Lattice artifacts of Gradient Flow quantities

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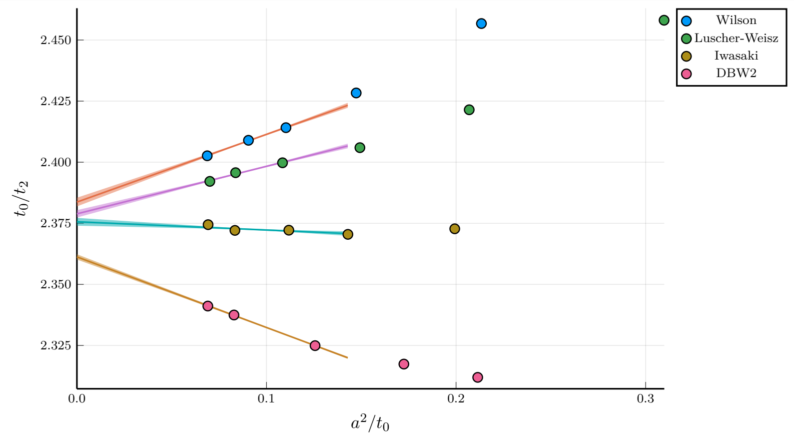
Motivation: Continuum extrapolation

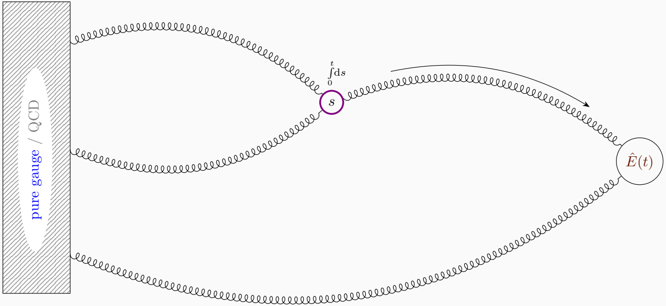


$$t^2\langle E(t)\rangle = \begin{cases} 0.3 & t = t_0 \\ 0.15 & t = t_2 \end{cases}$$

Plot provided by Alberto Ramos & Guilherme Catumba.

Gradient flow in the SymEFT

	Boundary theory	YM Gradient Flow	flowed observable
Lattice	$\hat{S}_{\mathrm{gauge}}[U] + \hat{S}_{\mathrm{ferm}}$	$\begin{cases} a^2 \partial_t V_\mu = \hat{\mathcal{F}}_\mu V_\mu \\ V_\mu(t=0,x) = U_\mu(x) \end{cases}$	$\hat{E}(t)$



SymEFT $\mathscr{L}[A] + a^n \delta \mathscr{L}[A]$ $\begin{cases} \partial_t B_\mu = [D_\nu, G_{\nu\mu}] + a^n \delta \mathcal{F}_\mu \\ B_\mu(t=0,x) = A_\mu(x) + a^n \delta A_\mu(x) \end{cases}$ $E(t) + a^n \delta E(t)$ $\mathbf{+O(a^{n+1})}$

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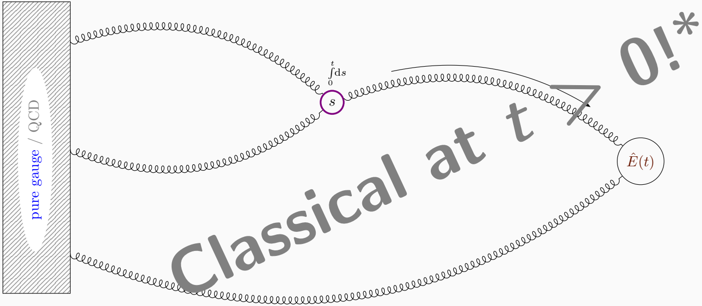
Lattice	$\hat{S}_{\mathrm{gauge}}[U]+\hat{S}_{\mathrm{ferm}}$	$\begin{Bmatrix} a^2\partial_t V_\mu = \hat{\mathcal{F}}_\mu V_\mu \\ V_\mu(t=0,x) = U_\mu(x) \end{Bmatrix}$	$\hat{E}(t)$



SymEFT $\mathscr{L}[A] + a^n\delta\mathscr{L}[A]$ $\quad \begin{Bmatrix} \partial_t B_\mu = [D_\nu, G_{\nu\mu}] + a^n\delta\mathcal{F}_\mu \\ B_\mu(t=0,x) = A_\mu(x) + a^n\delta A_\mu(x) \end{Bmatrix}$ $\quad E(t) + a^n\delta E(t) \quad +\mathrm{O}(a^{n+1})$

*Flowed fermions require renormalisation of the wavefunction [Lüscher, Weisz, 2011].

