Gradient flow renormalon subtraction and the hadronic tau decay series

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Zürich Gradient Flow Workshop
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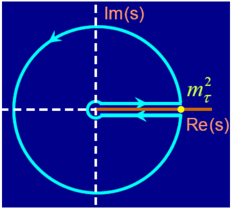
MB, H. Takaura, PoS RADCOR2023 (2024) 062
[2309.10853] and work in progress



Gradient flow regularization and the OPE

$$i\int d^4x\, e^{iqx}\,\langle J(x)J(0)\rangle = C_0(\alpha_s, Q/\mu) + C_{GG}(\alpha_s, Q/\mu)\,\frac{1}{Q^4}\,\langle\frac{\alpha_s}{\pi}G^2\rangle(\mu) + O(1/Q^6)$$

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- In $\overline{\text{MS}}$-like schemes, the short-distance coefficients and condensates are both ill-defined (renormalons and power-divergence subtraction).
- Any definition / subtraction of the divergent perturbative series implies a renormalization scheme for the quartic power-divergences of the operator $\langle\frac{\alpha_s}{\pi}G^2\rangle(\mu)$ and vice versa.

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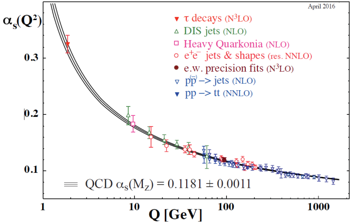
Proposition: Formulate OPE in terms of gradient-flow regularized operators at finite flow time

$$\Lambda_{\text{QCD}} \ll 1/\sqrt{8t} \ll Q$$

- "Wilsonian OPE" with gauge-invariant, hard cut-off, well-defined condensates ...
- ... in the continuum and on the lattice ($a \ll \sqrt{8t} \ll L$)
- Well-defined short-distance coefficients: Divergent series from IR renormalon disappear

α_s determinations and τ decay

1907.01435



Summary of α_s determinations (2203.08271, PDG update)

category	$\alpha_S(m_Z^2)$	relative $\alpha_S(m_Z^2)$ uncertainty
τ decays and low Q^2	0.1178 ± 0.0019	1.6%
$Q\overline{Q}$ bound states	0.1181 ± 0.0037	3.1%
PDF fits	0.1162 ± 0.0020	1.7%
e^+e^- jets & shapes	0.1171 ± 0.0031	2.6%
electroweak	0.1208 ± 0.0028	2.3%
hadron colliders	0.1165 ± 0.0028	2.4%
lattice	0.1182 ± 0.0008	0.7%
world average (without lattice)	0.1176 ± 0.0010	0.9%
world average (with lattice)	0.1179 ± 0.0009	0.8%

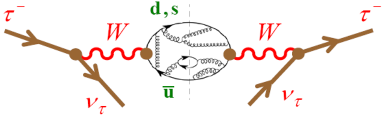
Hadronic τ decay width

- QCD PT + OPE condensates [Braaten (1989), Braaten, Narison, Pich (1992)]
- Precise: $\frac{\delta\alpha_s(M_Z)}{\alpha_s(M_Z)} \approx \frac{\alpha_s(M_Z)}{\alpha_s(M_\tau)} \times \frac{\delta\alpha_s(M_\tau)}{\alpha_s(M_\tau)}$
- Accuracy limited by a systematic discrepancy within *perturbation theory* – FOPT vs CIPT [MB, Jamin, 2008]

$$R_\tau \equiv \frac{\Gamma[\tau^- \to \text{hadrons}\,\nu_\tau(\gamma)]}{\Gamma[\tau^- \to e^-\overline{\nu}_e\nu_\tau(\gamma)]} = \frac{1-\mathcal{B}_e-\mathcal{B}_\mu}{\mathcal{B}_e} = R_{\tau,V}+R_{\tau,A}+R_{\tau,S} = 3.6381 \pm 0.0075$$

