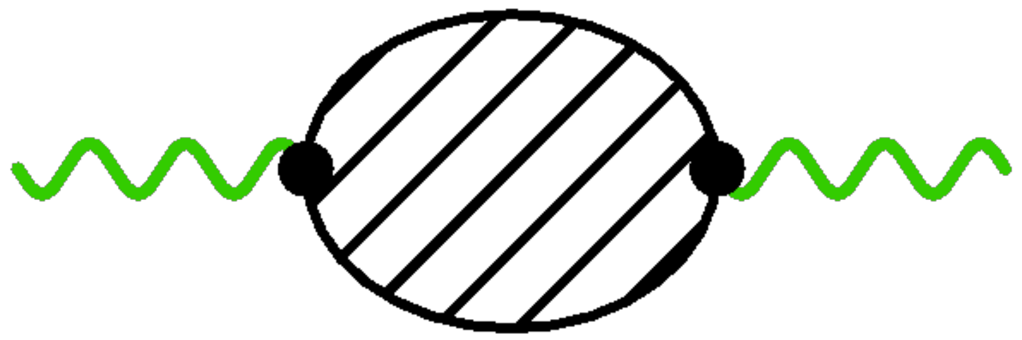

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→ see Martin Beneke's talk
Friday, 3pm

Application to effective field theories

Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

perturbation theory → C_n

lattice → $\langle \mathcal{O}_n \rangle$

match renormalization schemes?

Application to effective field theories

Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

perturbation theory

lattice

Instead: $$R = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

match renormalization schemes?

gradient flow renormalization

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Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

perturbation theory → ← lattice

Instead: $$R = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

match renormalization schemes?

gradient flow renormalization

$\langle \tilde{\mathcal{O}}_n(t) \rangle$ is UV finite $\Rightarrow \lim_{a\to 0} \langle \tilde{\mathcal{O}}_n(t) \rangle$ exists!

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match renormalization schemes?

gradient flow renormalization

$\langle \tilde{\mathcal{O}}_n(t) \rangle$ is UV finite $\Rightarrow \lim_{a\to 0} \langle \tilde{\mathcal{O}}_n(t) \rangle$ exists!

→ need $\tilde{C}(t)$

Small flow-time expansion

Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

Small flow-time expansion

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small flow-time expansion:

Lüscher, Weisz '11
Suzuki '13
Lüscher '13

$$\tilde{\mathcal{O}}_n(t) \overset{t\to 0}{\to} \sum_m \zeta_{nm}(t)\, \mathcal{O}_m$$

$$\tilde{C}_n(t) \overset{t\to 0}{\to} \sum_m C_m\, \zeta^{-1}_{mn}(t)$$

$\Rightarrow$ need $\zeta_{nm}(t)$ for small t $\Rightarrow$ perturbation theory

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Lüscher, Weisz '11
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⇒ need $\zeta_{nm}(t)$ for small t ⇒ perturbation theory

→ see Chris Monahan's talk (tomorrow, 9am)

Determining $\zeta(t)$

Matching: calculate a set of suitable Green's functions and solve for $\zeta_{nm}(t)$

$$\langle \tilde{\mathcal{O}}_n(t) \rangle \overset{t\to 0}{\to} \sum_m \zeta_{nm}(t)\, \langle \mathcal{O}_m \rangle$$

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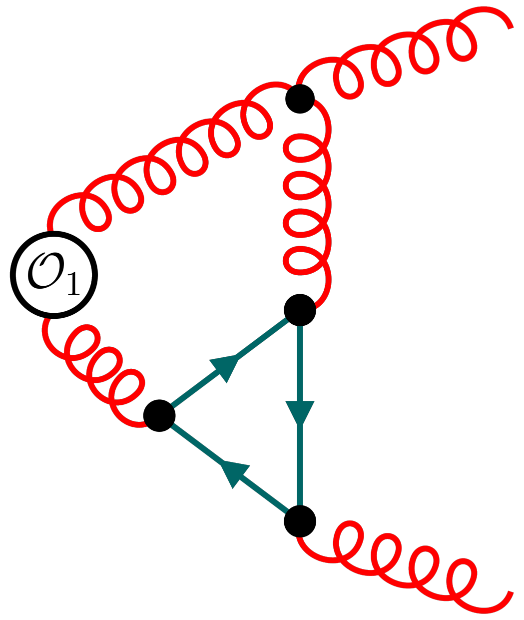

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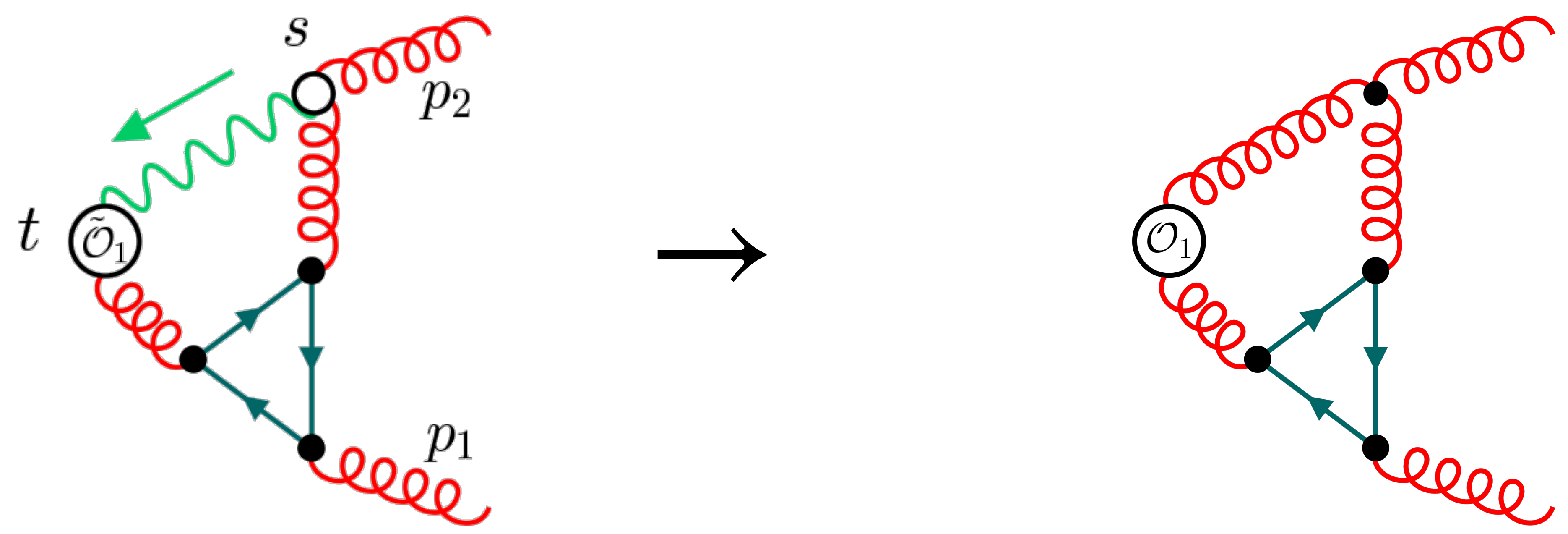

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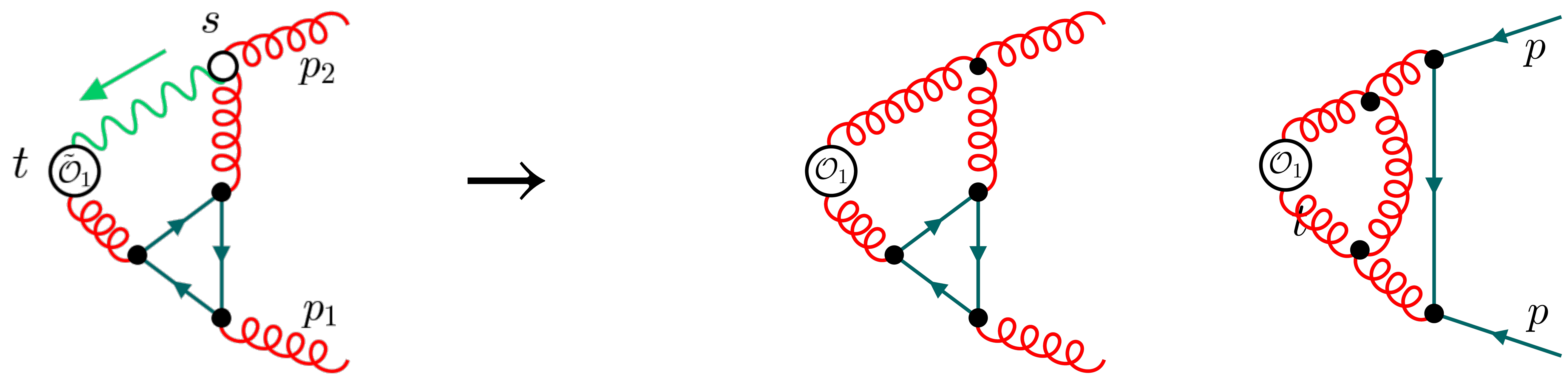

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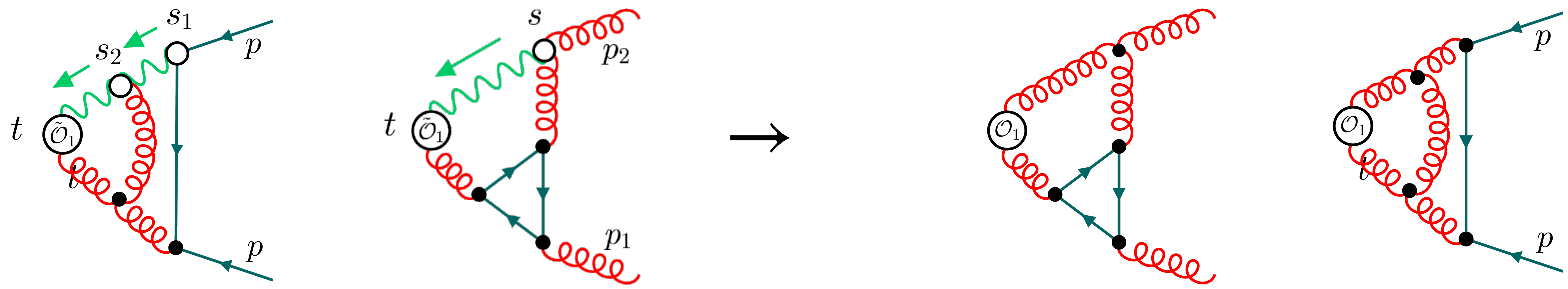

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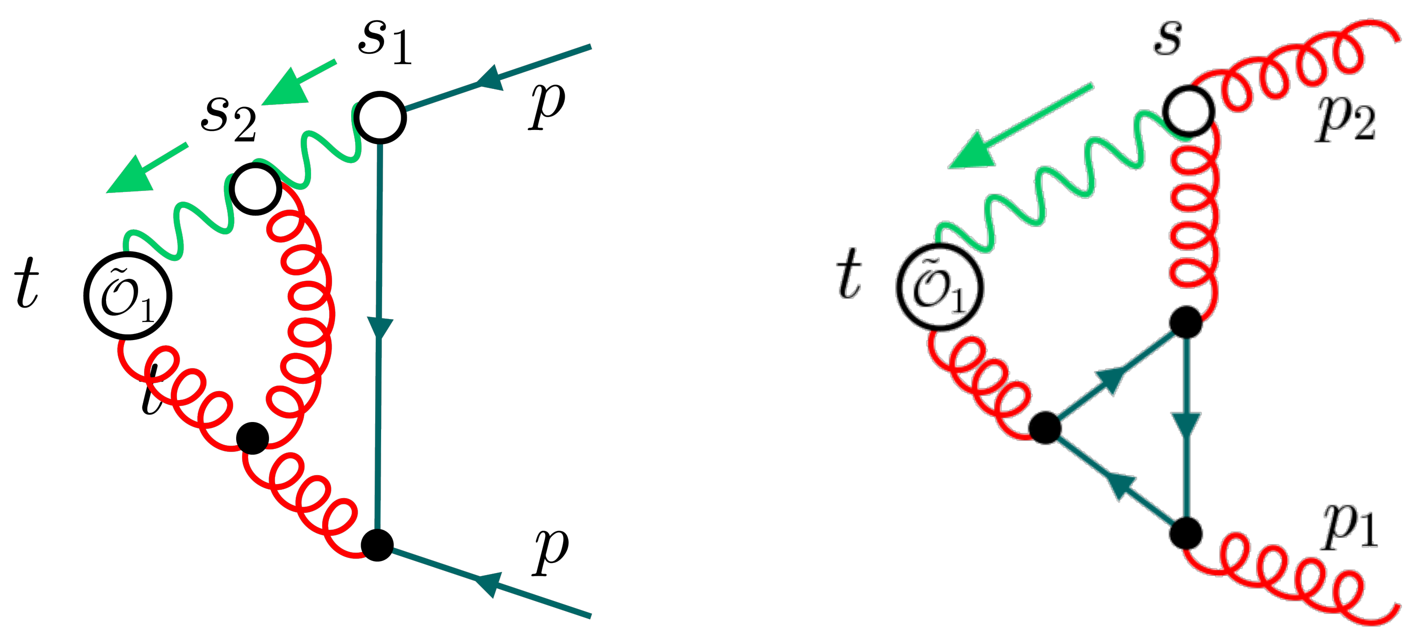
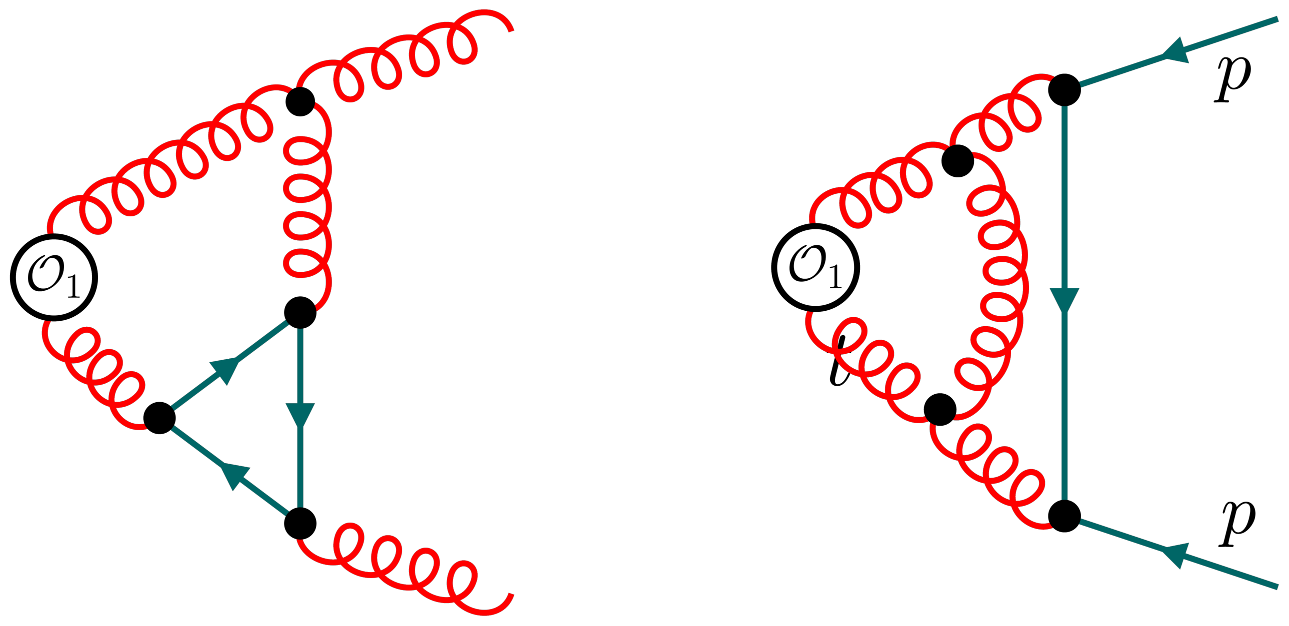


