Stout smearing and Wilson flow in lattice perturbation theory (and beyond)

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work with my former PhD student Max Ammer

Zürich Gradient Flow Workshop 2025 – 13 Feb 2025

Outline

- why stout smearing and gradient flow are closely related

[1] M. Ammer and S. Dürr, Phys.Rev.D 109, 014512 (2024) [2302.11261]
Calculation of c_{SW} at one-loop order for Brillouin fermions

[2] M. Ammer and S. Dürr, Phys.Rev.D 110, 054504 (2024) [2406.03493]
Stout smearing and Wilson flow in lattice perturbation theory

[3] M. Ammer and S. Dürr, forthcoming
One-loop c_{SW} in lattice perturbation theory for Wilson and Brillouin fermions with stout smearing or gradient flow

- application to stag./KW/BC eigenvalues (with Stefano Capitani) [2411.18237]

- application to topological susceptibility (with Gianluca Fuwa) [2501.08217]

Recap of stout smearing

$U_\mu(x) = V_\mu^{(0)}(x)$ original ("thin link") gauge field $[\forall\mu, \forall x]$

$V_\mu(x) = V_\mu^{(n)}(x)$ smeared ("fat link") gauge field [after n steps of smoothing]

$$V_\mu^{(0)}(x) \longrightarrow V_\mu^{(1)}(x) \longrightarrow V_\mu^{(2)}(x) \longrightarrow \ldots \longrightarrow V_\mu^{(n)}(x)$$

Problem: APE smearing gives $V_\mu(x) \notin SU(N_c)$, thus "needs backprojection"

Solution: stout smearing gives $V_\mu(x) \in SU(N_c)$, obey $0 \le \rho \le 0.125$ [hep-lat/0311018]

$$V_\mu^{(n)}(x) = e^{\mathrm{i}\rho Q_\mu^{(n-1)}(x)}\, V_\mu^{(n-1)}(x)$$

$$Q_\mu^{(n-1)}(x) = P_{\mathrm{TH}}\Big[\frac{1}{\mathrm{i}} S_\mu^{(n-1)}(x) V^{(n-1)\,\dagger}(x)\Big] \qquad \text{[in Lie algebra]}$$

$$S_\mu^{(k)}(x) = \sum_{\nu\neq\mu} \Big\{ V_\nu^{(k)}(x) V_\mu^{(k)}(x+\hat\nu) V_\nu^{(k)\dagger}(x+\hat\mu) + V_\nu^{(k)\dagger}(x-\hat\nu) V_\mu^{(k)}(x-\hat\nu) V_\nu^{(k)}(x+\hat\mu-\hat\nu) \Big\}$$

Recap of gradient flow

- **1st order integrator [Morningstar Peardon, hep-lat/0311018]**

$$X_0 = V_t, \quad \text{build } Q_0 \text{ from } X_0$$
$$V_{t+\rho a^2} = \exp(\mathrm{i}\rho Q_0) X_0$$

- **2nd order integrator [midpoint rule]**

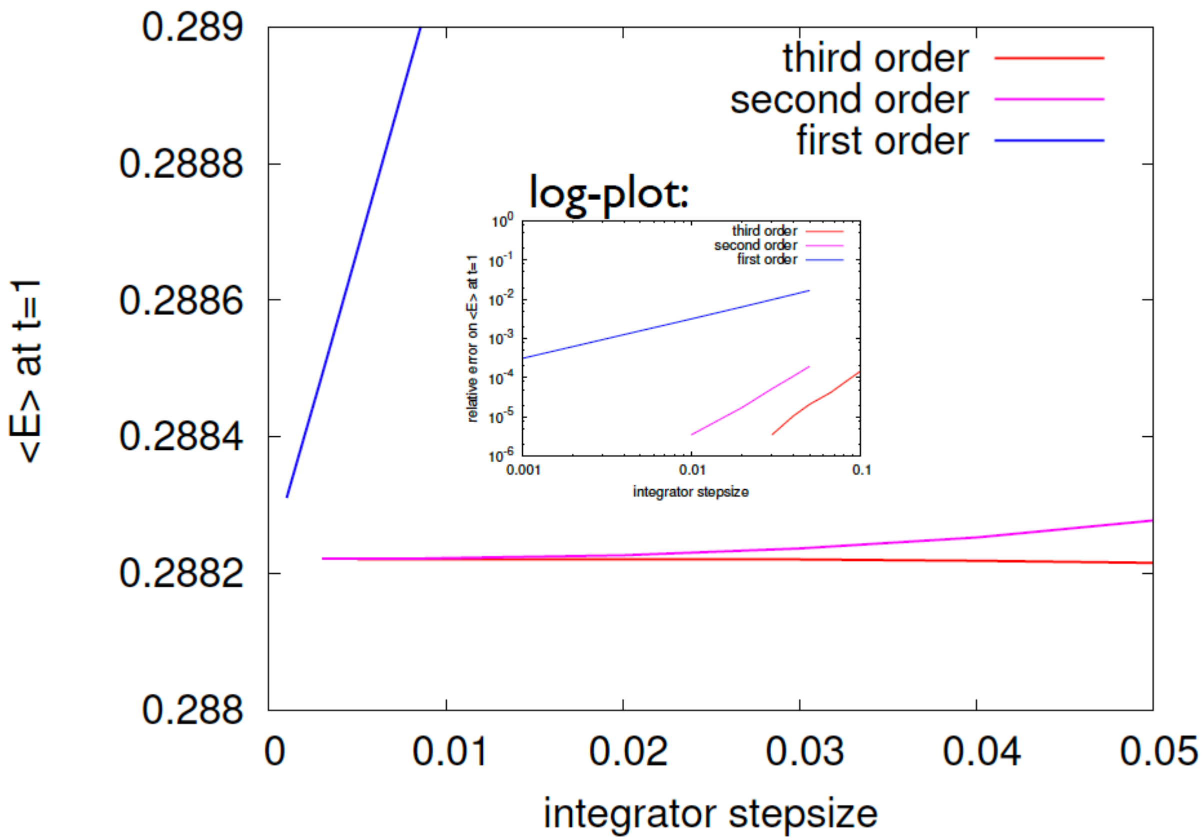
$$X_0 = V_t, \quad \text{build } Q_0 \text{ from } X_0$$
$$X_1 = \exp(\mathrm{i}\frac{\rho}{2} Q_0) X_0, \quad \text{build } Q_1 \text{ from } X_1$$
$$V_{t+\rho a^2} = \exp(\mathrm{i}\rho Q_1 - \mathrm{i}\frac{\rho}{2} Q_0) X_1 \quad \text{or} \quad \exp(\mathrm{i}\rho Q_1) X_0$$

- **3rd order integrator [Lüscher, arXiv:1006.4518]**

$$X_0 = V_t, \quad \text{build } Q_0 \text{ from } X_0$$
$$X_1 = \exp(\mathrm{i}\frac{\rho}{4} Q_0) X_0, \quad \text{build } Q_1 \text{ from } X_1$$
$$X_2 = \exp(\mathrm{i}\frac{8\rho}{9} Q_1 - \mathrm{i}\frac{17\rho}{36} Q_0) X_1, \quad \text{build } Q_2 \text{ from } X_2$$
$$V_{t+\rho a^2} = \exp(\mathrm{i}\frac{3}{4}\rho Q_2 - \mathrm{i}\frac{8}{9}\rho Q_1 + \mathrm{i}\frac{17}{36}\rho Q_0) X_2$$

- **Szabolcs Borsanyi: Methods of ... (Bad Honnef School 2023)**

Numerical flow integration



$$
\begin{aligned}
X_0 &= V_t, \\
X_1 &= \exp\left(\frac{1}{4} Z_0\right) X_0, \\
X_2 &= \exp\left(\frac{8}{9} Z_1 - \frac{17}{36} Z_0\right) X_1, \\
V_{t+\epsilon} &= \exp\left(\frac{3}{4} Z_2 - \frac{8}{9} Z_1 + \frac{17}{36} Z_0\right) X_2
\end{aligned}
$$

Lüscher: JHEP 1008 (2010) 071

$$
\begin{aligned}
X_0 &= V_t, \\
X_1 &= \exp\left(\frac{1}{2} Z_0\right) X_0, \\
V_{t+\varepsilon} &= \exp\left(Z_1\right) X_0
\end{aligned}
$$

midpoint method

$$
\begin{aligned}
X_0 &= V_t, \\
V_{t+\varepsilon} &= \exp\left(Z_0\right) X_0
\end{aligned}
$$

Morningstar&Peardon
Phys.Rev. D69 (2004) 054501

Continuum strategies with gradient flow

Flow time in lattice units defined through limit $n \to \infty, \rho \to 0$ with $n\rho = \text{fixed}$

$$t/a^2 \;\simeq\; n_\text{stout}\, \rho_\text{stout} \;\simeq\; \text{cumulative sum of used } \rho$$

whereupon t has dimension [area]=[length^2]=[mass^{-2}], while $\rho_\text{stout} \in \mathbb{R}$.