[HFLAV, 2206.07501]

$$\Gamma_{\tau\to\nu_\tau+\text{had}} \sim \text{Im}\left\{ \right.$$



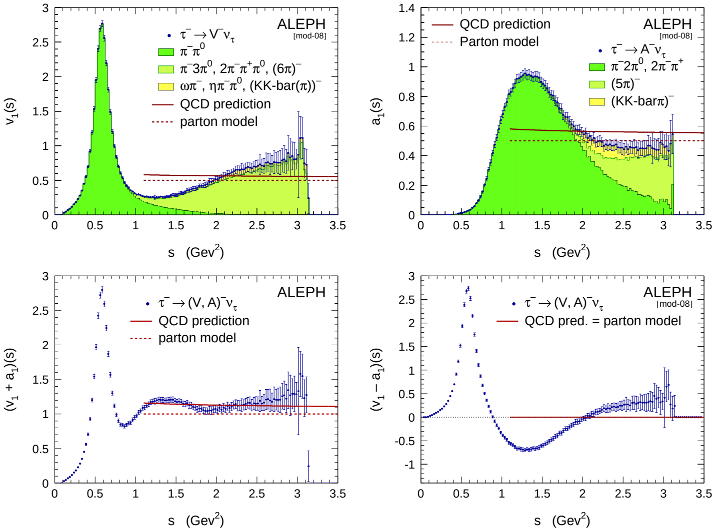
$$\left. \right\}$$

Focus on non-strange final states

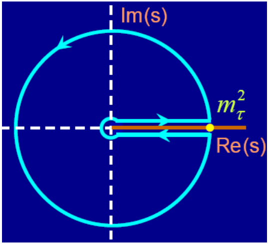
$$R_\tau = 12\pi \int_0^{M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left[\left(1+2\frac{s}{M_\tau^2}\right)\text{Im}\,\Pi^{(1)}(s)+\text{Im}\,\Pi^{(0)}(s)\right]$$

$$\Pi_{\mu\nu}^{V/A}(p) \equiv i\int dx\, e^{ipx}\,\langle\Omega|\,T\{J_\mu^{V/A}(x)\,J_\nu^{V/A}(0)^\dagger\}|\Omega\rangle = (p_\mu p_\nu - g_{\mu\nu}p^2)\,\Pi^{V/A,(1)} + p_\mu p_\nu\,\Pi^{V/A,(0)}$$

ALEPH spectral functions



The τ hadronic width in QCD



- Analyticity
- Condensate expansion
- Slightly Euclidean [$(1-x)^3$ suppression]
- $D^{(1+0)}(s) \equiv -s\,\frac{d}{ds}\left[\Pi^{(1+0)}(s)\right]$ (Adler fn)

$$\begin{aligned}
R_\tau &= 6\pi i \oint\limits_{|s|=M_\tau^2} \frac{ds}{M_\tau^2}\left(1-\frac{s}{M_\tau^2}\right)^2\left[\left(1+2\frac{s}{M_\tau^2}\right)\Pi^{(1)}(s)+\Pi^{(0)}(s)\right] \\
&= -i\pi \oint\limits_{|x|=1}\frac{dx}{x}(1-x)^3\left[3\,(1+x)\,D^{(1+0)}(M_\tau^2 x)+4\,D^{(0)}(M_\tau^2 x)\right] \\
&= N_c\, S_{\rm EW}\,|V_{ud}|^2\left[1+\delta^{(0)}+\delta'_{\rm EW}+\sum_{D\geq 2}\frac{C_D(s,\mu)\langle O_D(\mu)\rangle}{(-s)^{D/2}}\right]
\end{aligned}$$

[Braaten, Narison, Pich, 1992]

Condensate expansion

- D=2 m_q^2
  $\longrightarrow (3.1 \pm 8.6) \cdot 10^{-5}$
- D=4 $m_q^4, m_q\langle\bar{q}q\rangle, \langle\frac{\alpha_s}{\pi}GG\rangle$
  $\longrightarrow (6.3 \pm 3.3) \cdot 10^{-4}$