$p = m = 0$

$\to$



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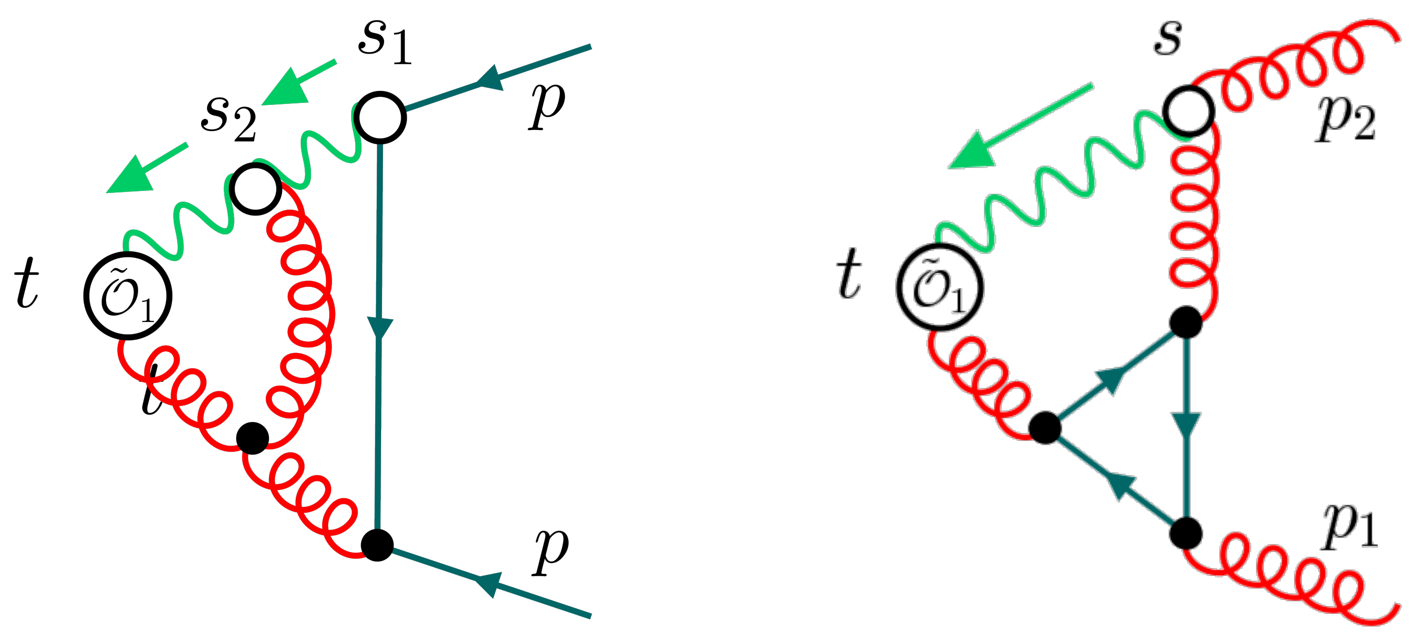
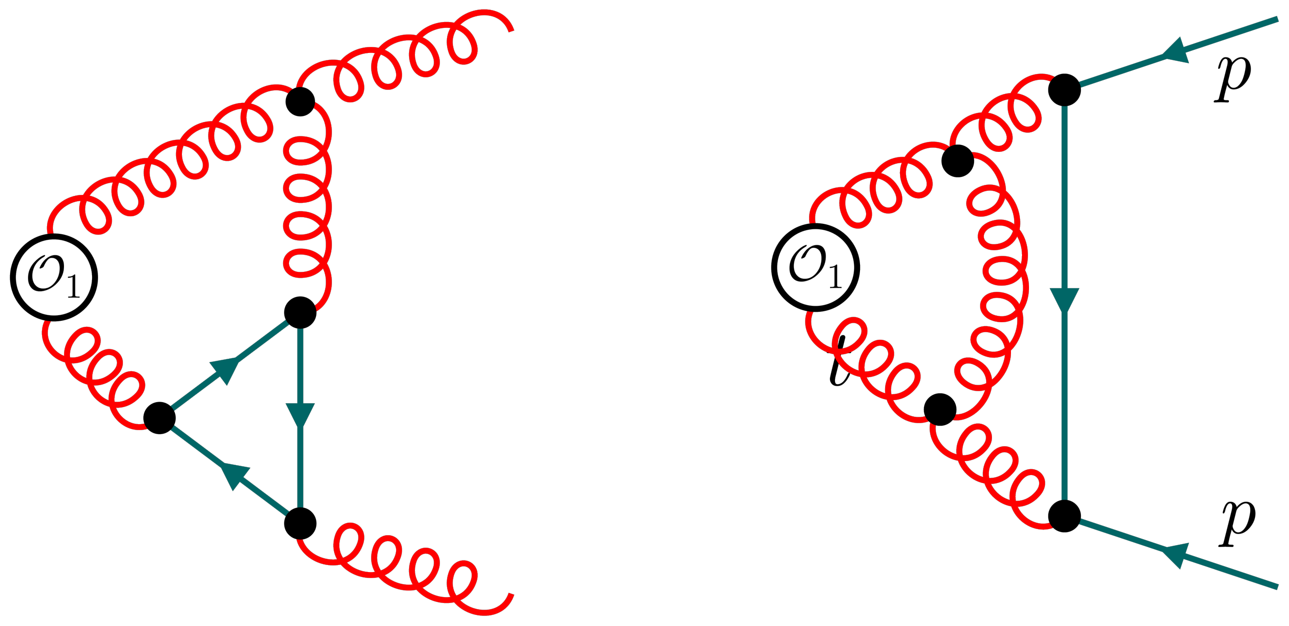


$\Big|_{p=m=0}$ $\rightarrow$



$\Big|_{p=m=0}$

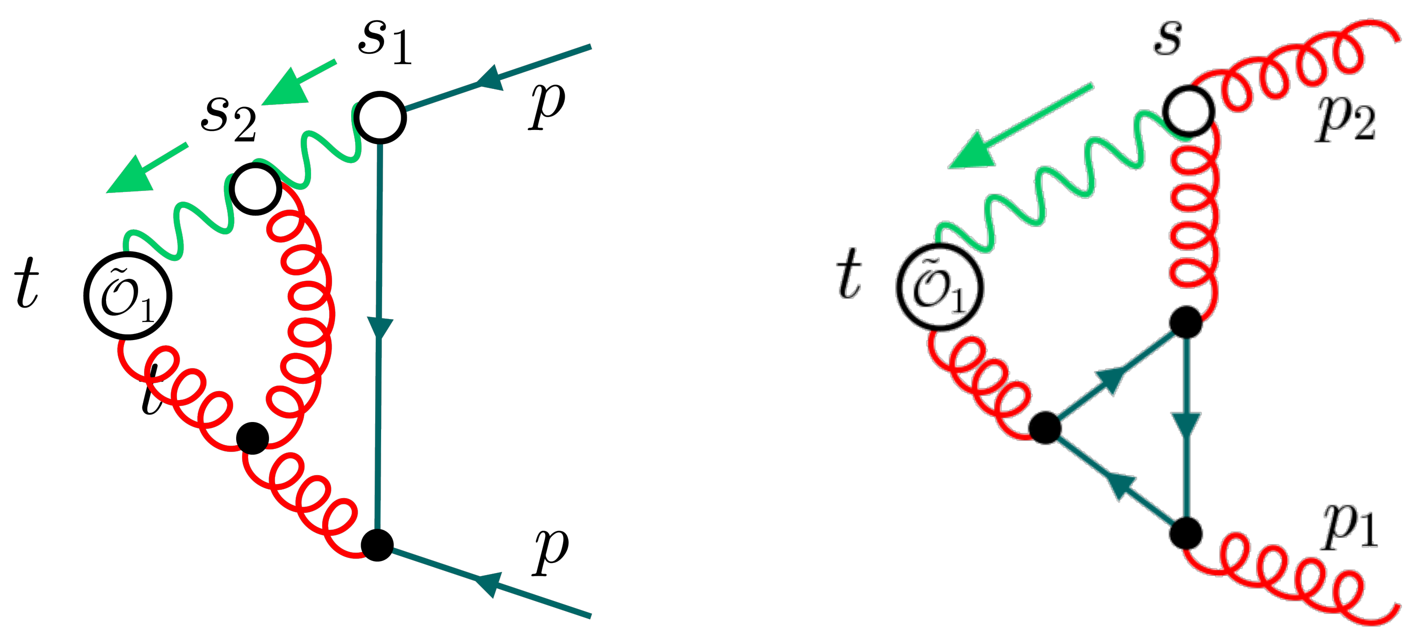
only tree-level diagrams survive on r.h.s.

Gorishnii, Larin, Tkachov '83

Determining $\zeta(t)$

Matching: calculate a set of suitable Green's functions and solve for $\zeta_{nm}(t)$

$$\langle \tilde{\mathcal{O}}_n(t) \rangle \overset{t\to 0}{\to} \sum_m \zeta_{nm}(t) \, \langle \mathcal{O}_m \rangle$$



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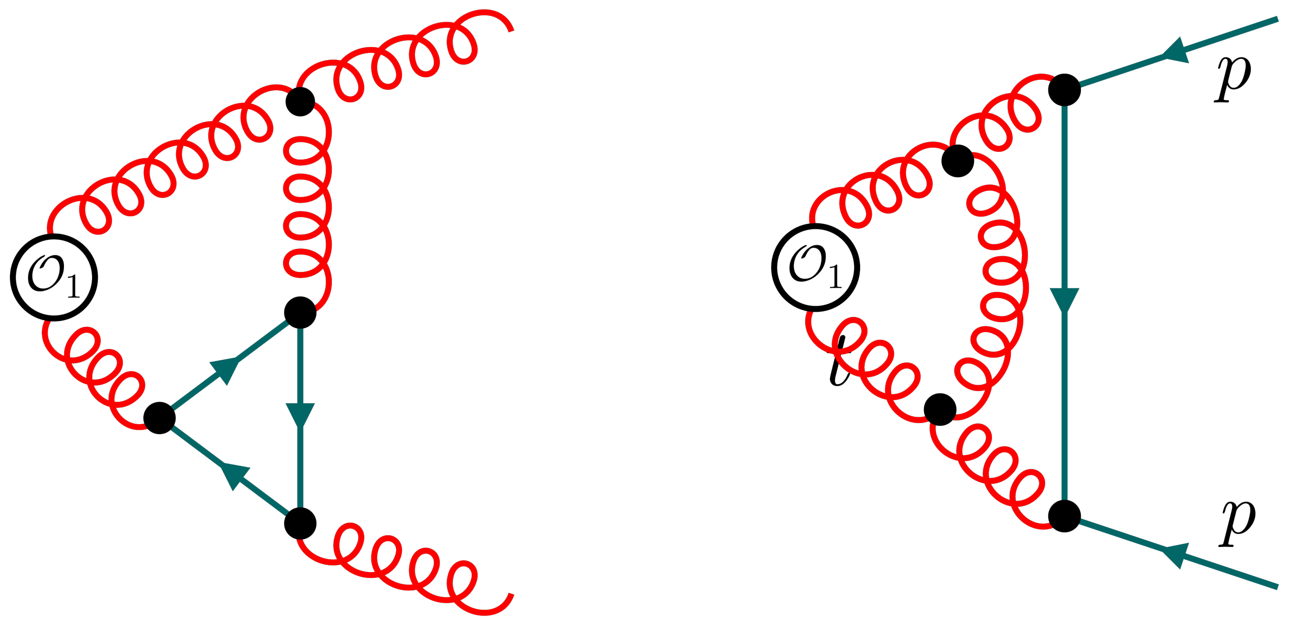

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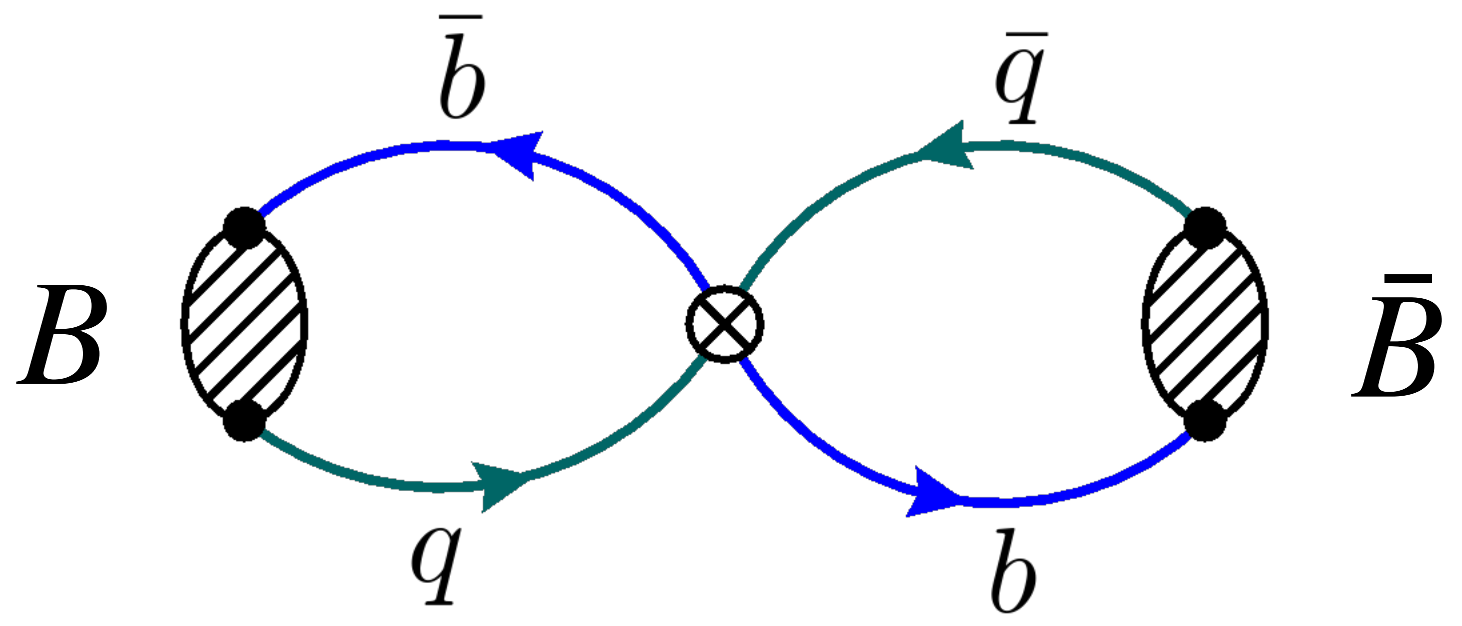
→ see Janosch Borgulat's talk (up next!)