Two options for "keeping flow time fixed" in continuum limit:
(1) keep t/a^2 ["flow-time in lattice units"] fixed as $a \to 0$
(2) keep t ["flow-time in physical units"] fixed as $a \to 0$

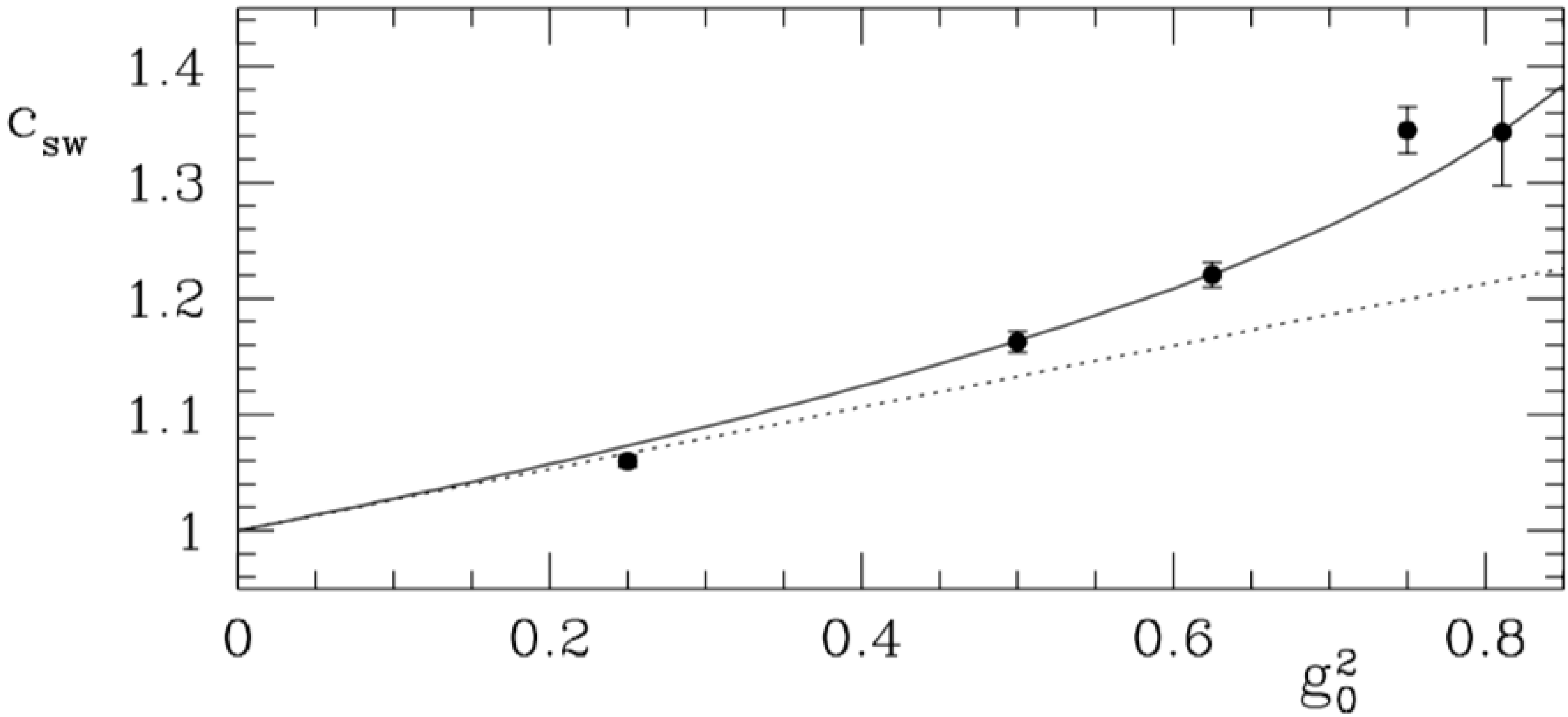
(1) CPU demand (number of calls to stout routine) stays *constant* under $a \to 0$
(2) CPU demand *proliferates* [scilicet $\propto a^{-2}$] at fixed $\rho \simeq 0.01$ under $a \to 0$

(1) diffusion radius $\sqrt{8t} \propto a$ goes to zero as $a \to 0$; "ultralocal modification"
(2) diffusion radius $\sqrt{8t} \propto r_0$ defines physical distance as $a \to 0$; "new regulator"

Strategy (2) opens option for defining new regularization and renormalization schemes ("flow schemes") for quantities like $\alpha_\text{st}^\text{GF}(1/\sqrt{8t})$ or $m_q^\text{GF}(1/\sqrt{8t})$ or ... [Lüscher]

Lattice perturbation theory

- calculate quantities in Symanzik improvement program, e.g. $m_{\rm crit}$ and $c_{\rm SW}$
- matching to continuum schemes, e.g. $m_q^{\rm stag,\,Sym,\,stout,\dots}(a^{-1}) \to m_q^{\rm RGI}$ or $m_q^{\overline{\rm MS}}(\mu)$



Lüscher et al. "Nonperturbative O(a) improvement of lattice QCD" [hep-lat/9609035]

Lattice gluon propagator

• Relate link variables $U_\mu(x) \in SU(N_c)$ to gluon fields $A_\mu(x) \in su(N_c)$ (hermitean)

$$U_\mu(x) = \exp(\mathrm{i}g_0 A_\mu(x)) \qquad \text{with} \qquad A_\mu(x) = \sum_{a=1}^{N_c^2-1} A_\mu^a(x) T^a$$

• Apply to desired actions and expand to requested order in g_0 to get Feynman rules. Two gluon actions (plaquette, Symanzik) and two fermion actions (Wilson, Brillouin).

• Gauge-fixing and Fadeev-Popov similar to continuum (work in covariant gauge).
• Haar measure $DU = e^{-S_\mathrm{meas}} DA$ from DA on $su(N_c)$ with known S_meas.
• All together perturbation theory for Yang-Mills theory from $S_\mathrm{glue} + S_\mathrm{meas} + S_\mathrm{gf} + S_\mathrm{gh}$

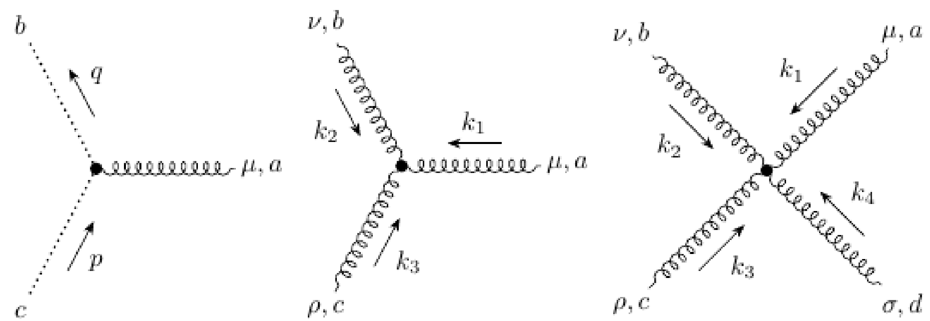


Figure 4.1: *Gauge vertices for one-loop lattice perturbation theory.*

$$\hat{k}_\mu = \frac{2}{a}\sin(\frac{a}{2}k_\mu)\ ,\ \hat{k}^2 = \sum_\mu \hat{k}_\mu^2$$

$$G_{\mu\nu}^{ab}(k) = \delta^{ab}\frac{1}{\hat{k}^2}\Big(\delta_{\mu\nu} - (1-\xi)\frac{\hat{k}_\mu\hat{k}_\nu}{\hat{k}^2}\Big)$$

$$G_\mathrm{gh}^{ab}(k) = \delta^{ab}\frac{1}{\hat{k}^2}$$

Lattice fermions: Naive/Wilson/Brillouin

- **Naive fermions**

$$D_{\rm nai}(x,y) = \sum_\mu \gamma_\mu \nabla_\mu(x,y) + m\delta_{x,y}$$

$$D_{\rm nai}(p) = \mathrm{i}\sum_\mu \gamma_\mu \frac{1}{a}\sin(ap_\mu) + m$$

$$= \mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + m \quad \text{with} \quad \bar{p}_\mu \equiv \frac{1}{a}\sin(ap_\mu)$$