  Suppression of the D=4 contribution due to the kinematic weight function
  $(1-x)^3(1+2x) = 1 - 2x + 2x^3 - x^4$
- D=6 $\langle\bar{q}q\bar{q}q\rangle, \langle\alpha_s G^3\rangle$
  $\longrightarrow (-4.8 \pm 2.9) \cdot 10^{-3}$ – dominant

  Factor 10 cancellation between V and A. This explains $R_{\tau,V-A} \approx 0.08$ ($\rightarrow$ Fig.)
- S+P $D^{(0)}(s)$ contribution dominated by the calculable pion pole contribution
  $\longrightarrow (-2.64 \pm 0.05) \cdot 10^{-3}$

Non-perturbative terms very small [3.5% of perturbative contribution!] due to V+A cancellation and kinematic suppression

$$\delta_{\rm PC} = (-6.8 \pm 3.5) \cdot 10^{-3}$$

Nevertheless, the gluon condensate will play an important role in the following.

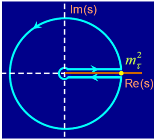
FO and CI perturbation theory

$$D_{V,A}^{(1+0)}(s) = \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} a_\mu^n \sum_{k=1}^{n+1} k\, c_{n,k}\, \ln^{k-1}\frac{-s}{\mu^2} = \frac{N_c}{12\pi^2}\sum_{n=0}^{\infty} c_{n,1}\, a_Q^n$$

$$= \frac{N_c}{12\pi^2}\left[1 + a_Q + 1.64 a_Q^2 + 6.37 a_Q^3 + 49.08 a_Q^4 + \ldots\right] \qquad (a_Q = \alpha_s(Q)/\pi)$$

[5-loop $c_{4,1}$: Baikov, Chetyrkin, Kühn, 2008]

$$R_\tau = -i\pi \oint_{|x|=1} \frac{dx}{x}\,(1-x)^3\left[3\,(1+x)\,D^{(1+0)}(M_\tau^2 x) + 4\,D^{(0)}(M_\tau^2 x)\right]$$



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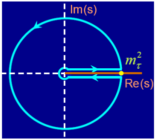


FOPT $\qquad \delta^{(0)}_{\rm FO} = \sum_{n=1}^{\infty} a(M_\tau^2)^n \sum_{k=1}^{n} k\, c_{n,k}\, J_{k-1} \qquad J_l \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}\,(1-x)^3\,(1+x)\,\ln^l(-x)$

CIPT $\qquad \delta^{(0)}_{\rm CI} = \sum_{n=1}^{\infty} c_{n,1}\, J_n^a(M_\tau^2) \qquad J_n^a(M_\tau^2) \equiv \frac{1}{2\pi i}\oint_{|x|=1}\frac{dx}{x}\,(1-x)^3\,(1+x)\, a^n(-M_\tau^2 x)$

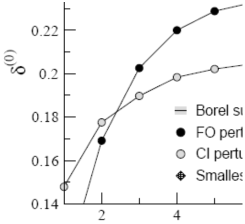
[Le Diberder, Pich, 1993] - Sums π^2 terms

The problem [MB, Jamin, 2008]

Numerical series expansions for $\alpha_s(M_\tau^2) = 0.34$.
(We will often use the estimate $c_{5,1} = 283 \pm 283$.)

$$
\begin{array}{ccccccccc}
 & & \alpha_s^1 & \alpha_s^2 & \alpha_s^3 & \alpha_s^4 & \alpha_s^5 & & \\
\delta_{\rm FO}^{(0)} & = & 0.1082 & + 0.0609 & + 0.0334 & + 0.0174 & (+ 0.0088\,) & = & 0.2200\ (0.2288) \\
\delta_{\rm CI}^{(0)} & = & 0.1479 & + 0.0297 & + 0.0122 & + 0.0086 & (+ 0.0038\,) & = & 0.1984\ (0.2021)
\end{array}
$$

- FO/CI difference *increases* by adding more orders. Systematic problem.
- Difference in α_s value is larger than the error of each individual method.