Symanzik Effective Theory (SymEFT) [Symanzik, 1980, 1981, 1983a,b]

Reformulate as $4+1$-dimensional theory to have a local effective Lagrangian [Lüscher, Weisz, 2011]

$$\mathscr{L}_{\mathrm{eff}}(t,x) = [\mathscr{L} + a^{n_{\min}} \sum_i \omega_i(g_0)\mathcal{O}_i](0,x)\delta(t) - 2\mathrm{tr}(L_\mu[\mathcal{F} + a^2\delta\mathcal{F}]_\mu)(t,x) + \mathrm{O}(a^{n_{\min}+1}, a^3)\,.$$

- Corrections to the flow are purely classical [Lüscher, Weisz, 2011] and can be removed e.g. via the “Zeuthen flow” [Ramos, Sint, 2016].
- Corrections to the boundary action require renormalisation.
- Modification of the gluon EOM

$$\frac{1}{g_0^2}[D_\nu, F_{\nu\mu}](0,x) = T^a\bar{\Psi}\gamma_\mu T^a\Psi(0,x) - L_\mu(0,x)$$

introduces **additional** on-shell operators on the boundary contributing to flowed quantities [Ramos, Sint, 2016].

Example: ratio of flow-time scales

$$\frac{\hat{t}_0(a)}{\hat{t}_2(a)} = \frac{t_0}{t_2}\Bigg\{1 - a^{n_{\min}} \sum_i \hat{c}_i [2b_0\bar{g}^2(1/a)]^{\hat{\Gamma}_i} \int \mathrm{d}^4x \frac{\langle \mathcal{B}_{i;\mathrm{RGI}}(x) E(t_2)\rangle}{2\langle E(t_2)\rangle} + a^2 \frac{\langle \delta E(t_2)\rangle}{2\langle E(t_2)\rangle}$$

Classical!

$$+a^2 \int_0^\infty \mathrm{d}t \int \mathrm{d}^4x \frac{\langle \mathrm{tr}[L_\mu \delta\mathcal{F}_\mu](t,x) E(t_2)\rangle}{\langle E(t_2)\rangle} - (t_2 \to t_0) + \mathrm{O}(a^{n_{\min}+1}, a^3)\Bigg\}$$

where $\hat{\Gamma}_i = (\gamma_0^{\mathcal{B}})_i/(2b_0) + n_i$ can be obtained from 1-loop running of the operators $\mathcal{O}_i$

$$\mu \frac{\mathrm{d}\mathcal{O}_{i;\overline{\mathrm{MS}}}}{\mathrm{d}\mu} = -\bar{g}^2(\mu)\left[\gamma_0^{\mathcal{O}} + \mathrm{O}(\bar{g}^2)\right]_{ij} \mathcal{O}_{j;\overline{\mathrm{MS}}}$$

and a change of basis $\mathcal{O} \to \mathcal{B}$ s.t. $\gamma_0^{\mathcal{B}}$ is diagonal.

Full on-shell basis

$$\mathcal{O}_1 = \frac{1}{g_0^2}\mathrm{tr}([D_\mu, F_{\nu\rho}][D_\mu, F_{\nu\rho}]),$$

$$\mathcal{O}_2 = \frac{1}{g_0^2}\sum_\mu \mathrm{tr}([D_\mu, F_{\mu\nu}][D_\mu, F_{\mu\nu}]),$$

$$\mathcal{O}_3 = \frac{1}{g_0^2}\,\mathrm{tr}\big([D_\mu, G_{\mu\rho}][D_\nu, G_{\nu\rho}]\big)\Big|_{t=0}$$

$$\begin{pmatrix}\mathcal{O}_1\\ \mathcal{O}_2\\ \mathcal{O}_3\end{pmatrix}_{\overline{\mathrm{MS}}} = \begin{pmatrix} Z_{11} & 0 & Z_{13}\\ Z_{21} & Z_{22} & Z_{23}\\ 0 & 0 & Z_{33}\end{pmatrix}\begin{pmatrix}\mathcal{O}_1\\ \mathcal{O}_2\\ \mathcal{O}_3\end{pmatrix}$$

[NH et al., 2020]

[NH, 2021] $Z_{33} = 1 + \mathrm{O}(g^4)$ (Ph.D. thesis)

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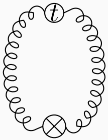

- Only one renormalisation condition required, here $\langle E(t)\tilde{\mathcal{O}}(0)\rangle$.
- Strategy gives 1-loop matrix elements affecting $\langle E(t)\rangle$, **BUT** no 1-loop matching available!