- **Wilson fermions**

$$D_{\rm W}(x,y) = \sum_\mu \gamma_\mu \nabla_\mu(x,y) - \frac{ra}{2}\sum_\mu \triangle_\mu(x,y) + m\delta_{x,y}$$

$$D_{\rm W}(p) = \mathrm{i}\sum_\mu \gamma_\mu \frac{1}{a}\sin(ap_\mu) + \frac{r}{a}\sum_\mu \{1-\cos(ap_\mu)\} + m$$

$$= \mathrm{i}\sum_\mu \gamma_\mu \bar{p}_\mu + \frac{ra}{2}\sum_\mu \hat{p}_\mu^2 + m \quad \text{with} \quad \hat{p}_\mu \equiv \frac{2}{a}\sin(\frac{ap_\mu}{2})$$

- **Brillouin fermions**

$$
\begin{aligned}
D_{\mathrm{B}}(x,y) &= -\frac{\lambda_0}{2} r\delta(x,y) + \sum_{\mu=\pm1}^{\pm4} \Big(\rho_1\gamma_\mu - \frac{\lambda_1}{2}r\Big) W_\mu(x)\delta(x+\hat\mu,y) \\
&+ \sum_{\mu,\nu=\pm1}^{\pm4} \Big(\rho_2\gamma_\mu - \frac{\lambda_2}{4}r\Big) W_{\mu\nu}(x)\delta(x+\hat\mu+\hat\nu,y) \\
&+ \sum_{\mu,\nu,\rho=\pm1}^{\pm4} \Big(\frac{\rho_3}{2}\gamma_\mu - \frac{\lambda_3}{12}r\Big) W_{\mu\nu\rho}(x)\delta(x+\hat\mu+\hat\nu+\hat\rho,y) \\
&+ \sum_{\mu,\nu,\rho,\sigma=\pm1}^{\pm4} \Big(\frac{\rho_4}{6}\gamma_\mu - \frac{\lambda_4}{48}r\Big) W_{\mu\nu\rho\sigma}(x)\delta(x+\hat\mu+\hat\nu+\hat\rho+\hat\sigma,y)
\end{aligned}
$$

with $|\mu| \neq |\nu|$ and $|\mu|,|\nu|,|\rho|$ pairwise unequal and $|\mu|,|\nu|,|\rho|,|\sigma|$ pairwise unequal.

$W_\mu(x)$ is smoothed link in μ-dir, $W_{\mu\nu}(x)$ in $\mu\nu$-dir, ... ,$W_{\mu\nu\rho\sigma}(x)$ in $\mu\nu\rho\sigma$-dir.

$(\rho_1,\rho_2,\rho_3,\rho_4) = \frac{1}{432}(64,16,4,1)$ and $(\lambda_0,...,\lambda_4) = \frac{1}{64}(-240,8,4,2,1)$ gives Brillouin.

$(\rho_1,\rho_2,\rho_3,\rho_4) = \frac{1}{2}(1,0,0,0)$ and $(\lambda_0,...,\lambda_4) = (-8,1,0,0,0)$ reproduces Wilson.

Propagators: Wilson/Brillouin

Compact notation for Wilson/Brillouin fermion propagator:

$$S_\mathrm{W} = \frac{-\mathrm{i}\sum_\mu \gamma_\mu \sin(ak_\mu) + 2r\sum_\nu \sin^2(\frac{a}{2}k_\nu)}{\sum_\mu \sin^2(ak_\mu) + 4r^2[\sum_\nu \sin^2(\frac{a}{2}k_\nu)]^2} = \frac{-\mathrm{i}\sum_\mu \gamma_\mu \bar{s}(k_\mu) + 2rs^2}{\bar{s}^2 + [2rs^2]^2}$$

$$S_\mathrm{B} = \frac{-\frac{\mathrm{i}}{27}\sum_\mu\{\gamma_\mu \bar{s}_\mu(k_\mu)\prod_{\nu\neq\mu}(\bar{c}(k_\nu)+2)\} + 2r[1-c^2(k_1)c^2(k_2)c^2(k_3)c^2(k_4)]}{\frac{1}{729}\sum_\mu\{\bar{s}^2(k_\mu)\prod_{\nu\neq\mu}(\bar{c}(k_\nu)+2)^2\} + 4r^2[1-c^2(k_1)c^2(k_2)c^2(k_3)c^2(k_4)]^2}$$

Standard abbreviations for trigonomentric functions (at $a=1$):

$$s_\mu = \sin(\frac{1}{2}k_\mu)\ ,\ c_\mu = \cos(\frac{1}{2}k_\mu)\ ,\ s^2 = \sum_\mu s(k_\mu)^2$$

$$\bar{s}_\mu = \sin(k_\mu)\ ,\ \bar{c}_\mu = \cos(k_\mu)\ ,\ \bar{s}^2 = \sum_\mu \bar{s}(k_\mu)^2$$

In addition clover improvement term (implicit fundamental color index summed over):

$$-\frac{c_\mathrm{SW}}{2}\sum_x \sum_{\mu<\nu} \bar{\psi}(x)\sigma_{\mu\nu}F_{\mu\nu}(x)\psi(x)$$

Vertices: Wilson/Brillouin/Clover

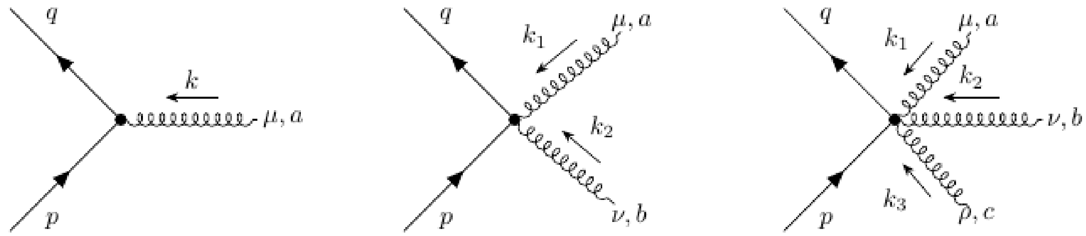


Figure 4.2: *Momentum assignments for the vertices with one, two and three gluons.*

Vertices for Wilson fermion:

$$\begin{aligned}
V_{1\mathrm{W}}{}^{a}_{\mu}(p,q) &= -g_0 T^a[\mathrm{i}\gamma_\mu c(p_\mu+q_\mu)+rs(p_\mu+q_\mu)] \\
V_{2\mathrm{W}}{}^{ab}_{\mu\nu}(p,q) &= \frac{1}{2}g_0^2 T^a T^b \delta_{\mu\nu}[\mathrm{i}\gamma_\mu s(p_\mu+q_\mu)-rc(p_\mu+q_\mu)] \\
V_{3\mathrm{W}}{}^{abc}_{\mu\nu\rho}(p,q) &= \frac{1}{6}g_0^3 T^a T^b T^c \delta_{\mu\nu}\delta_{\mu\rho}[\mathrm{i}\gamma_\mu c(p_\mu+q_\mu)+rs(p_\mu+q_\mu)]
\end{aligned}$$

Vertices for Brillouin fermion are significantly longer [arXiv:2302.11261].
Vertices for Clover term:

$$\begin{aligned}
V_{1\mathrm{C}}{}^{a}_{\mu}(p,q) &= \mathrm{i}g_0 T^a c_{\mathrm{SW}} \sum\nolimits_{\nu} \sigma_{\mu\nu} c(p_\mu-q_\mu)\bar{s}(p_\nu-q_\nu)/2 \\
V_{2\mathrm{C}}{}^{ab}_{\mu\nu}(p,q) &= g_0^2 T^a T^b c_{\mathrm{SW}} \ldots \\
V_{3\mathrm{C}}{}^{abc}_{\mu\nu}(p,q) &= \mathrm{i}g_0^3 T^a T^b T^c c_{\mathrm{SW}} \ldots
\end{aligned}$$

Self-energy: Wilson/Brillouin

- Tadpole and sunset diagrams

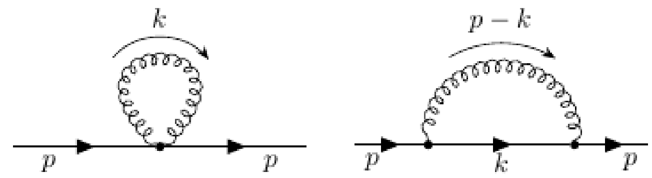


Figure 4.3: *Tadpole (left) and sunset (right) diagrams of the quark self energy.*