Problem is connected with systematic pattern of coefficients in higher perturbative orders.

General structure: Several components of "renormalon" factorial divergence of form

$$\sum_n c_{n,1}\alpha_s^{n+1} \stackrel{n\gg 1}{=} \sum_n \alpha_s^{n+1} K(-a\beta_0)^n n!\, n^b \left(1+\frac{s_1}{n}+O(1/n^2)\right)$$

related to the structure of the OPE ($2a = d =$ dimension of operator, b anomalous dimension. Stokes constant K is truly non-perturbative) and singularities of the Borel transform.

Asymptotics of PT, Borel transform

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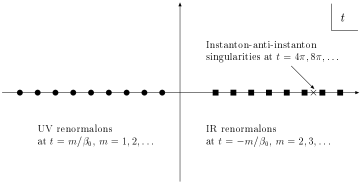
- UV renormalon at $u = -\beta_0 t = -1$, leading singularity for Adler function and R_τ for very large orders (sign alternation)
- Leading IR renormalon singularity at $u = 2$, related to the gluon condensate.

  Especially simple structure, only one operator (GG).



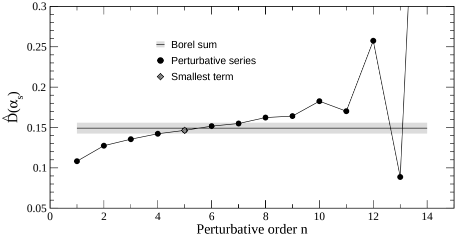
Expect fixed sign series in intermediate orders and sign-alternation only asymptotically – in the $\overline{\text{MS}}$ scheme.

Borel function model [MB, Jamin, 2008]

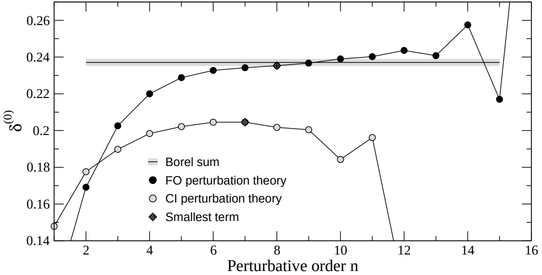
Incorporate the knowledge of asymptotic behaviour into an Ansatz for the Adler function that reproduces known $c_{n,1}$ to $n = 4$ and $c_{5,1} = 283$.
Then compare FOPT/CIPT.

$$B[D](u) = B[D_1^{\rm UV}](u) + B[D_2^{\rm IR}](u) + B[D_3^{\rm IR}](u) + d_0^{\rm PO} + d_1^{\rm PO} u$$

- Fit Stokes constants K_p for $u = -1, 2, 3$ to $c_{3,1}$, $c_{4,1}$ and $c_{5,1}$, and adjust $d_{0,1}^{\rm PO}$ to reproduce $c_{1,1}$ and $c_{2,1}$.
- Pole ansatz works well already at $n = 2$ ($d_1^{\rm PO}$ small). Apparently the series is very regular.



FO vs CI for R_τ



- FO converges to Borel sum
- CI converges more quickly than FO at low orders, but never reaches the Borel sum.
- At $n = 4, 5$ FO is close to the true result, CI too small $\Rightarrow$ α_s from CI too large.