Example: Meson mixing



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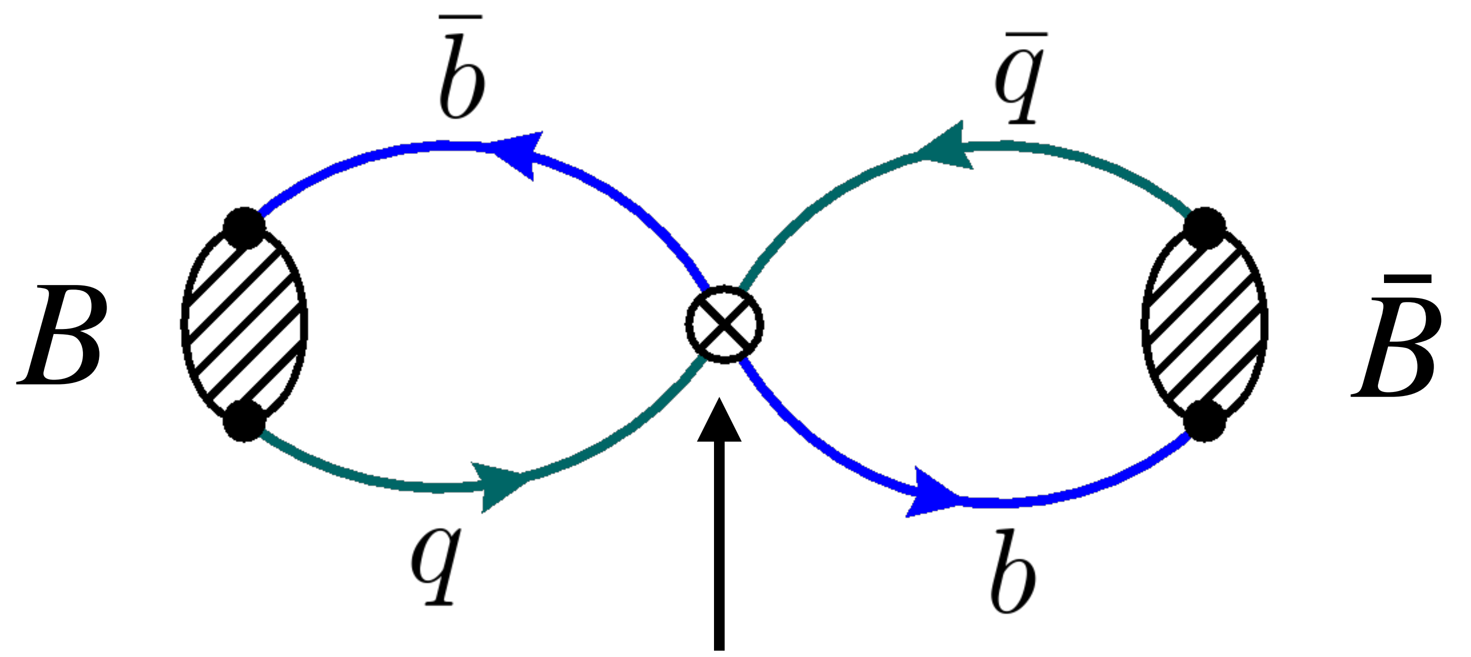

$$H_{\text{eff}} \sim \sum_n C_n \mathcal{O}_n$$

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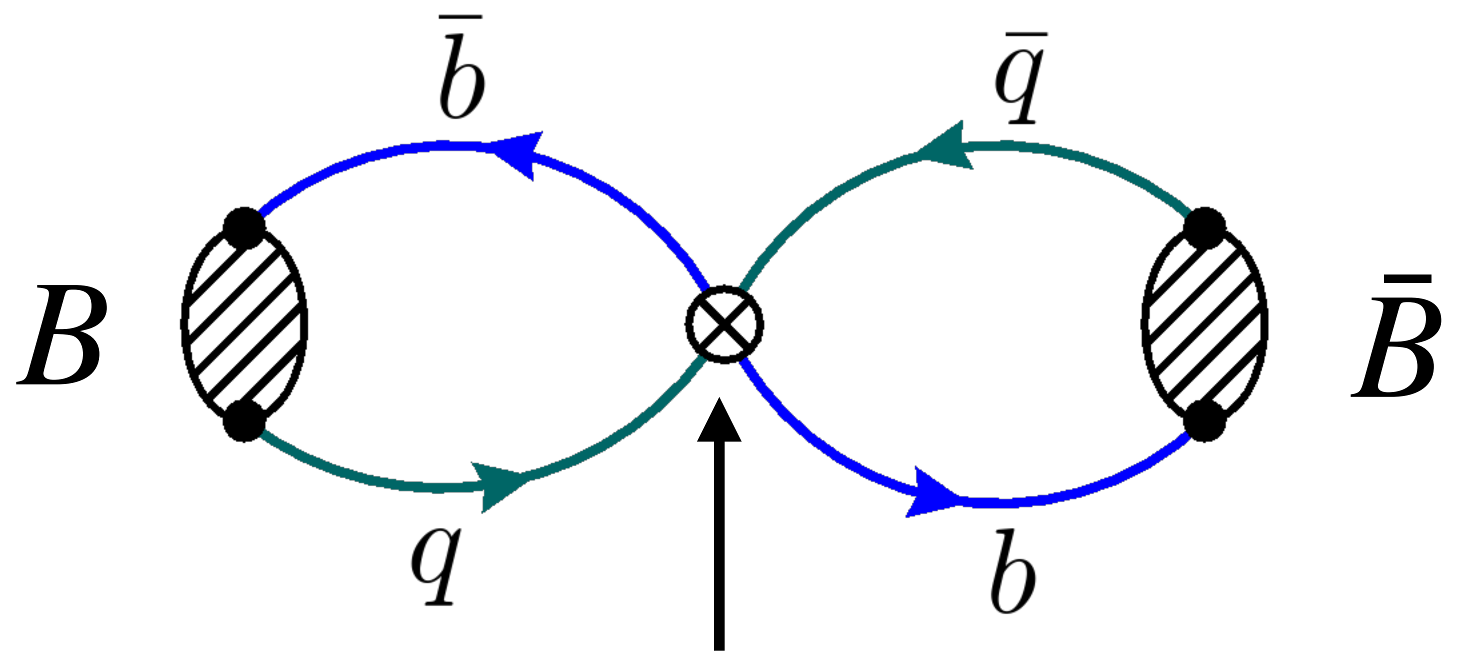

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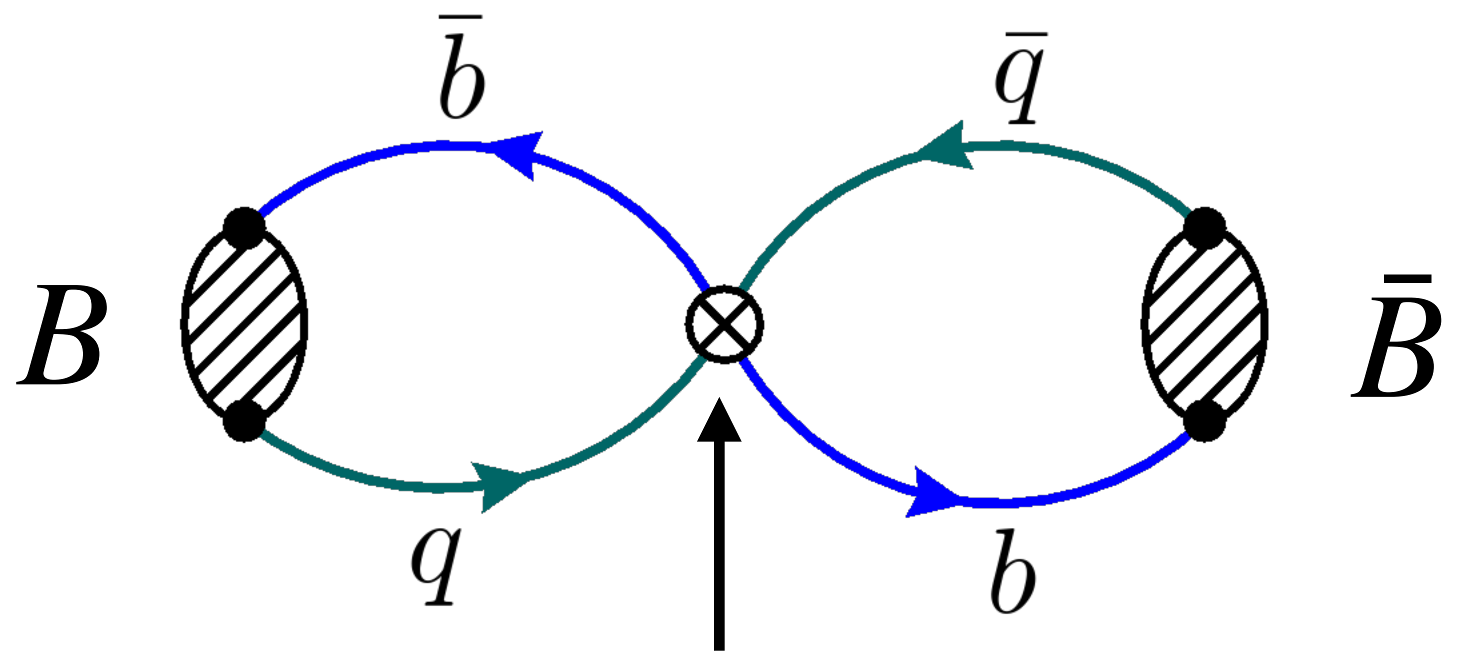

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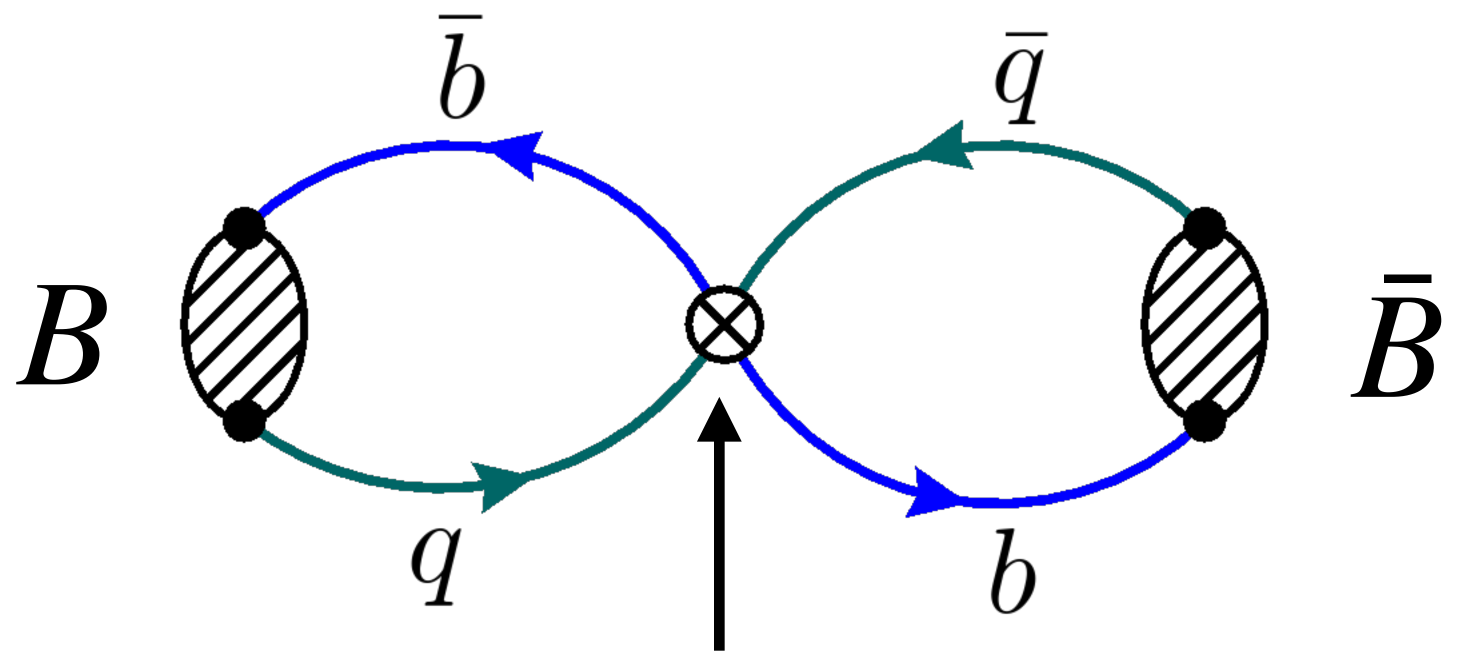