- Expand to order $1/a$

$$\Sigma = \frac{g_0^2 C_F}{16\pi^2}\Big(\frac{\Sigma_0}{a} + \Sigma_1\,\mathrm{i}\,\not{p} + O(a)\Big) = m_{\mathrm{crit}} + O(a,p)$$

- Evaluate tadpole and sunset integrals (Feynman gauge, i.e. $\xi = 1$)

$$g_0^2 C_F \Sigma_0^{(\mathrm{tadpole})} = \int_{-\pi}^{\pi} \frac{\mathrm{d}^4 k}{(2\pi)^4} \sum_{\mu,\nu,a} \Big[G_{\mu\nu}(k) V_{2\,\mu\nu}^{\;aa}(p,p,k,-k)\Big]_{p=0}$$

$$g_0^2 C_F \Sigma_0^{(\mathrm{sunset})} = \int_{-\pi}^{\pi} \frac{\mathrm{d}^4 k}{(2\pi)^4} \sum_{\mu,\nu,a} \Big[V_{1\,\mu}^{\;a}(p,k) G_{\mu\nu}(p-k) S(k) V_{1\,\nu}^{\;a}(k,p)\Big]_{p=0}$$

- **Result for self-energy of Wilson/Brillouin fermion on plaq/Sym glue**

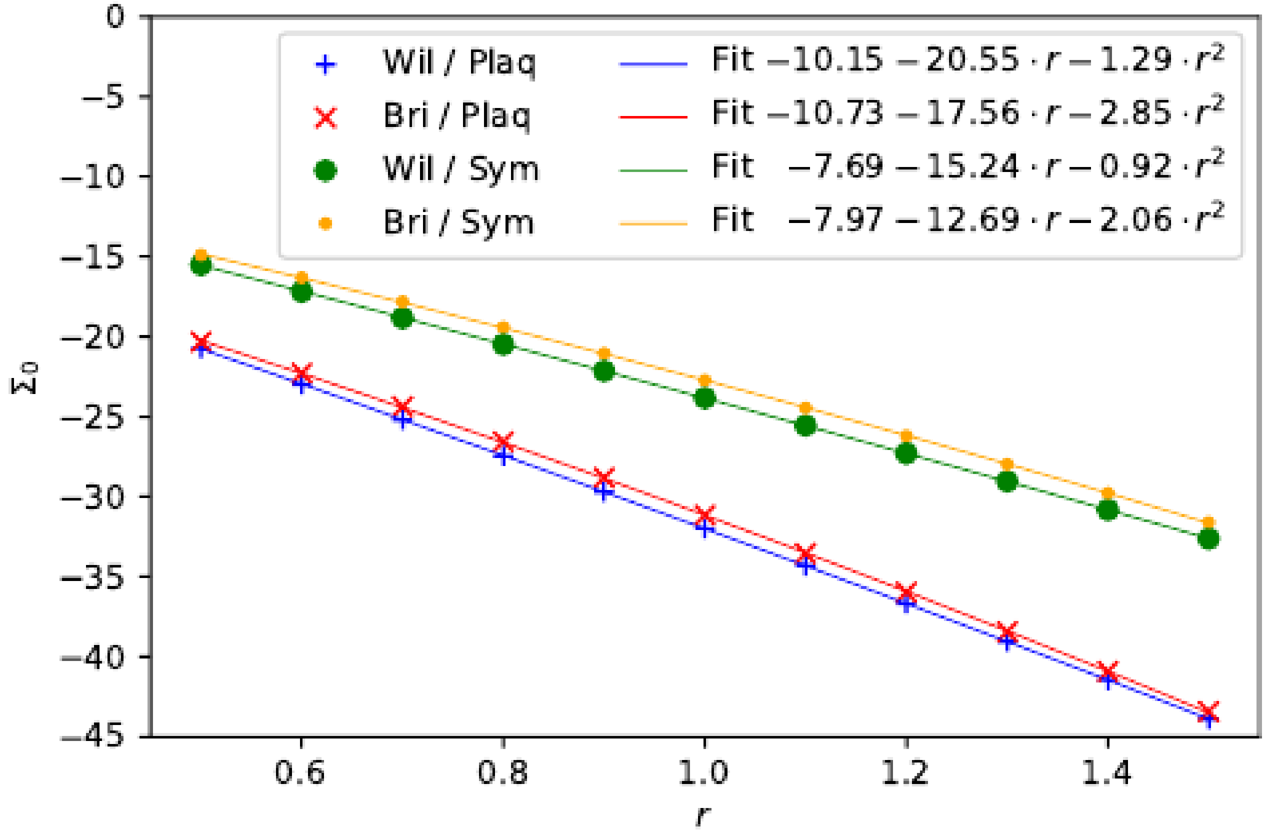


Figure 5.1: *Self energy Σ_0 of Wilson and Brillouin fermions as a function of r.*

$\longrightarrow$ additive mass shift significantly reduced by Symanzik glue (instead of plaquette)

$\longrightarrow$ additive mass shift minimally reduced by Brillouin fermion (instead of Wilson)

$c_{\rm SW}$ at tree level: Wilson/Brillouin fermion

- Perturbative expansion is $c_{\rm SW} = c_{\rm SW}^{(0)} + g_0^2 c_{\rm SW}^{(1)} + O(g_0^4)$
- At tree level only qqg vertex appears

$$\Lambda^{(0)}{}^a_\mu = V_1{}^a_\mu(p,p) = -g_0 T^a \Big\{ \mathrm{i}\gamma_\mu \Big[2\rho_1 + 12\rho_2 + 24\rho_3 + 16\rho_4 \Big]$$

$$+ a \Big[\frac{r}{2}(p_\mu + q_\mu)(\lambda_1 + 6\lambda_2 + 12\lambda_3 + 8\lambda_4) + \frac{\mathrm{i}}{2} c_{\rm SW}^{(0)} \sum\nolimits_\nu \sigma_{\mu\nu}(p_\nu - q_\nu) \Big] + O(a^2) \Big\}$$

- On-shell condition is

$$\bar{u}(q)\Lambda^{(0)}{}^a_\mu u(p) = -g_0 T^a \bar{u}(q) \Big\{ \mathrm{i}\gamma_\mu \Big[2\rho_1 + 12\rho_2 + 24\rho_3 + 16\rho_4 \Big]$$

$$+ \frac{a}{2} \Big[r(\lambda_1 + 6\lambda_2 + 12\lambda_3 + 8\lambda_4) - c_{\rm SW}^{(0)} \Big] (p_\nu + q_\nu) + O(a^2) \Big\} u(p)$$

- Since both $2\rho_1 + 12\rho_2 + 24\rho_3 + 16\rho_4 = 1$ and $\lambda_1 + 6\lambda_2 + 12\lambda_3 + 8\lambda_4 = 1$ necessary for legal action [satisfied by W/B], improvement condition is $c_{\rm SW}^{(0)} = r$ for W/B.

c_{SW} at one-loop level: Wilson/Brillouin fermion

- One-loop vertex function

$$g_0^3 \Lambda^{(1)}{}^a_\mu = -g_0^3 T^a \Big[\gamma_\mu F_1 + a\not{q}\gamma_\mu F_2 + a\gamma_\mu \not{p} F_3 + a(p_\mu + q_\mu) G_1 + a(p_\mu - q_\mu) H_1\Big]$$

- On shell [F_2, F_3 not contributing, $H_1 = 0$ due to symmetry, Aoki Kuramashi 2003]

$$g_0^3 \bar{u}(q)\Big[\mathrm{i}\gamma_\mu F_1 + \frac{a}{2}(p_\mu + q_\mu)(c_{\mathrm{SW}}^{(1)} - 2G_1)T^a\Big]u(p)$$

- Improvement condition is $c_{\mathrm{SW}}^{(1)} = 2G_1$ with G_1 from $\Lambda^{(1)}{}^a_\mu$ via 6 diagrams:

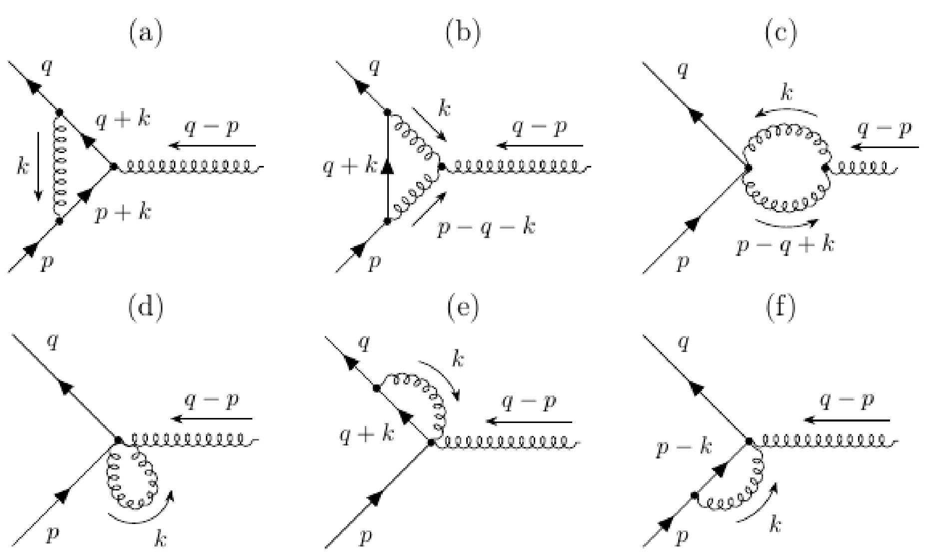


Figure 4.4: *The six one-loop diagrams contributing to the vertex function.*

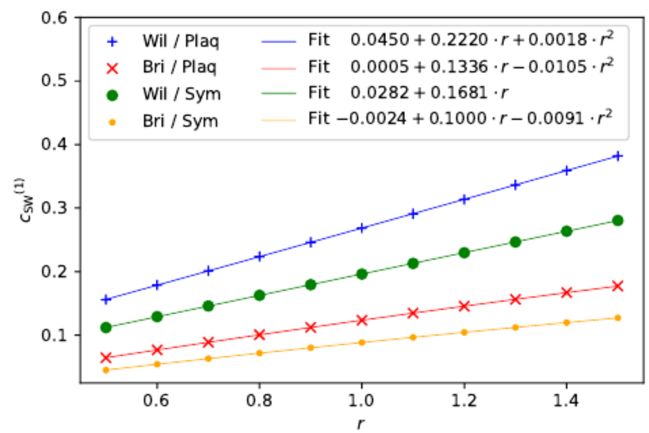


Figure 5.2: *The one-loop values of $c_{\mathrm{SW}}^{(1)}$ for Wilson and Brillouin fermions with $N_c = 3$ as a function of r.*

- Brillouin fermion brings major reduction
- Symanzik glue brings minor reduction

- **Divergence structure – Wilson results agree with Aoki Kuramashi 2003**

- sum of all diagrams is finite

- individually only (d) is finite, other five are IR-divergent

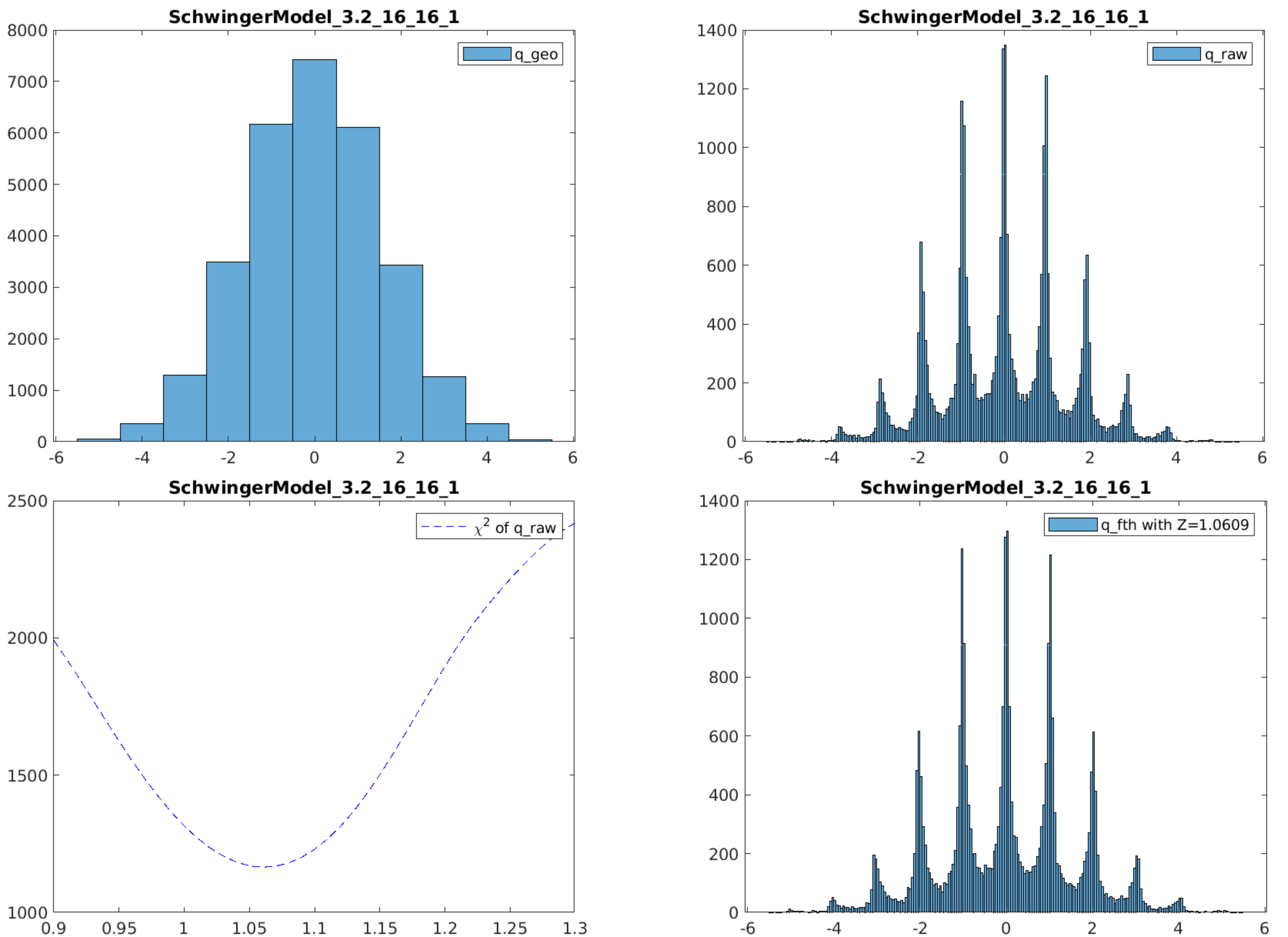
- regulate (a,b,c,e,f) by subtracting log. div. lattice integral with appropriate prefactor

$$\mathcal{B}_2 = \int_{-\pi}^{\pi} \frac{d^4k}{(2\pi)^4} \frac{1}{(\hat{k}^2)^2} \qquad \text{gives} \qquad \mathcal{B}_2(\mu) = \frac{1}{16\pi^2}\Big[-\ln(\mu^2) + F_0 - \gamma_E\Big] + O(\mu^2)$$

diag.	$\propto \mathcal{B}_2$	Wilson/plaq	Brillouin/plaq	Wilson/Sym	Brillouin/Sym
(a)	$-1/3$	0.009852153(1)	0.0100402212(1)	0.01048401(1)	0.0108335(1)
(b)	$-9/2$	0.125895883(1)	0.098371668(1)	0.1285594(1)	0.102829(1)
(c)	$+9/2$	-0.124125079(1)	-0.100558858(1)	-0.1337781(1)	-0.1098254(1)
(d)	0	0.297394534(1)	0.142461144(1)	0.2354388(1)	0.1120815(1)
(e)	$+1/6$	-0.020214623(1)	-0.013344189(1)	-0.022229808(1)	-0.013659(1)
(f)	$+1/6$	-0.020214623(1)	-0.013344189(1)	-0.022229808(1)	-0.013659(1)
sum	0	0.26858825(1)	0.12362580(1)	0.1962445(1)	0.088601(1)