Inconsistency of CIPT

$$\delta_{\text{FO}}^{(0)} = \sum_{n=1}^{\infty} \left[c_{n,1} + g_n\right] a(M_\tau^2)^n \qquad g_n = \sum_{k=2}^{n} k\, c_{n,k} J_{k-1}$$

- For the leading IR contribution ($u = 2$) there are *large cancellations* in going from Adler function to R_τ related to suppression of the gluon condensate contribution: [MB, 1993]

$$\frac{c_{n,1} + g_n}{c_{n,1}} \propto 1/n^2$$

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- CIPT does not respect these cancellations, problem with OPE. For general spectral function moments, the poor behaviour of CIPT disappears when one assumes that there is no $u = 2$ singularity. [MB, Jamin 2008; MB, Boito, Jamin, 2012]

- ... many papers ... [Caprini, Fischer, 2009 – , Cvetic et al., 2010, ...]

- Asymptotic behaviour of the CIPT series is inconsistent with the OPE. [Hoang, Regner, 2021; Gracia, Hoang, Mateu, 2023]

- Solution to the CIPT problem: Applying renormalon subtraction to the $u = 2$ singularity of the Adler function makes CIPT and FOPT agree. [Benitez-Rathgeb, Boito, Hoang, Jamin, 2022: MB, Takaura, 2023]

Here: continuum version ("Wilson flow" on lattice)
Define "flowed" gluon field $B_\mu(t,x)$ by

$$\partial_t B_\mu = \tilde{D}_\nu \tilde{G}_{\nu\mu} + \xi_0 \tilde{D}_\mu \partial_\nu B_\nu, \qquad B_\mu|_{t=0} = A_\mu$$

$t =$ flow "time", $\tilde{G}_{\mu\nu}$, $\tilde{D}_\mu$ usual definitions but with B_μ.

Interpretation: Smeared gluon field over distance $\sqrt{8t}$. LO solution

$$B_\mu(t,x) = \int d^d y\, K(t,x-y)\, A_\mu(x) \qquad K(t,z) = \frac{e^{-z^2/(4t)}}{(4\pi t)^{d/2}}$$

Action density

$$E(t) = \frac{g^2}{4} G^A_{\mu\nu}(t) G^{A\mu\nu}(t)$$

Its expectation value, $\langle E(t) \rangle$, can be regarded as a gauge-invariant non-perturbative definition of the gluon condensate with cut-off $\Lambda_{\rm UV} \propto 1/\sqrt{t}$, which can replace the ill-defined $\langle \frac{\alpha_s}{\pi} G^2 \rangle(\mu)$ in the $\overline{\rm MS}$-OPE.

OPE of the action density / subtracted Adler function

Small flow-time expansion $t \ll 1/\Lambda^2_{\rm QCD}$ [Lüscher, 1006.4518; Harlander, Neumann 1606.03756 (NNLO)]

$$\frac{1}{\pi^2}\langle E(t)\rangle = \frac{\widetilde{C}_0(t)}{t^2} + \widetilde{C}_{GG}(t)\langle \frac{\alpha_s}{\pi} GG\rangle + \mathcal{O}(t \times \text{dim-6})$$

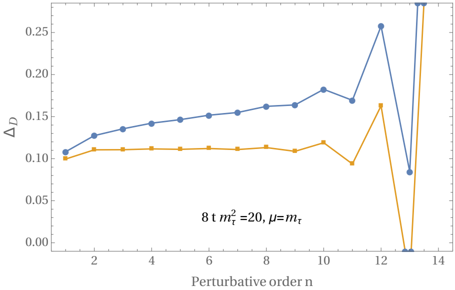
$\widetilde{C}_1$ known to NLO [Lüscher, 1006.4518] and NNLO [Harlander, Neumann 1606.03756], $\widetilde{C}_{GG}(t)$ to NNLO [Harlander, Kluth, Lange, 1808.09837]

Eliminate $\langle \frac{\alpha_s}{\pi} G^2\rangle(\mu)$ in the standard OPE for the Adler function

$$D(Q) = \underbrace{\left[C_0(Q) - \frac{1}{t^2 Q^4}\frac{C_{GG}(Q)}{\widetilde{C}_{GG}(t)} \times \widetilde{C}_0(t)\right]}_{u\,=\,2 \text{ renormalon cancels}} + \frac{1}{Q^4}\underbrace{\frac{C_{GG}(Q)}{\widetilde{C}_{GG}(t)}\frac{1}{\pi^2}\langle E(t)\rangle}_{\text{non-perturbatively defined}} + O(1/Q^6)$$

Only need fixed-order calculations to obtain a better-behaved expansion.