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bag parameter

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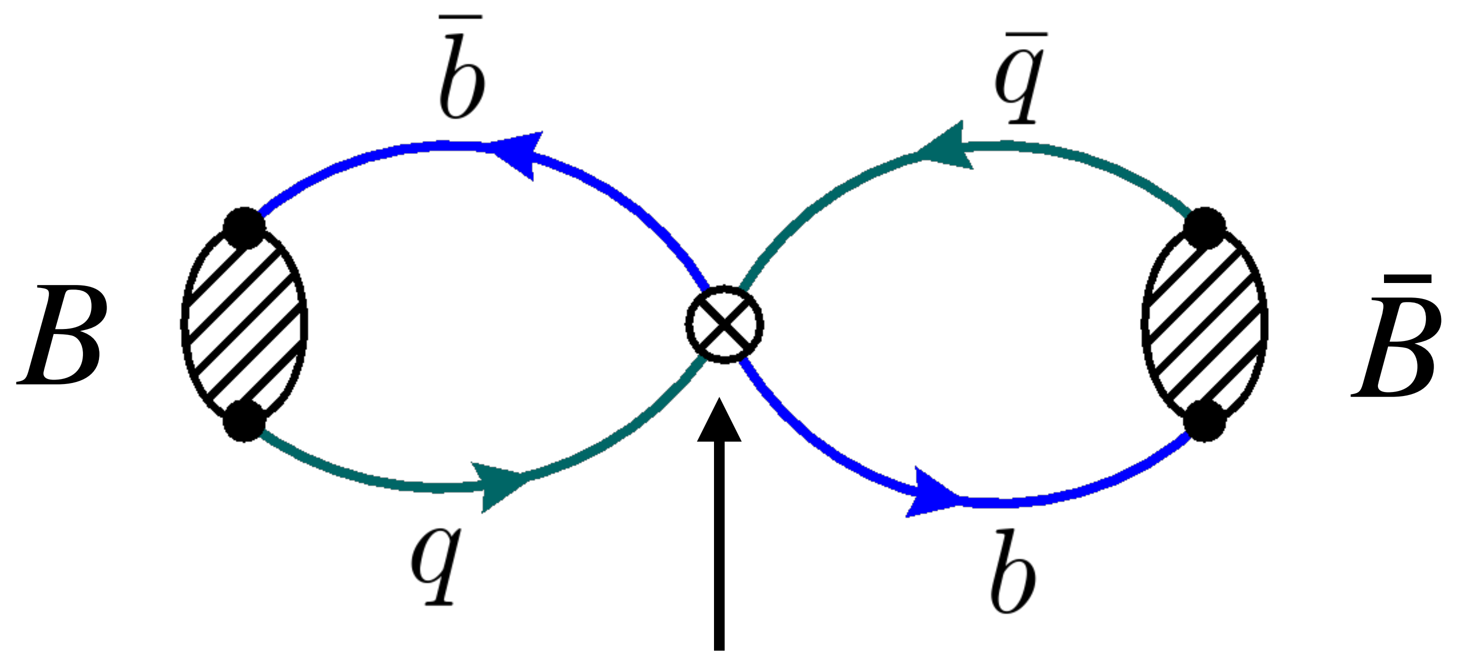

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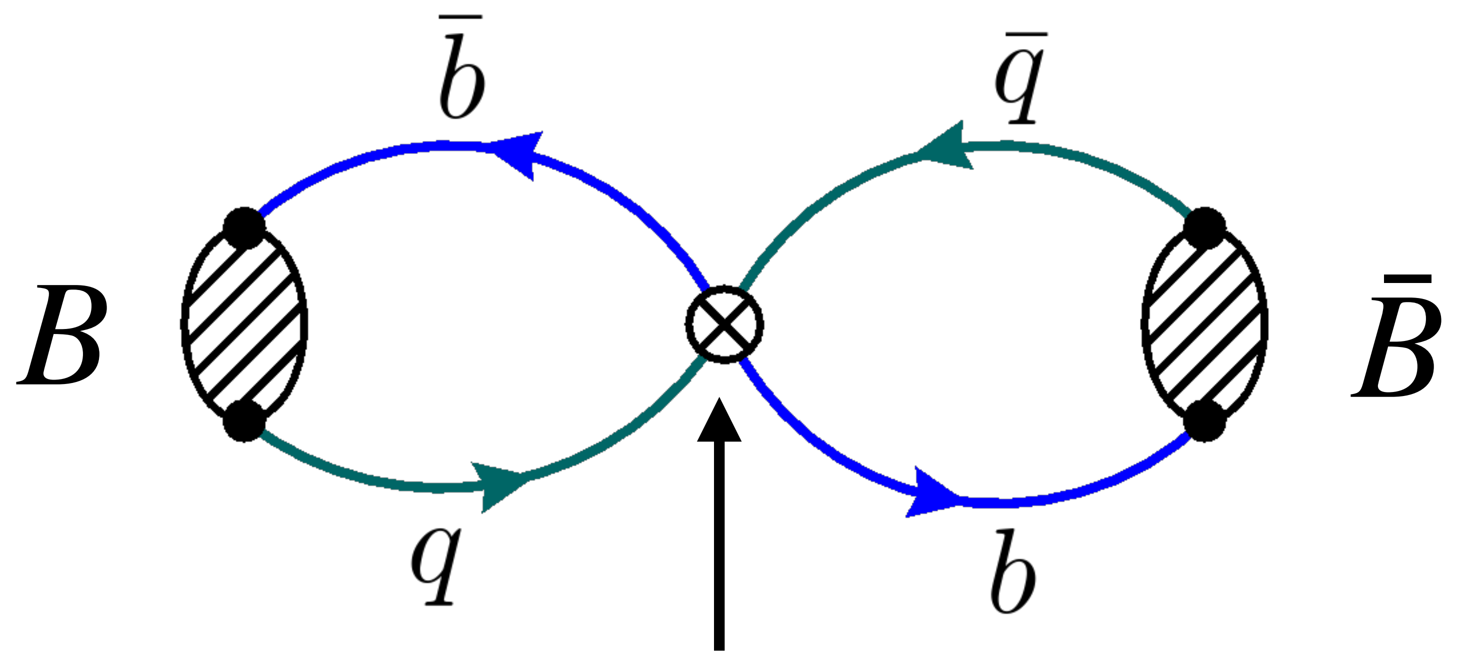

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perturbative

→ see Jonas Kohnen's talk (today, 2:30pm)

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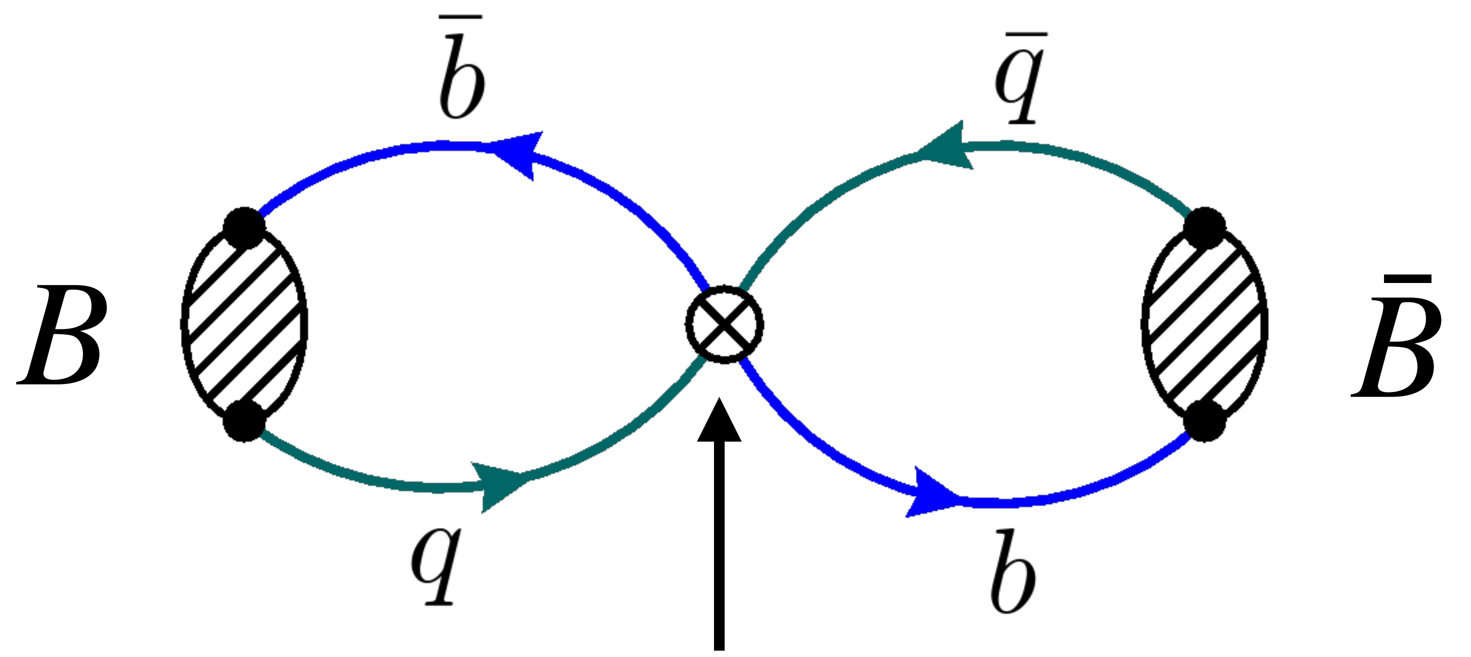

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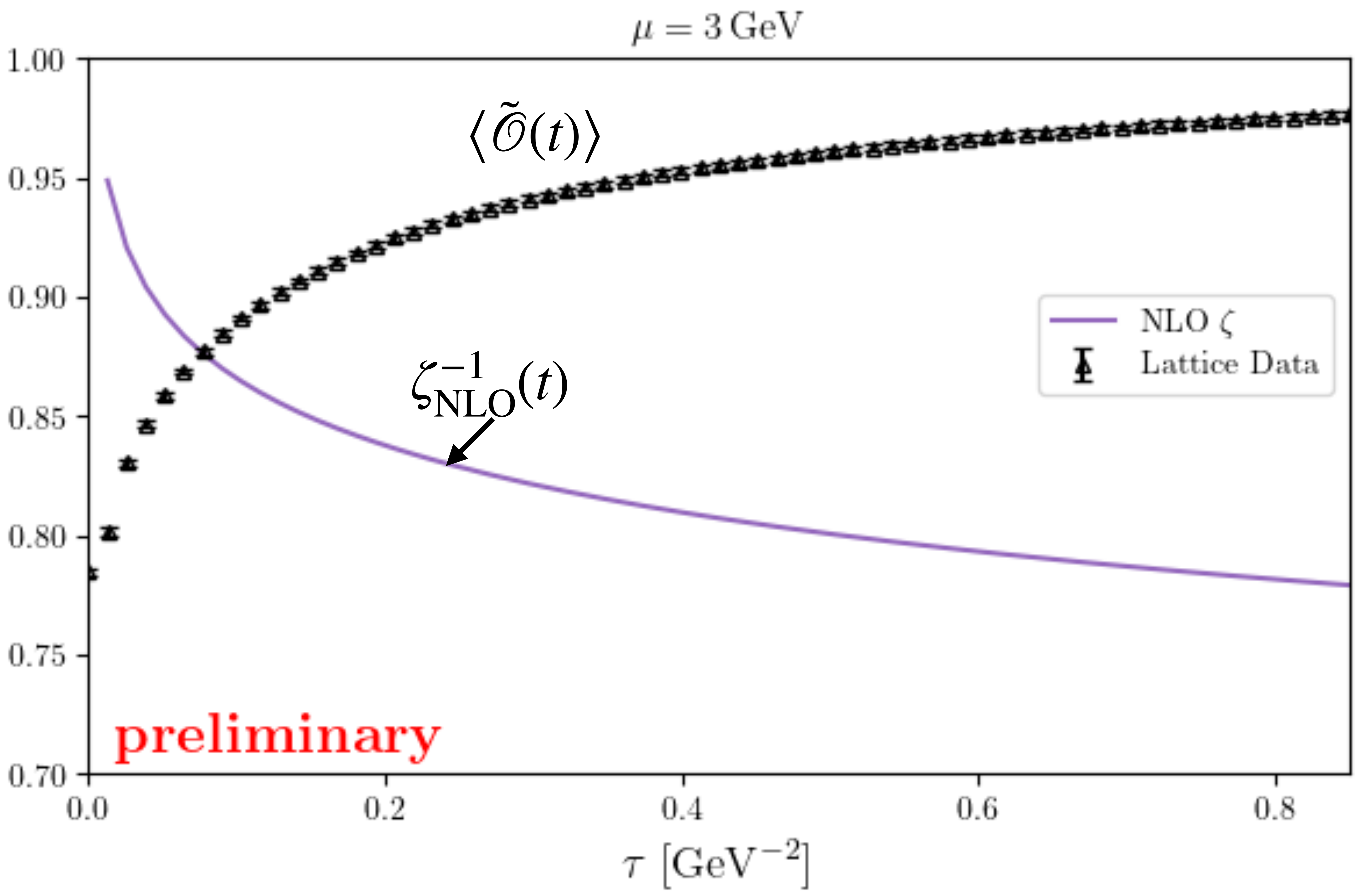
perturbative ($\zeta^{-1}(t)$) — lattice ($\langle B | \tilde{\mathcal{O}}_1(t) | B\rangle$)

→ see Jonas Kohnen's talk (today, 2:30pm)

→ see Matthew Black's talk (today, 2pm)