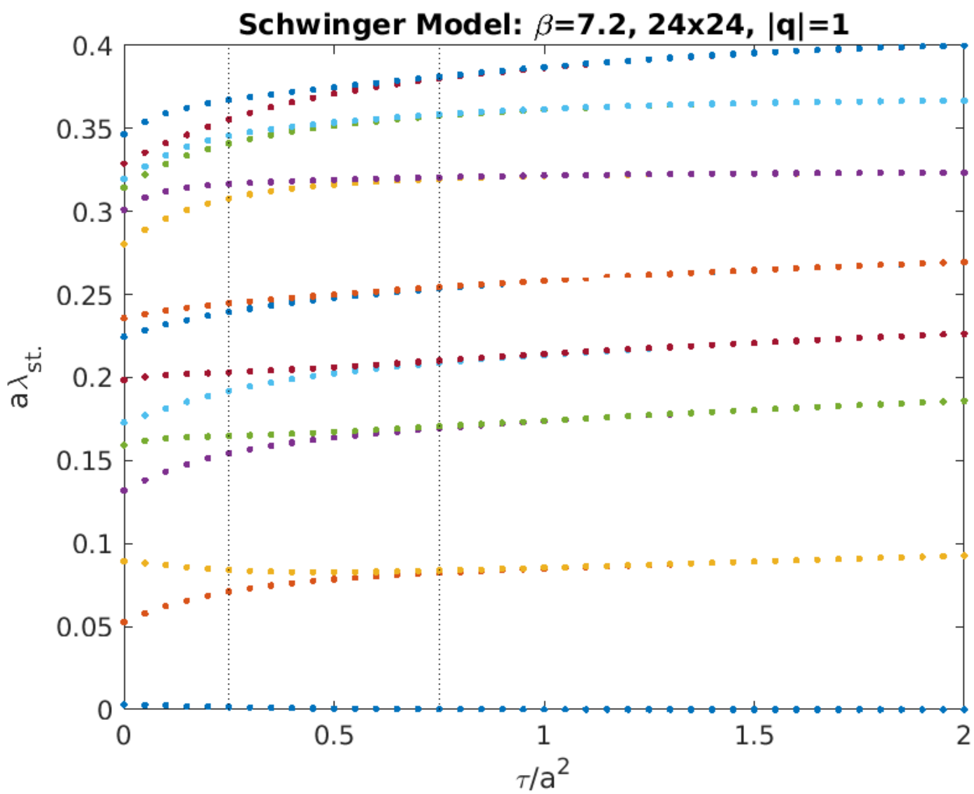
More details (dependence on r, N_c, ...) in [arXiv:2302.11261]
Stout-smearing and gradient-flow details in [arXiv:2406.03493]
Combination to appear (hopefully) soon [arXiv:25??.?????]

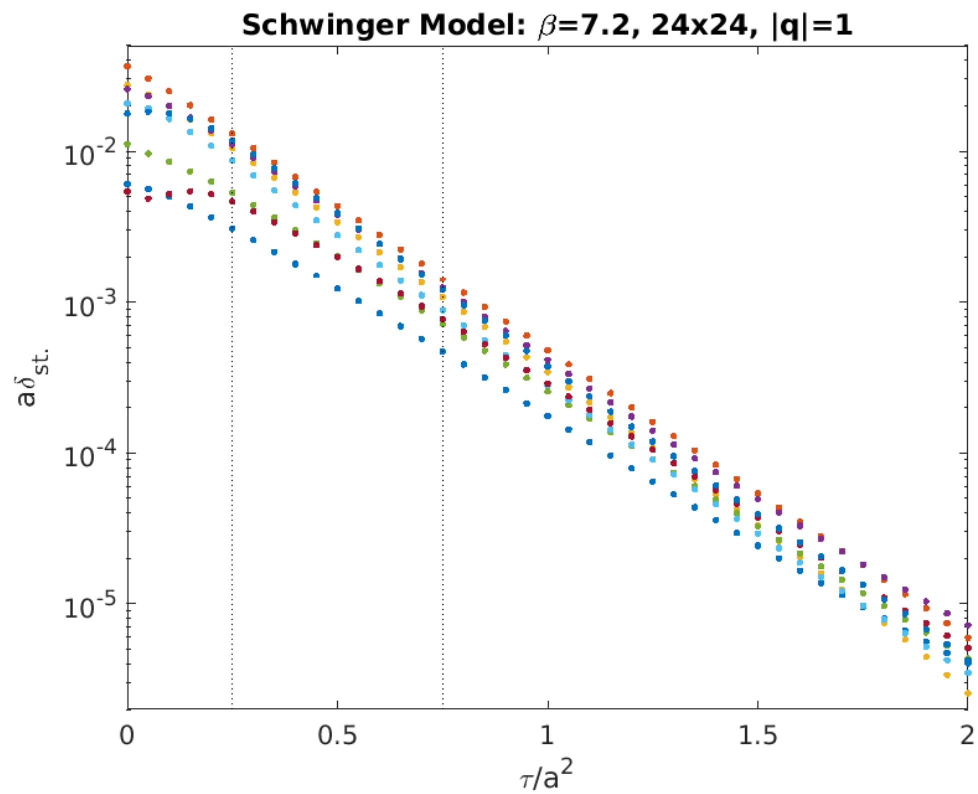
Schwinger Model: topological charge distribution



Taste splittings: $a\delta_{\rm stag}$ under gradient flow

Eigenvalues $\pm i\lambda_1, ..., \pm i\lambda_{15}$ of $D_{\rm stag}$ on a $|q|=1$ configuration at $(\beta, L/a) = (7.2, 24)$ versus gradient flow time τ (everything in lattice units).
Note that λ_1 pairs with $-\lambda_1$ for $|q| = 1$, while $\lambda_2 \simeq \lambda_3$ and $\lambda_4 \simeq \lambda_5$ pair, and so on.
Splittings defined with proper pairing: $\delta_1 = 2\lambda_1$, $\delta_2 = \lambda_3 - \lambda_2$ and so on for $|q| = 1$.





$\longrightarrow$ for $D_{\rm stag}$ splittings decrease exponentially with the gradient flow time

Topological susceptibility in SU(3) theory in 4D

Gluonic definition of the field-strength tensor $F_{\mu\nu}(x) = F_{\mu\nu}(x)^a T^a$ with $T^a = \lambda^a/2$ via clover-plaquette. Based on this *local* topological charge density is

$$q_{\rm nai}(x) = \frac{1}{4\pi^2}\mathrm{Tr}[F_{12}(x)F_{34}(x) - F_{13}(x)F_{24}(x) + F_{14}(x)F_{23}(x)]$$

In this case the topological susceptibility [with $V = Na^4$ the physical box volume]

$$\chi_{\rm top} = Z_q^2(\beta)\chi_{\rm nai} + M(\beta) \qquad \text{with} \qquad \chi_{\rm nai} = \frac{a^4}{N}\sum\nolimits_{x,y\in\Lambda} q_{\rm nai}(x)q_{\rm nai}(y)$$

renormalizes multiplicatively and additively [Campostrini,DiGiacomo,Alles,...]. Still

$$q_{\rm ren} = \mathrm{round}(Z_q(\beta)q_{\rm nai}) \qquad \text{with} \qquad q_{\rm nai} = a^4\sum\nolimits_{x\in\Lambda} q_{\rm nai}(x)$$

is a gluonic definition of the *global* topological charge distribution P_q which renormalizes only multiplicatively [CP symmetry]. Based on it one may define

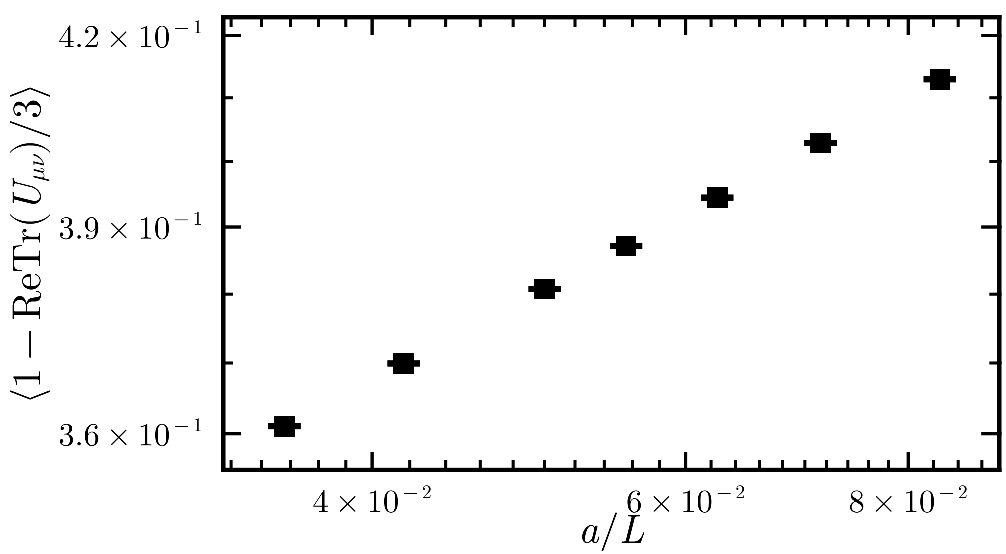
$$\chi_{\rm top} = \langle q_{\rm ren}^2\rangle/V$$