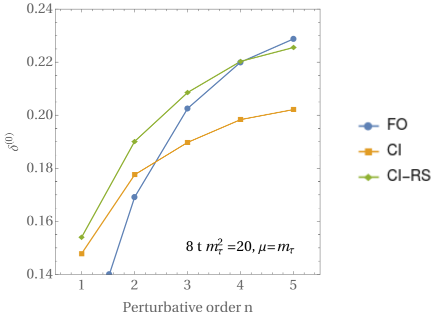
Adler function series with gradient-flow subtraction



Unsubtracted (blue) and subtracted (orange, $8tm_\tau^2 = 20$) Adler function perturbation series (unit operator) truncated at order n, $\sum_{k=1}^{n} c_{k1} a(m_\tau)^k$.

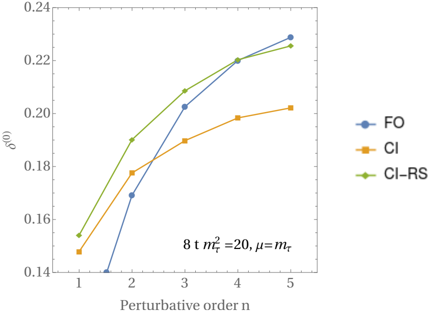
(t-dependence cancels when the non-perturbative gluon condensate is added.)

Hadronic τ decay with gradient-flow subtraction



$$8tm_\tau^2 = 20$$

FO and subtracted FO identical for NLO $C_{GG}(Q)/\widetilde{C}_{GG}(t)$.



$8tm_\tau^2 = 20$

FO and subtracted FO identical for NLO $C_{GG}(Q)/\widetilde{C}_{GG}(t)$.

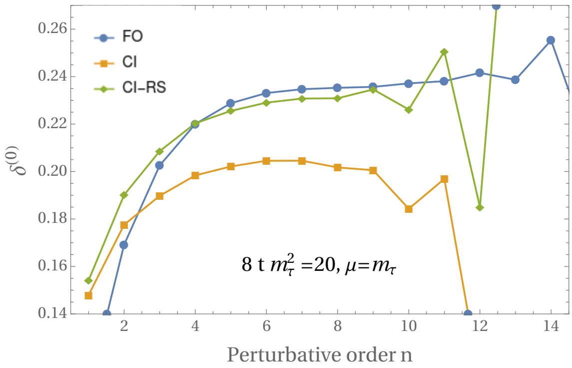
Renormalon model

The gradient flow action density has no UV renormalons. Ansatz (in practice for the entire subtraction term

$$B[E](u) \; = \; B[E_2^{\rm IR}](u) + B[E_3^{\rm IR}](u) + e_0^{\rm PO} + e_1^{\rm PO} u$$

Three unknowns (one fixed by $u = 2$ cancellation). Matches the available three exactly known coefficients of $\widetilde{C}_0$

Hadronic τ decay with gradient-flow subtraction [MB, Takaura, 2309.10853]



- CI and FO approach now similar values.
  (How well, depends on choice of t.)
- CI converges more quickly than FO at low orders, now to the correct value.

Conclusion

- FOPT/CIPT resolved by gluon-condensate renormalon subtraction.
  FOPT α_s value from τ decay is the correct one.

- The gradient flow subtraction scheme works in low orders without explicit knowledge of asymptotic behaviour at a low subtraction scale.