Bag parameter



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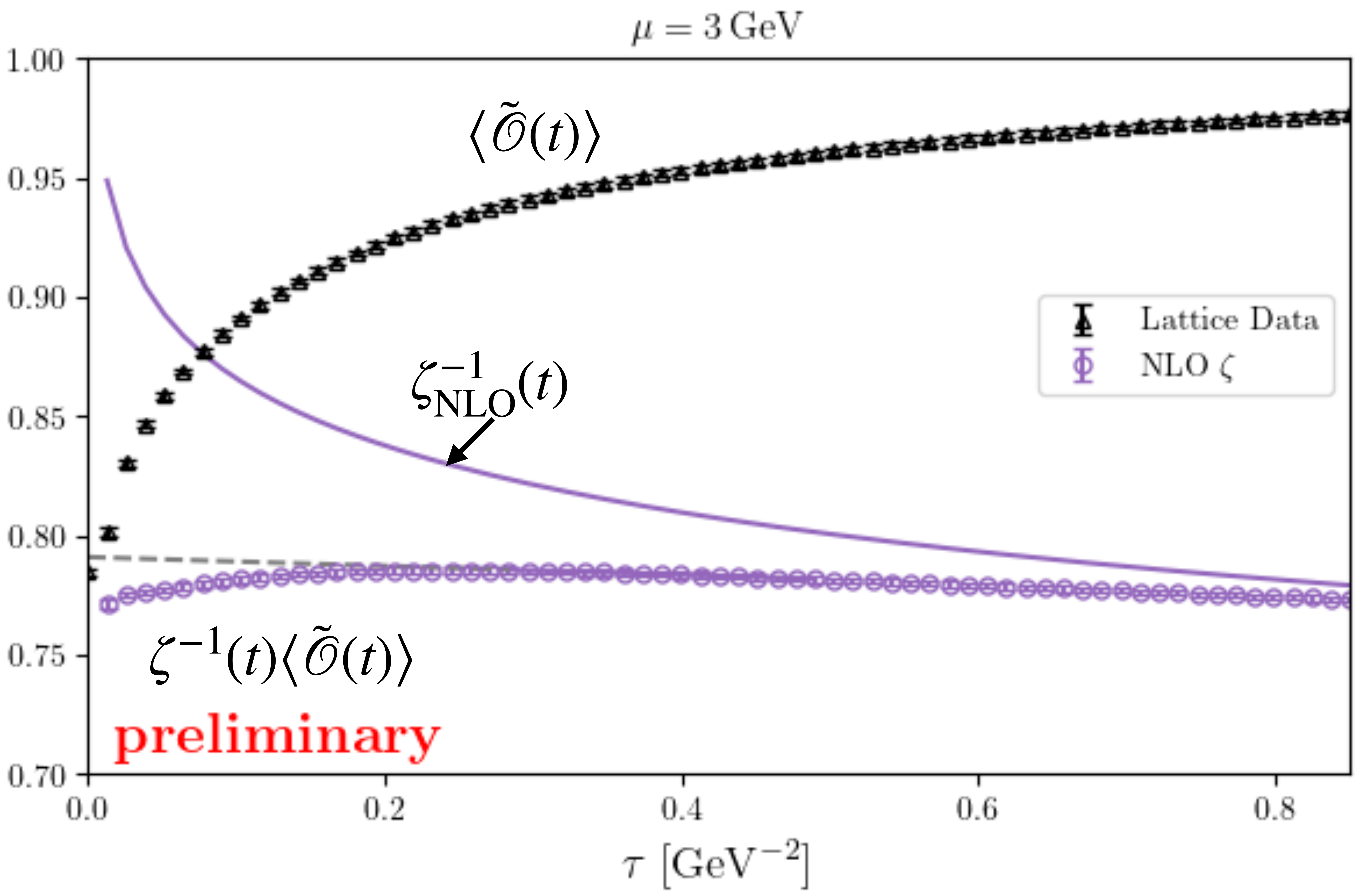

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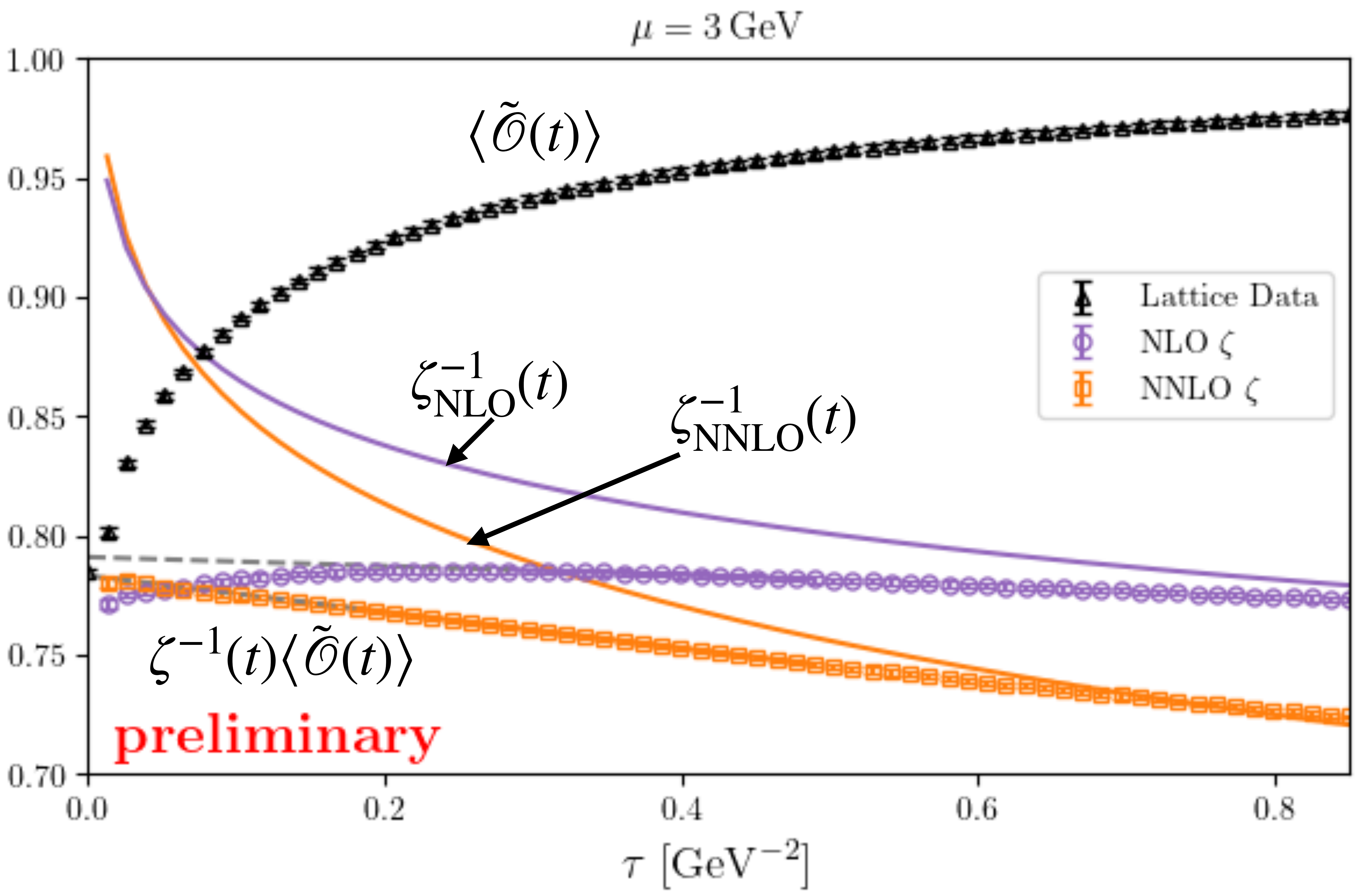

Black, RH, Lange, Rago, Shindler, Witzel (2023)

CRC TRR 257

→ see Matthew Black's talk (today, 2pm)

Other applications

electric dipole operators

NLO: Mereghetti, Monahan, Rizik, Shindler, Stoffer (2022)
Crosas, Monahan, Rizik, Shindler, Stoffer (2023)
NNLO: Borgulat, RH, Rizik, Shindler (2022)

→ see Andrea Shindler's talk (tomorrow, 4pm)

hadronic vacuum polarization

NNLO: RH, Lange, Neumann (2020)

quark bilinears

NLO: Hieda, Suzuki (2016)
NNLO: Borgulat, RH, Kohnen, Lange (2023)

…

Application to effective field theories

Observable: $$R = \sum_n C_n \langle \mathcal{O}_n \rangle$$

perturbation theory

lattice

Instead: $$R = \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$$

match renormalization schemes?

gradient flow renormalization

$\langle \tilde{\mathcal{O}}_n(t) \rangle$ is UV finite $\Rightarrow \lim_{a\to 0} \langle \tilde{\mathcal{O}}_n(t) \rangle$ exists!

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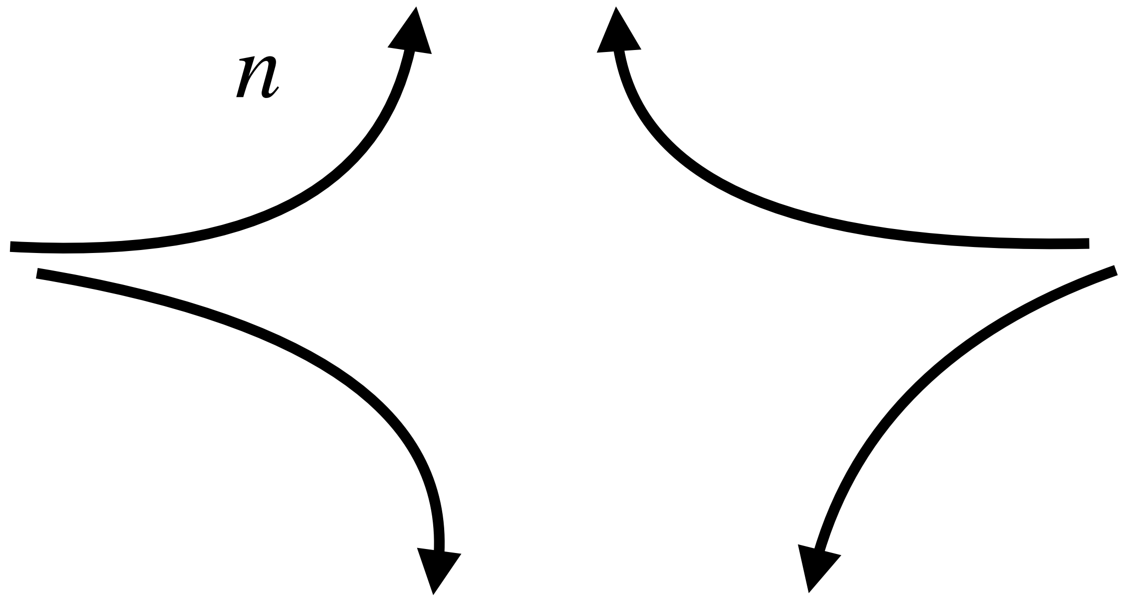

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application: energy-momentum tensor Suzuki '13

parton density functions Shindler '24

Energy momentum tensor

Suzuki '13
Suzuki, Makino '14

Entropy density:

$$\varepsilon + p = -\frac{4}{3}\left\langle T_{00}(x) - \frac{1}{4}T_{\mu\mu}(x)\right\rangle$$

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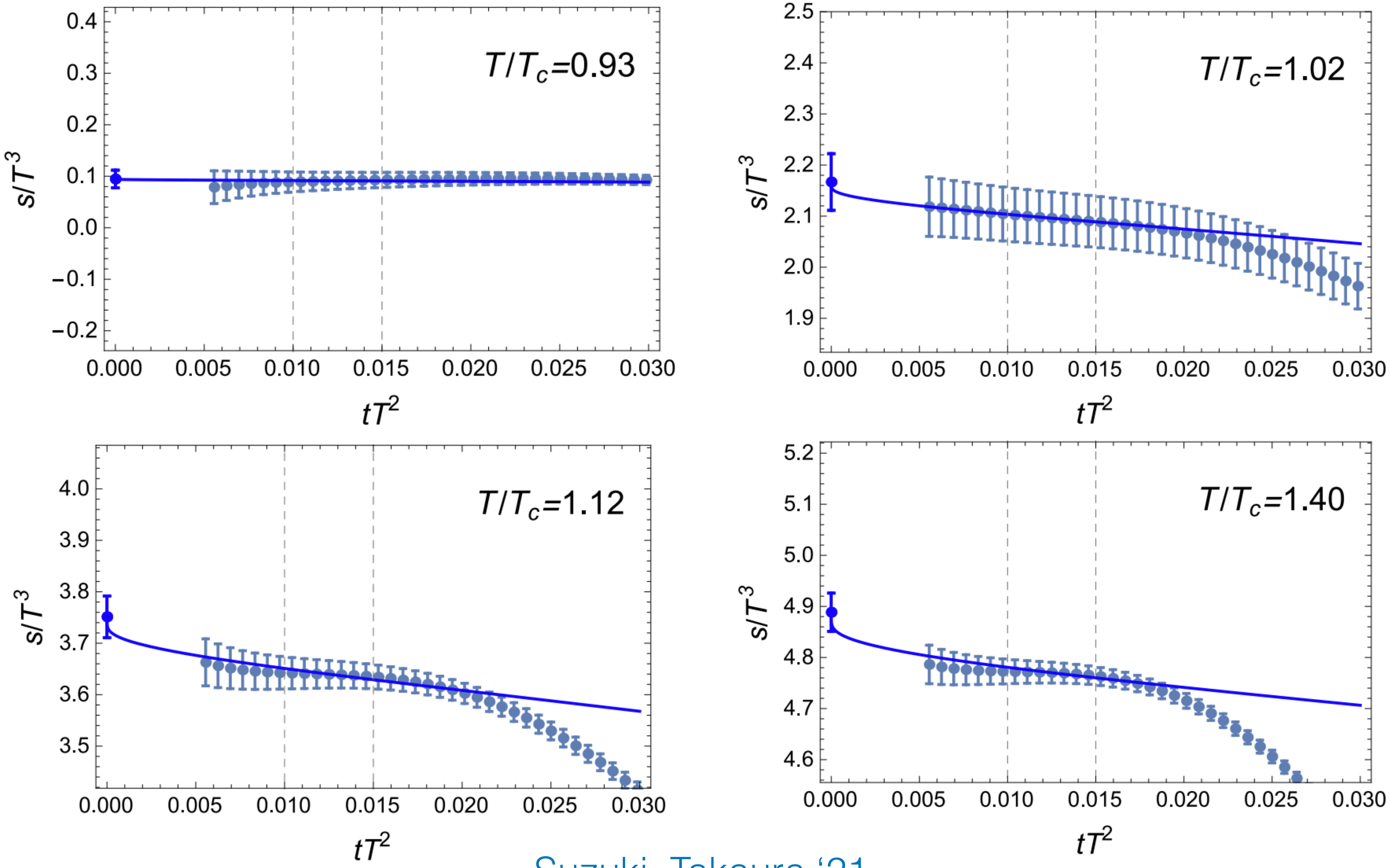
NLO