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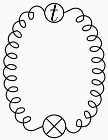
$\tilde{E}(t,0)$



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$$\hat{\gamma}_1 = \frac{7}{11}, \quad \hat{\gamma}_2 = \frac{63}{55}, \quad \hat{\gamma}_3 = 0$$

Vanishing anomalous dimension to all orders?

Preliminary

Additional boundary operator

$$\int \mathrm{d}^4x\, \mathcal{O}_3 \overset{\text{EOM}}{=} -\int \mathrm{d}^4x\ \mathrm{tr}\big(L_\mu[D_\nu, G_{\nu\mu}]\big)\big|_{t=0}$$

$$\overset{?}{=} \int \mathrm{d}^4x \int_0^\tau \mathrm{d}s\, \partial_s \mathrm{tr}\big(L_\mu[D_\nu, G_{\nu\mu}]\big) - \int \mathrm{d}^4x\ \mathrm{tr}\big(L_\mu[D_\nu, G_{\nu\mu}]\big)\big|_{t=\tau}$$

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$$\underset{\text{IBP}}{\overset{\tau\to\infty}{=}} \int \mathrm{d}^4x \int_0^\infty \mathrm{d}s\,\mathrm{tr}\big(L_\alpha[D_\beta,[D_\gamma,[D_\rho, G_{\nu\mu}]]]\big)$$

Use EOMs from [Ramos, Sint, 2016]

$$\times\,(4\delta_{\alpha\mu}\delta_{\beta\gamma}\delta_{\rho\nu} - 4\delta_{\alpha\mu}\delta_{\gamma\nu}\delta_{\beta\rho} + 2\delta_{\alpha\mu}\delta_{\beta\nu}\delta_{\gamma\rho} + \delta_{\alpha\beta}\delta_{\gamma\mu}\delta_{\rho\nu})$$

$\overset{?}{\Rightarrow}$ **Absorb operator into definition of the flow, i.e., purely classical and thus $Z_{33} \equiv 1$.**

Full QCD

- Significantly enlarged operator basis with powers $\hat{\gamma}_i$ for the boundary theory with various possible choices for the fermion action: GW, Wilson [NH, 2023], Staggered [NH, 2025], ...
- Yang-Mills Gradient flow requires (at least) **two** additional operators in the on-shell basis

$$\mathcal{O}_{n+1} = \mathrm{tr}\big(L_\mu[D_\nu, G_{\nu\mu}]\big)\Big|_{t=0}, \quad \mathcal{O}_{n+2} = \bar{\Psi}\gamma_\mu L_\mu \Psi\Big|_{t=0}.$$

$\rightsquigarrow$ If previous argument is correct: Both operators are purely classical.

- We now need **two** renormalisation conditions for the two operators.

What is missing for the full QCD computation?

We now need **two** additional renormalisation conditions for the two operators.

- What would be a suitable condition beside $\langle E(t)\tilde{\mathcal{O}}(0)\rangle$?
- Preferably on-shell renormalisation conditions, because we have too many EOMs: gluon EOM, fermion EOM, flow equation, flow equation of the Lagrange multiplier, ...
- How do I deal with an external on-shell Lagrange multiplier field? Amputate?

Conclusion

- Logarithmic corrections $a^2[2b_0\bar{g}^2(1/a)]^{\hat{\Gamma}_i}$ to the classical a^2-scaling of flowed observables in pure gauge are mild $\min_i \hat{\Gamma}_i \geq 0$ [NH et al., 2020; NH, 2021].
- For full QCD there are many more powers but so far no serious problems encountered for $N_\mathrm{f} \leq 4$ [NH, 2023, 2025].
  $\rightsquigarrow$ Still need to work out the 1-loop renormalisation of the **two** additional operators on the $t = 0$ boundary.
- Classical a^2-improvement of the flow and flowed observable theoretically cleaner (limits sources of lattice artifacts). **BUT** absence of accidental cancellations may even enlarge the overall lattice artifacts.
- Classically perfect actions (Urs Wenger today 15:30) should ensure TL-improvement to all orders in the lattice spacing. Requires use of classically perfect flow [Ramos, Sint, 2016].

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