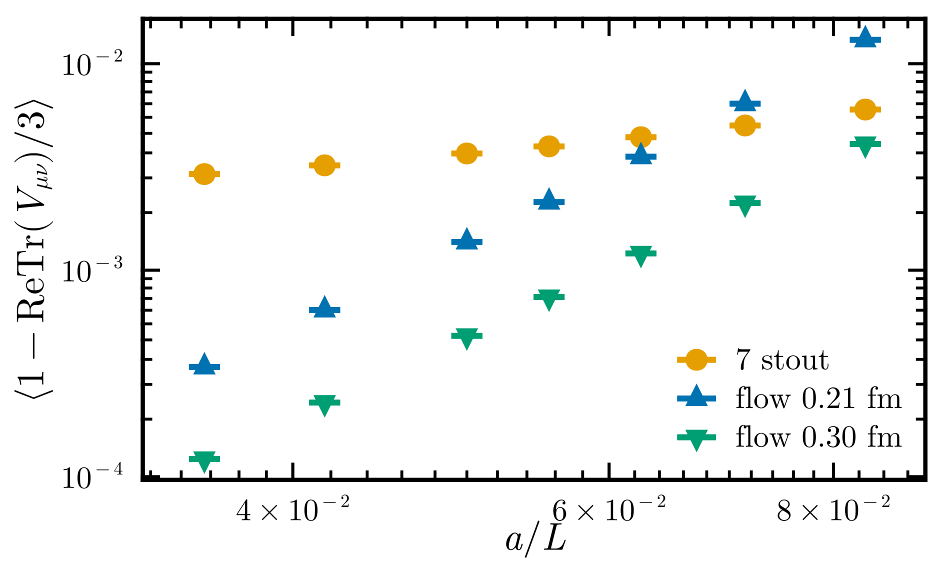
and higher moments without further renormalization [P_q is fully renormalized quantity].

Fixed t/a^2 versus fixed t in fixed $V = (2.4783\, r_0)^4$

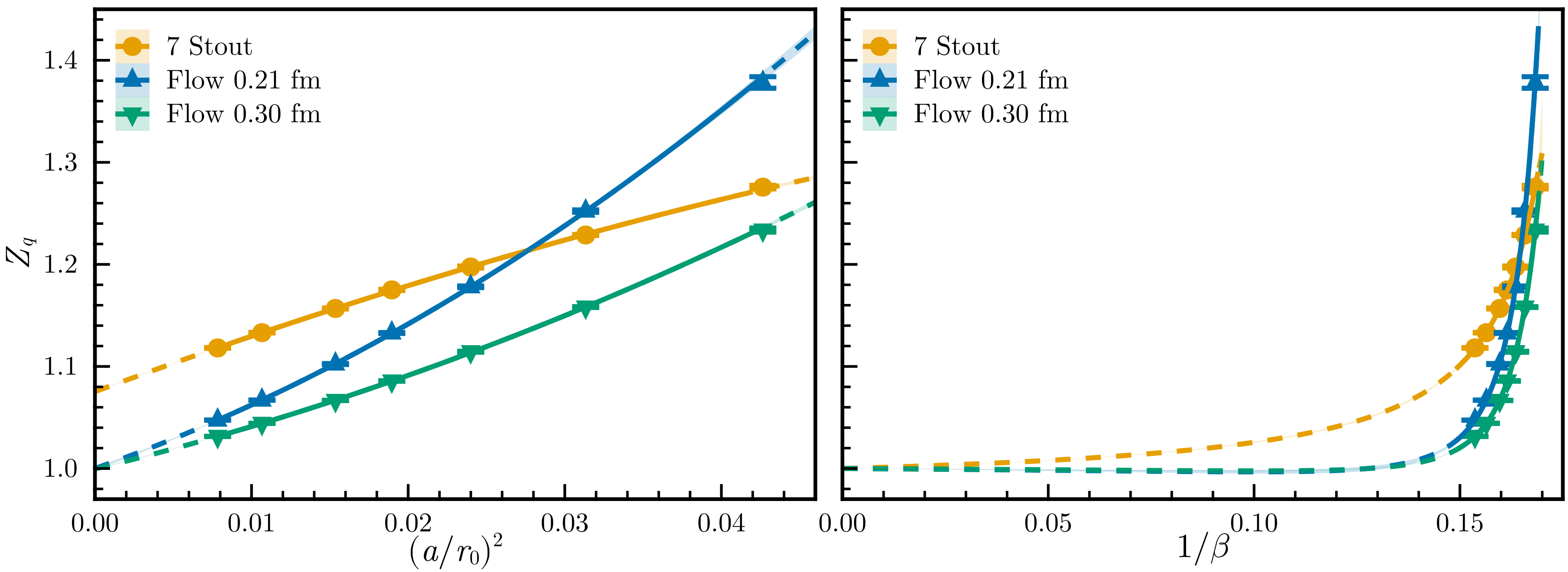
L/a	β	r_0/a	a [fm]	7 stout	flow 0.21 fm	flow 0.30 fm
12	5.9421	4.8420	0.101	7×0.12	$9 \times 0.06 = 0.54$	$9 \times 0.12 = 1.08$
14	6.0314	5.6490	0.087	7×0.12	$12 \times 0.06125 = 0.735$	$12 \times 0.1225 = 1.47$
16	6.1142	6.4560	0.076	7×0.12	$16 \times 0.06 = 0.96$	$16 \times 0.12 = 1.92$
18	6.1912	7.2630	0.067	7×0.12	$20 \times 0.06075 = 1.215$	$20 \times 0.1215 = 2.43$
20	6.2629	8.0700	0.061	7×0.12	$25 \times 0.06 = 1.5$	$25 \times 0.12 = 3.00$
24	6.3929	9.6841	0.051	7×0.12	$36 \times 0.06 = 2.16$	$36 \times 0.12 = 4.32$
28	6.5079	11.298	0.043	7×0.12	$49 \times 0.06 = 2.94$	$49 \times 0.12 = 5.88$

"7 stout" keeps flow time in lattice units at $t/a^2 = 0.84$ to give $\sqrt{8t} = \sqrt{6.72}\, a \to 0$.
"flow 0.21 fm" sets flow time to $t/a^2 = (N/4)^2 0.06$ to give $\sqrt{8t} = 0.429\, r_0 \simeq 0.21$ fm.
"flow 0.30 fm" sets flow time to $t/a^2 = (N/4)^2 0.12$ to give $\sqrt{8t} = 0.607\, r_0 \simeq 0.30$ fm.