Suzuki, Takaura '21

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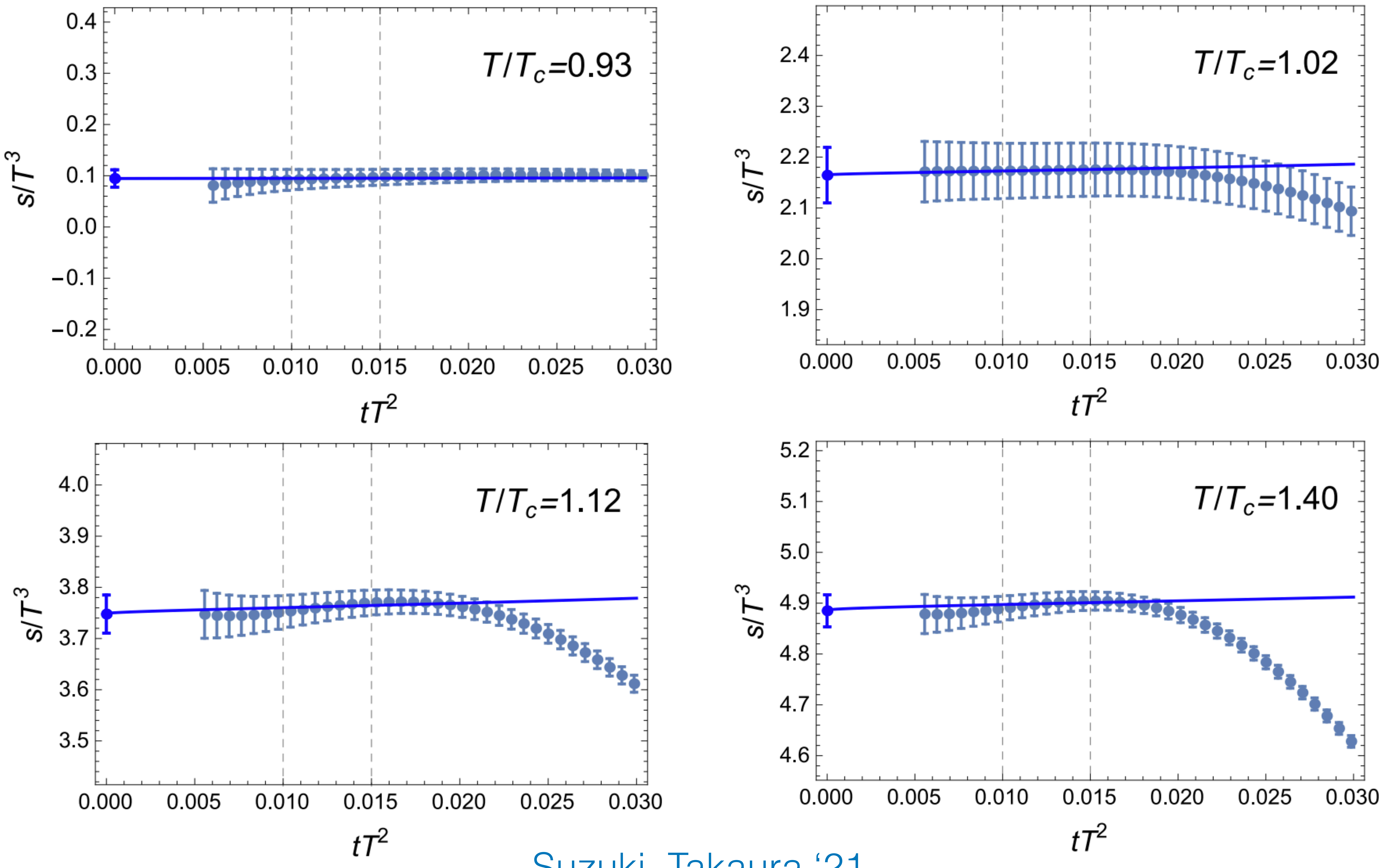
NNLO

RH, Kluth, Lange '18



Suzuki, Takaura '21

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perturbation theory → lattice

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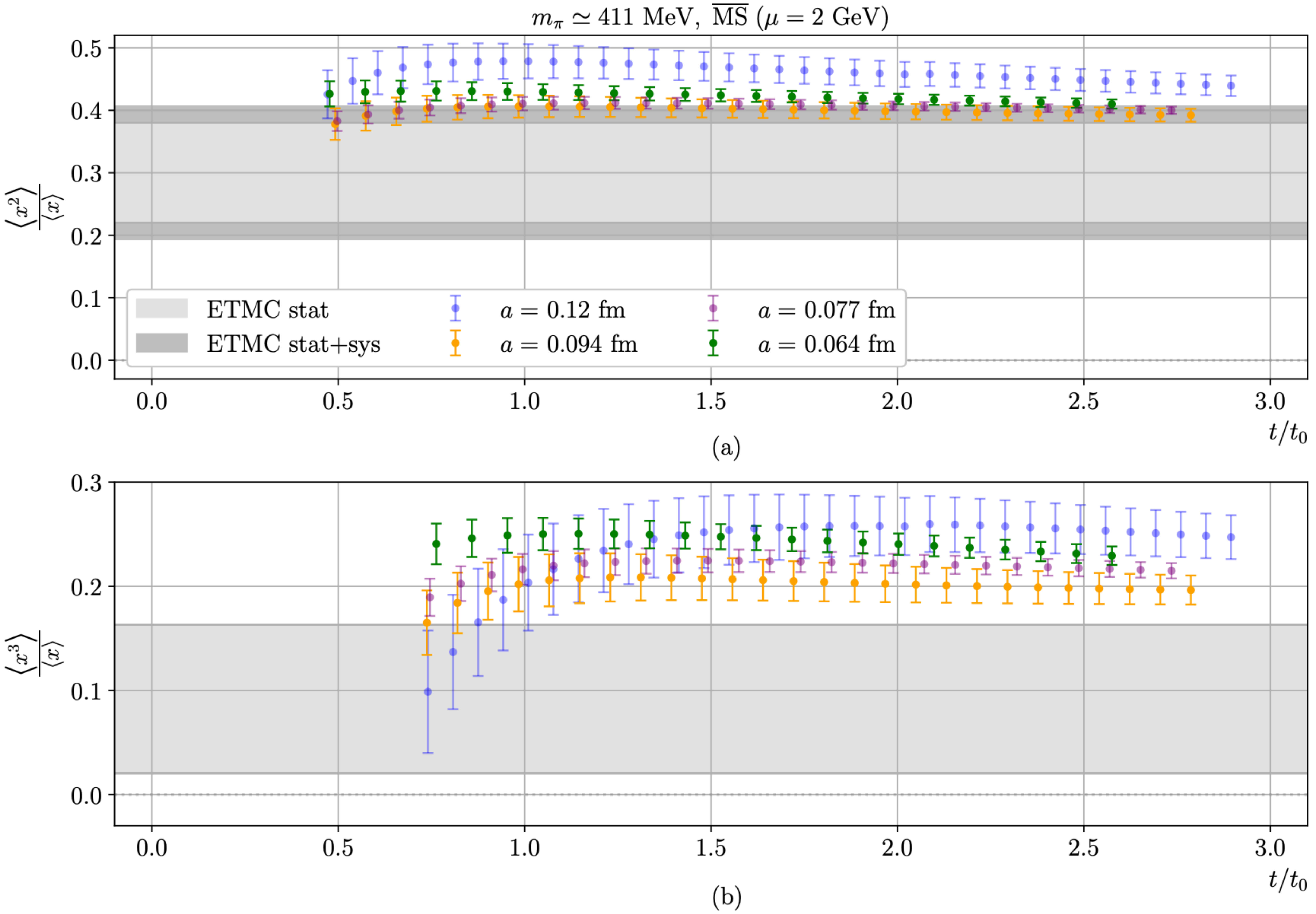
Parton densities

Shindler '24



Francis et al. '24

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$\overline{\text{MS}}$ renormalization of composite operators

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needs renormalization:

$$= \sum_n (C\mathbf{Z})_n (\mathbf{Z}^{-1}\mathcal{O})_n = \sum_n C_n^{\mathrm{R}} \mathcal{O}_n^{\mathrm{R}}$$

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(SFTX)

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UV finite

$\Rightarrow$ calculation of $\zeta(t)$ also determines Z in $\overline{\text{MS}}$ scheme!

$\overline{\text{MS}}$ renormalization of composite operators

application: check for four-quark operators

RH, Lange (2022)
RH, Kohnen, Lange (in prep)

Buras, Gorbahn, Haisch, Nierste (2006)
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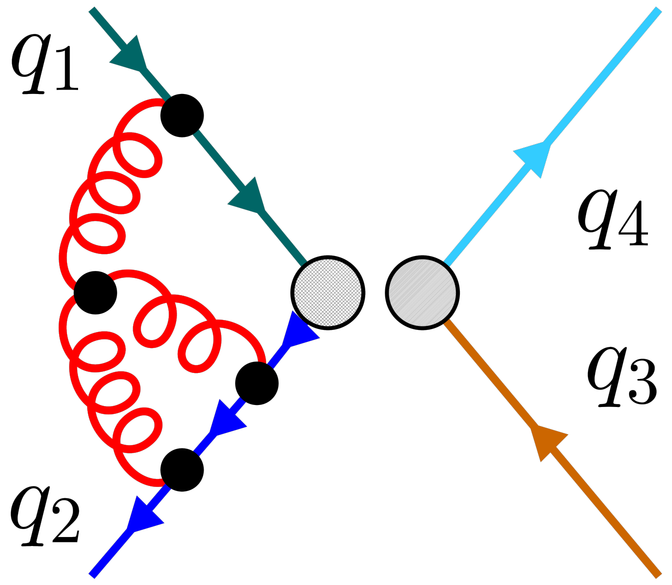

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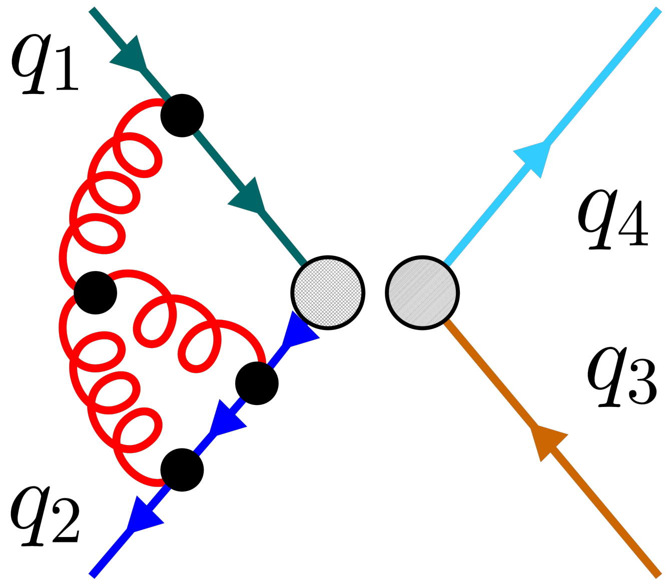


goal: SMEFT dim-6 renormalization at 2-loop level

→ need flowed Standard Model

i.e. flow equations, Feynman rules, …

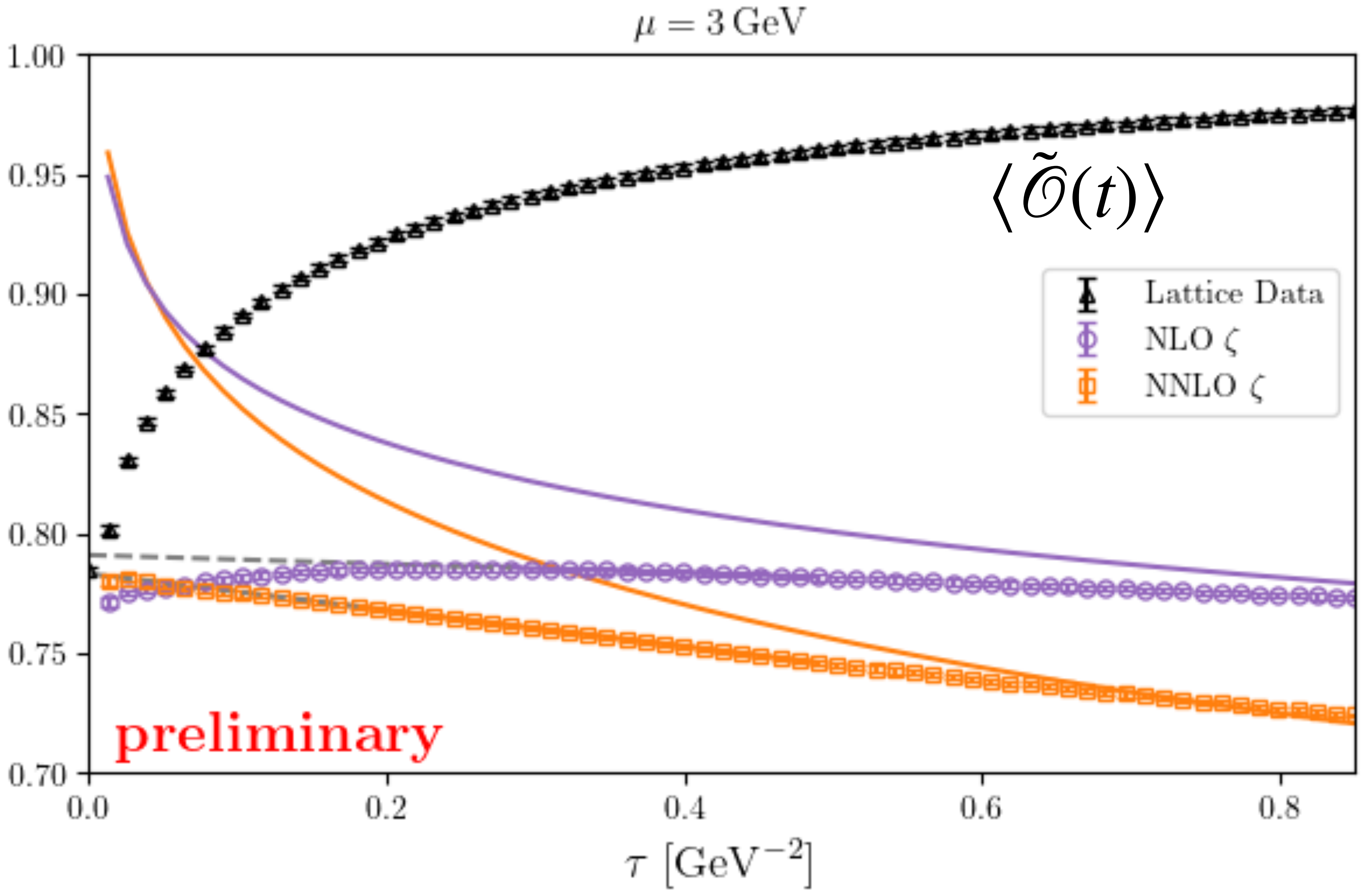
fermion field renormalizations Z_χ

→ see Janosch Borgulat's talk (up next!)