- It is possible to consistently add the leading power correction from the gluon condensate. The flowed action density can be computed on the lattice (t does not have to be very small).
  Optimal t window needs to be investigated.

- The gradient flow separates the continuum limit $a \to 0$ on the lattice from the cut-off scale $1/\sqrt{t}$ defining the renormalon subtraction.

The proposal is general and applies to HQET matrix elements, DIS twist-four moments etc. in the same way

Backup slides

Large-order asymptotics of PT

General structure of large-order behaviour is (believed to be) known.
Several components of factorial divergence of form

$$c_{n,1} \stackrel{n\gg 1}{=} \alpha_s^{n+1} K(a\beta_0)^n n!\, n^b \left(1 + \frac{s_1}{n} + O(1/n^2)\right)$$

Borel transform

$$F \sim \sum_{n=0}^{\infty} c_{n,1}\alpha_s^{n+1} \implies B[F](t) = \sum_{n=0}^{\infty} c_{n,1}\,\frac{t^n}{n!} \implies F(\alpha) = \int_0^\infty dt\, \mathrm{e}^{-t/\alpha}\, B[F](t)$$

$$c_{n,1} = K a^n \Gamma(n+1+b) \iff B[F](t) = \frac{K\Gamma(1+b)}{(1-at)^{1+b}}.$$

Minimal term at $n \approx 1/(|a\beta_0|\alpha_s(Q))$ of size

$$\Delta \approx e^{-1/(|a\beta_0|\alpha_s(Q))} \approx \left(\frac{\Lambda^2}{Q^2}\right)^{1/a} \approx \text{size of ambiguity}$$

IR renormalons and condensates

- IR renormalons – from small loop momentum
  Fixed-sign, singularity structure related to higher-dim operators in the OPE [David, 1984; Mueller, 1985; Zakharov, 1992; MB, 1993]

OPE $$\Pi(Q) = C_0(\alpha_s, Q/\mu) + \frac{1}{Q^d}\, C_d(\alpha_s, Q/\mu)\, \langle \frac{\alpha_s}{\pi} G^2 \rangle(\mu) + O(1/Q^6)$$

from $k < \mu \ll Q$ $$C_0^{\rm IR}(\alpha_s, Q/\mu) = \frac{1}{Q^d}\, C_d(\alpha_s, Q/\mu)\, \mu^d M(\alpha_s) + O(1/Q^6)$$

Ansatz $$M(\alpha_s) = \sum_n \alpha_s^{n+1} K (a\beta_0)^n n!\, n^b \left(1 + \frac{s_1}{n} + O(1/n^2)\right)$$

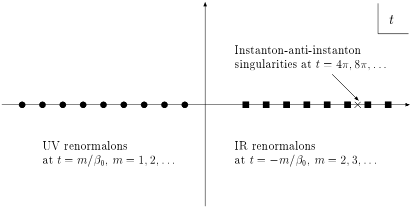
- Location of singularity related to operator dimension, $a = d/2$
- Nature of singularity related to operator anomalous dimension and β function including susb-leading $1/n^k$ terms:
  $$b = \frac{d\beta_1}{2\beta_0^2} - \frac{\gamma_0}{2\beta_0}, \qquad s_1 = f(\beta_2, \gamma_1, ...)$$
- Normalization = Stokes constant is the only unknown (non-perturbative)

Borel plane singularities of the Adler function

IR renormalons

- Leading singularity at $u = 2$, since no D=2 operator.

  Especially simple structure, only one operator (GG).
- Sub-leading $u = 3$ related to many D=6 operators.



UV renormalons

- $u = -1$, leading singularity for Adler function and R_τ for very large orders (sign alternation)
- No sign-alternation seen in the real Adler function and R_τ series. c_{-1} must be small in the $\overline{\text{MS}}$ scheme – true in the bubble diagram ("large-β_0") approximation.

Expect fixed sign series in intermediate orders and sign-alternation only asymptotically – in the $\overline{\text{MS}}$ scheme.