The gradient flow scheme

$$\Delta\Gamma \sim \sum_n C_n^{\rm R}(\mu)\,\langle \mathcal{O}_n^{\rm R}\rangle(\mu)$$

$$= \sum_n C_n^{\rm R}(\mu)\left[\zeta^{-1}(t,\mu)\langle\tilde{\mathcal{O}}_n\rangle(t)\right]$$



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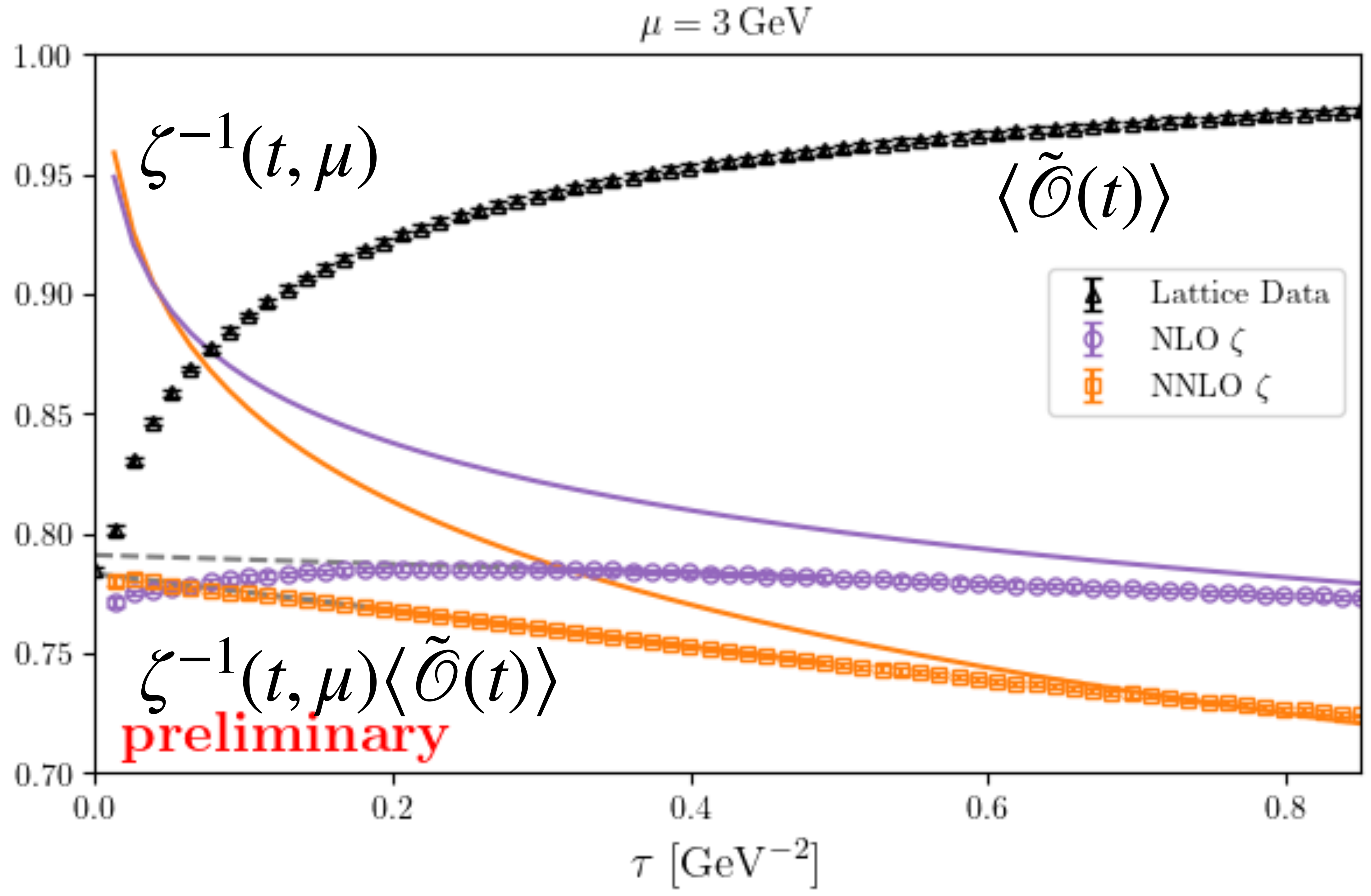

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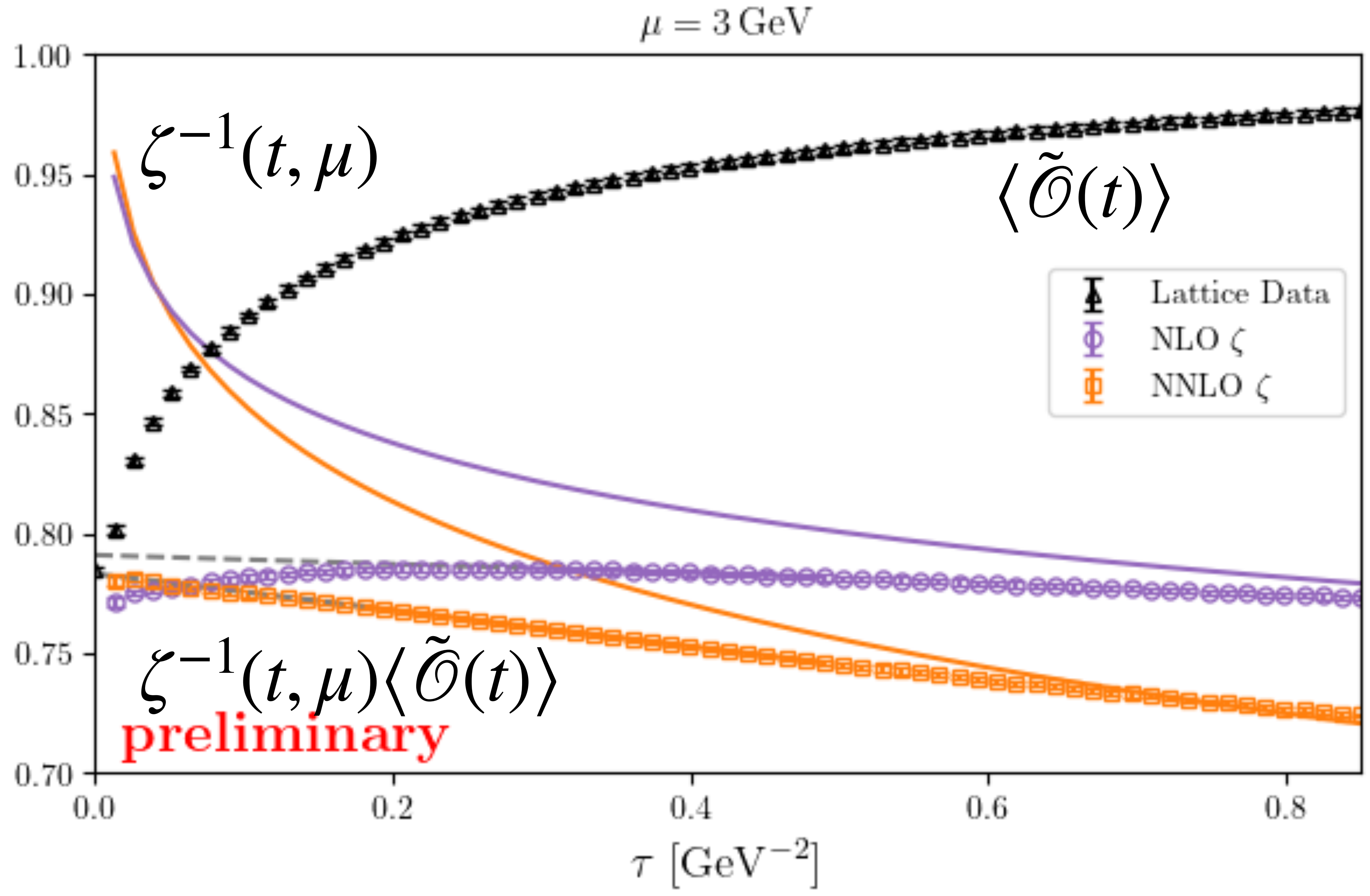

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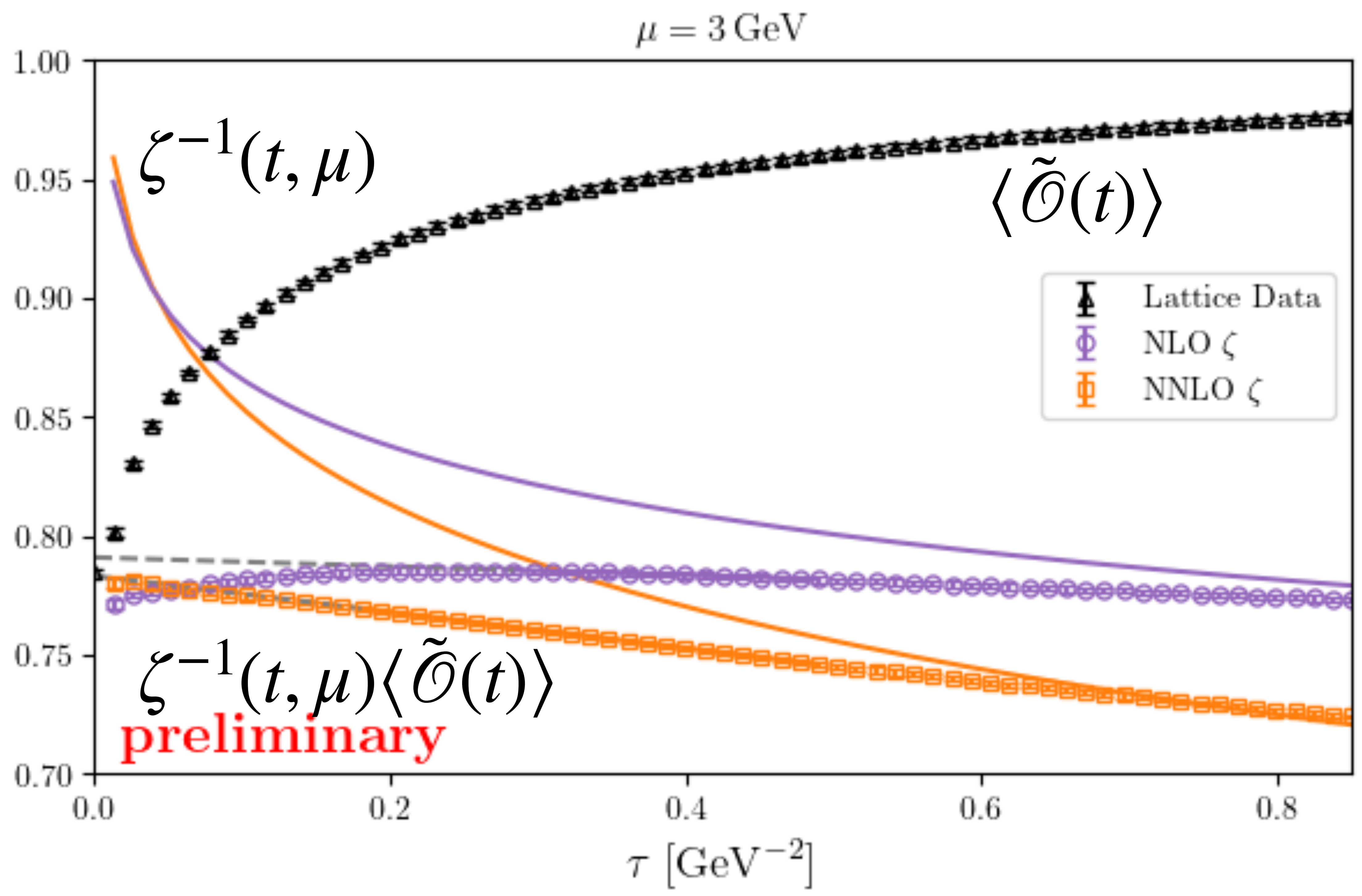

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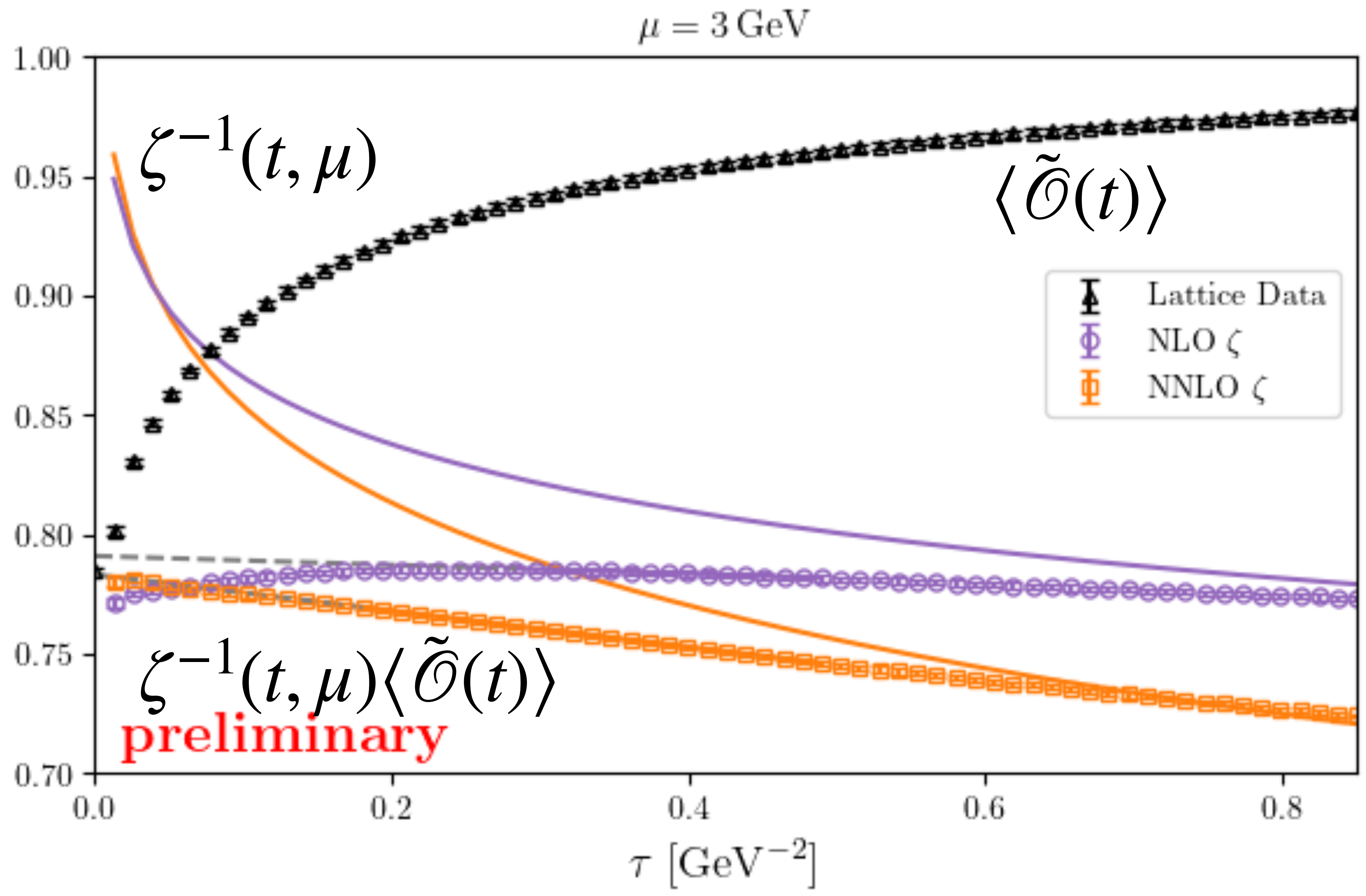

[...] Hence, no conversion to the MS scheme is needed any more, the advantage being that the renormalization scheme employed is well-defined beyond perturbation theory.

Ammer, Dürr '24

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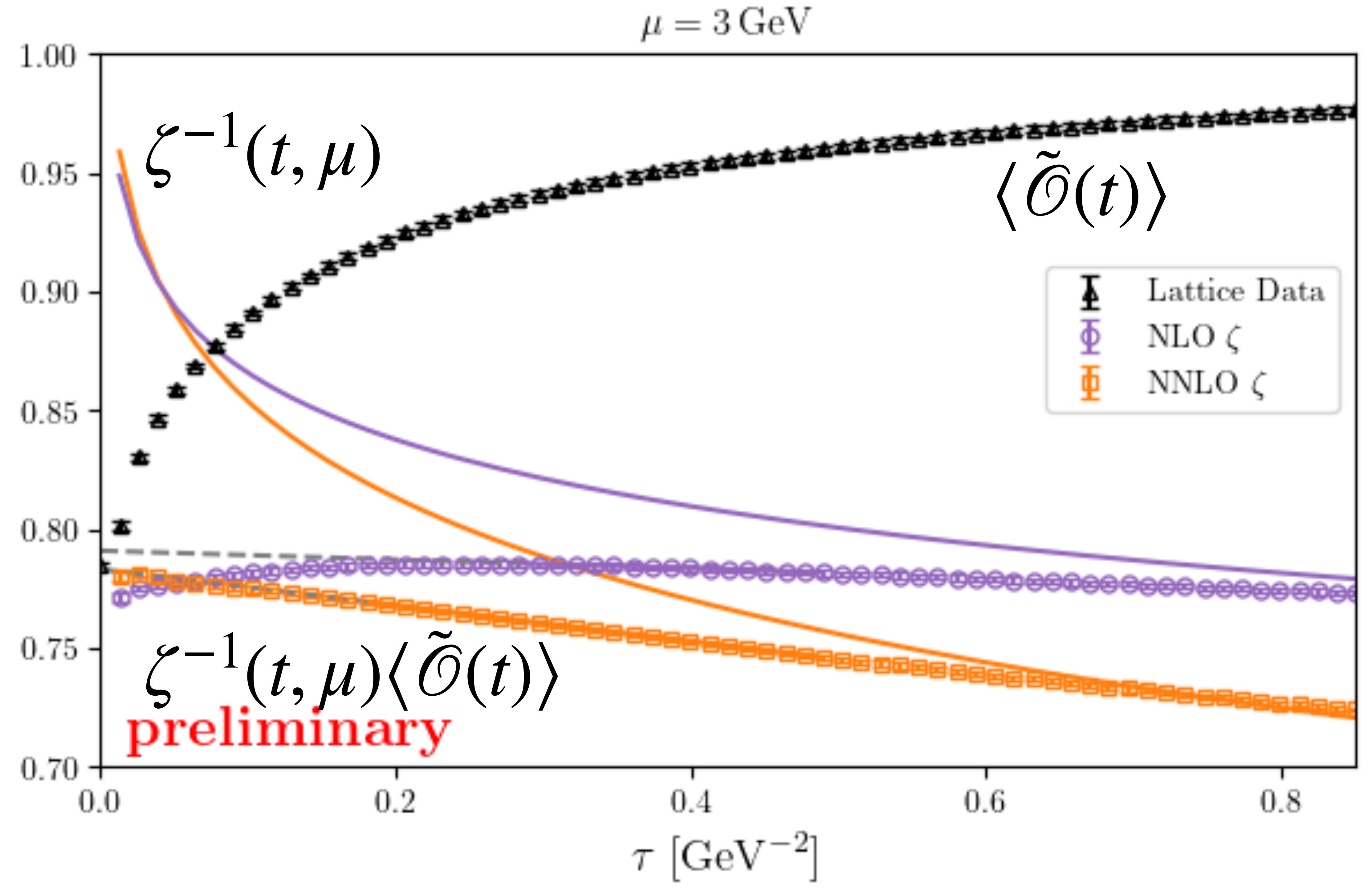

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Ammer, Dürr '24

→ see Stefan Dürr's talk (tomorrow, 4pm)

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GF	$\overline{\text{MS}}$

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The GF scheme

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GF	$\overline{\text{MS}}$
$= \sum_n (C\,\zeta^{-1}(t))_n \langle \zeta(t)\,\mathcal{O} \rangle_n$	$= \sum_n (C\,Z^{-1})_n \langle Z\,\mathcal{O} \rangle_n$

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$$\mathscr{L}_{\rm eff} = \sum_n C_n \langle \mathcal{O}_n \rangle$$

GF	$\overline{\rm MS}$
$= \sum_n (C\,\zeta^{-1}(t))_n \, \langle \zeta(t)\,\mathcal{O}\rangle_n$	$= \sum_n (C\,Z^{-1})_n \langle Z\,\mathcal{O}\rangle_n$
$= \sum_n \tilde{C}_n(t)\,\langle \tilde{\mathcal{O}}_n(t)\rangle$	$= \sum_n C_n^{\rm R}(\mu)\,\langle \mathcal{O}_n^{\rm R}\rangle(\mu)$
$t\frac{d}{dt}\tilde{C}(t) = \tilde{C}(t)\,\tilde{\gamma}$	$\mu\frac{d}{d\mu}C^{\rm R}(\mu) = C^{\rm R}(\mu)\,\gamma$

The GF scheme

$$\mathscr{L}_{\rm eff} = \sum_n C_n \langle \mathcal{O}_n \rangle$$

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$t\frac{d}{dt}\tilde{C}(t) = \tilde{C}(t)\,\tilde{\gamma}$	$\mu\frac{d}{d\mu}C^{\rm R}(\mu) = C^{\rm R}(\mu)\,\gamma$
$\tilde{\gamma} = -\,t\frac{d}{dt}\ln \zeta(t)$	$\gamma_{nm} = -\,\mu\frac{d}{d\mu}\ln Z$

The GF scheme

$$\mathscr{L}_{\text{eff}} = \sum_n C_n \langle \mathcal{O}_n \rangle$$

GF	$\overline{\text{MS}}$
$= \sum_n (C\,\zeta^{-1}(t))_n \langle \zeta(t)\,\mathcal{O} \rangle_n$	$= \sum_n (C\,Z^{-1})_n \langle Z\,\mathcal{O} \rangle_n$
$= \sum_n \tilde{C}_n(t) \langle \tilde{\mathcal{O}}_n(t) \rangle$	$= \sum_n C_n^{\text{R}}(\mu) \langle \mathcal{O}_n^{\text{R}} \rangle(\mu)$
$t\frac{d}{dt}\tilde{C}(t) = \tilde{C}(t)\,\tilde{\gamma}$	$\mu\frac{d}{d\mu}C^{\text{R}}(\mu) = C^{\text{R}}(\mu)\,\gamma$
$\tilde{\gamma} = -\,t\frac{d}{dt}\ln\zeta(t)$	$\gamma_{nm} = -\,\mu\frac{d}{d\mu}\ln Z$

RH, Lange, Neumann ‘20

Borgulat, Felten, RH, Kohnen ‘25

The GF scheme

GF	$\overline{\text{MS}}$
$t\frac{d}{dt}\tilde{C}(t) = \tilde{C}(t)\,\tilde{\gamma}$	$\mu\frac{d}{d\mu}C^{\mathrm{R}}(\mu) = C^{\mathrm{R}}(\mu)\,\gamma$
$\tilde{\gamma} = -\,t\frac{d}{dt}\ln\zeta(t)$	$\gamma_{nm} = -\,\mu\frac{d}{d\mu}\ln Z$

The GF scheme

GF	$\overline{\text{MS}}$
$t\frac{d}{dt}\tilde{C}(t) = \tilde{C}(t)\,\tilde{\gamma}$	$\mu\frac{d}{d\mu}C^{\text{R}}(\mu) = C^{\text{R}}(\mu)\,\gamma$
$\tilde{\gamma} = -\,t\frac{d}{dt}\ln\zeta(t)$	$\gamma_{nm} = -\,\mu\frac{d}{d\mu}\ln Z$
$\tilde{C}(t) = \tilde{C}(t_0)\,\exp\left[\int_{\tilde{\alpha}_s(t_0)}^{\tilde{\alpha}_s(t)}\frac{dx}{x}\frac{\tilde{\gamma}(x)}{\tilde{\beta}(x)}\right]$	$C^{\text{R}}(\mu) = C^{\text{R}}(\mu_0)\,\exp\left[\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)}\frac{d\mu}{\mu}\frac{\gamma(x)}{\beta(x)}\right]$

The GF scheme

GF	$\overline{\text{MS}}$
$t\frac{d}{dt}\tilde{C}(t) = \tilde{C}(t)\,\tilde{\gamma}$	$\mu\frac{d}{d\mu}C^{\mathrm{R}}(\mu) = C^{\mathrm{R}}(\mu)\,\gamma$
$\tilde{\gamma} = -\,t\frac{d}{dt}\ln\zeta(t)$	$\gamma_{nm} = -\,\mu\frac{d}{d\mu}\ln Z$
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$t\frac{d}{dt}\tilde{\alpha}_s(t) = \tilde{\beta}\,\tilde{\alpha}_s(t)$	$\mu\frac{d}{d\mu}\alpha_s(\mu) = \beta\,\alpha_s(\mu)$

The GF scheme

GF	$\overline{\text{MS}}$
$t\frac{d}{dt}\tilde{C}(t) = \tilde{C}(t)\,\tilde{\gamma}$	$\mu\frac{d}{d\mu}C^{\mathrm{R}}(\mu) = C^{\mathrm{R}}(\mu)\,\gamma$
$\tilde{\gamma} = -t\frac{d}{dt}\ln\zeta(t)$	$\gamma_{nm} = -\mu\frac{d}{d\mu}\ln Z$
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$t\frac{d}{dt}\tilde{\alpha}_s(t) = \tilde{\beta}\,\tilde{\alpha}_s(t)$	$\mu\frac{d}{d\mu}\alpha_s(\mu) = \beta\,\alpha_s(\mu)$

→ see Anna Hasenfratz's talk (Friday, 9am)

Omissions

static QCD force Brambilla, Chung, Vairo, Wang '22
Brambilla, Leino, Mayer-Steudte, Vairo '24

→ see Julian Mayer-Steudte's talk (Friday, 2:30pm)

…

Omissions

static QCD force Brambilla, Chung, Vairo, Wang ‘22
Brambilla, Leino, Mayer-Steudte, Vairo ‘24

→ see Julian Mayer-Steudte’s talk (Friday, 2:30pm)

off-light-cone Wilson lines Brambilla, Wang ‘24

→ see Xiangpeng Wang’s talk (Friday, 11am)

…

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off-light-cone Wilson lines   Brambilla, Wang '24

→ see Xiangpeng Wang's talk (Friday, 11am)

$\langle \bar{\chi}(t)\chi(t) \rangle$ with mass effects

→ see Hiromasa Takaura's talk (today, 11am)
→ see Robert Mason's talk (today, 11:30am)

…

Omissions

static QCD force Brambilla, Chung, Vairo, Wang '22
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$\langle \bar{\chi}(t)\chi(t) \rangle$ with mass effects
→ see Hiromasa Takaura's talk (today, 11am)
→ see Robert Mason's talk (today, 11:30am)

strong CP problem Dragos, Shindler, de Vries, Yousif '19
→ see Andrea Shindler's talk (tomorrow, 4pm)

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