Impact of smoothing strategy on Z_q

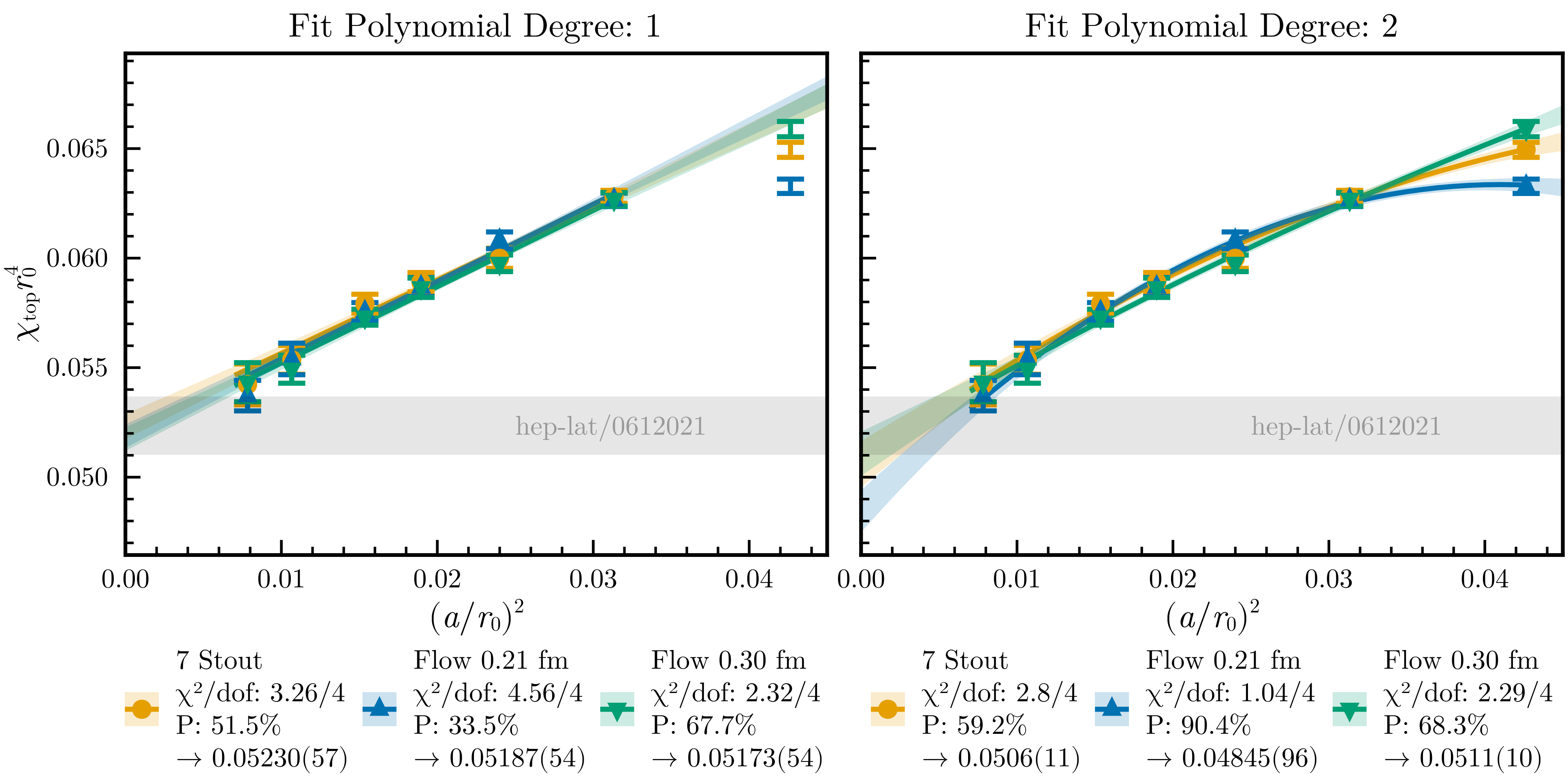


Z_q factors involved, with quadratic fits in $(a/r_0)^2$ (left) and rational fits in g_0^2 (right). All smoothing strategies yield consistent values; final result (extrapolations next slide)

$$\chi_{\rm top}^{1/4} = \frac{0.4769(18)}{0.4757(64)\,\mathrm{fm}} = 197.8(0.7)(2.7)\,\mathrm{MeV}$$

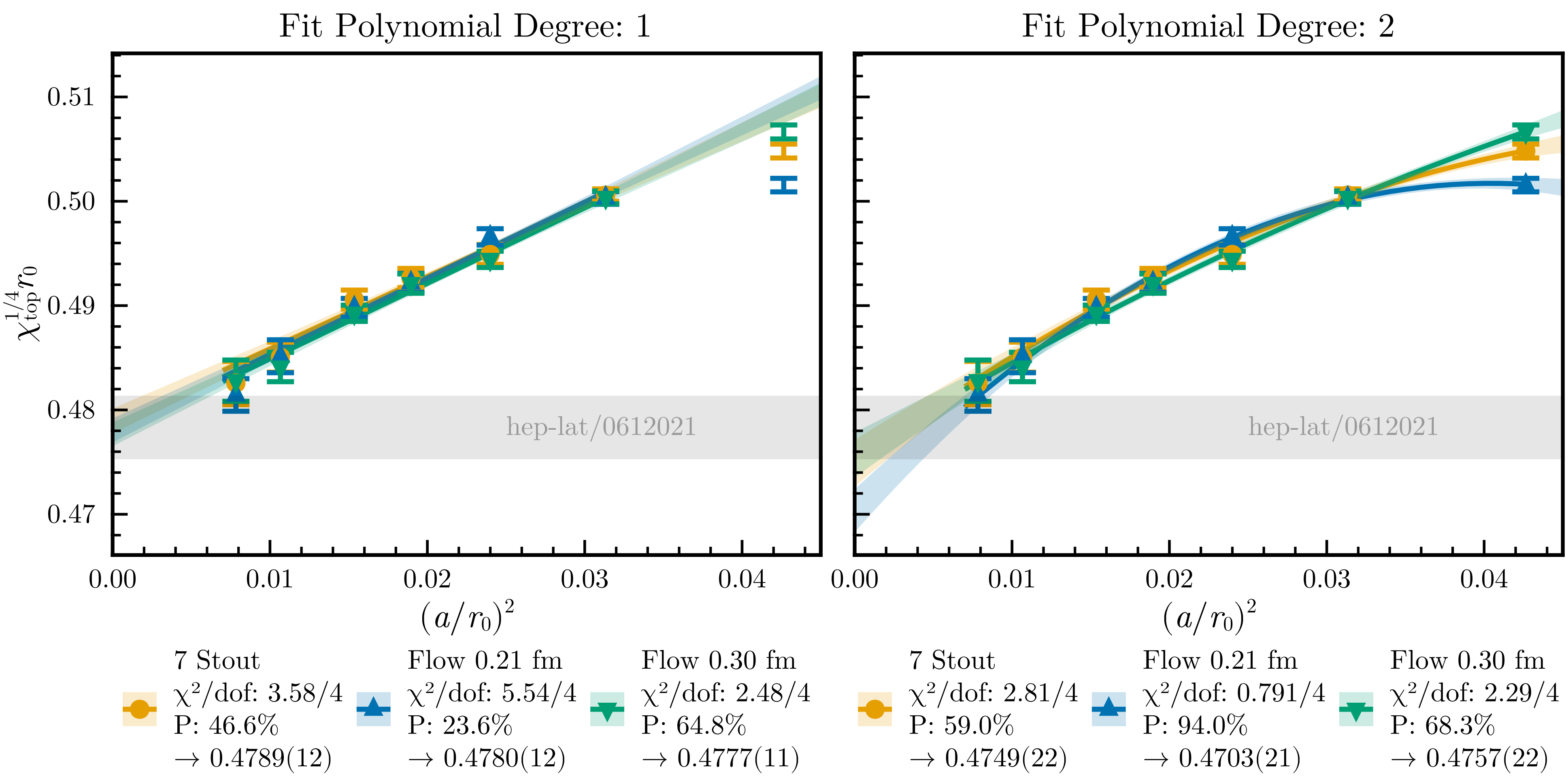
with $18^2 = \mathrm{stat}^2 + \mathrm{syst}^2$ and r_0 from [Asmussen:2024hfw]. For details: 2501.08217

Continuum extrapolation of $\chi_{\rm top}\, r_0^4$



	$\chi_{\rm top} r_0^4$	$\chi_{\rm top}^{1/4} r_0$	combined
7 stout	0.05194(68)(149)=$[0.4774(16)(34)]^4$	0.4780(14)(37)	0.4776(15)(35)(01)
flow 0.21 fm	0.05105(64)(342)=$[0.4753(15)(80)]^4$	0.4761(14)(77)	0.4759(15)(79)(01)
flow 0.30 fm	0.05159(64)(030)=$[0.4766(15)(07)]^4$	0.4773(13)(13)	0.4769(14)(10)(03)

Continuum extrapolation of $\chi_{\rm top}^{1/4} r_0$



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flow 0.30 fm	0.05159(64)(030)=$[0.4766(15)(07)]^4$	0.4773(13)(13)	0.4769(14)(10)(03)

Summary

- stout smearing is ingenious tool to smoothen gauge field, keeping links unitary
- gradient flow is defined by using stout as 1st-order integrator to flow time $t/a^2 = n\rho$
- two legitimate strategies for keeping flow time fixed in continuum limit:
  (1) keep t/a^2 ["flow-time in lattice units"] fixed as $a \to 0$
  (2) keep t ["flow-time in physical units"] fixed as $a \to 0$
- two different physical situations:
  (1) diffusion radius $\sqrt{8t} \propto a$ goes to zero as $a \to 0$; "ultralocal modification"
  (2) diffusion radius $\sqrt{8t} \propto r_0$ defines physical distance as $a \to 0$; "new regulator"
- lattice perturbation theory expands $U_\mu(x) = \exp(\mathrm{i}g_0 A_\mu(x))$ in powers of g_0
- examples were Σ_0 and c_{SW} for Wilson/Brillouin fermions (soon with stout/flow)
- gradient flow exhibits staggered eigenvalue pairs/quartets (invisible at $t/a^2 = 0$)
- "7 stout" and "flow 0.21 fm" and "flow 0.30 fm" give consistent results for $\chi_{\mathrm{top}} r_0^4$

Schwinger Model: QED in 2D with any N_f

SM at $N_f\!=\!0$ simulated with Metropolis/overrelax/instanton-hit/parity-hit.
Topological charge autocorrelation time is $O(1)$ at any β [arXiv:1203.2560].

Wilson gauge action per site:

$s_{\mathrm{wil}}(x) = 1\!-\!\mathrm{Re}(U(x)) = 1\!-\!\cos(\theta(x))$

Plaquette at position $x=(x_1,x_2)$:

$U(x) = U_1(x)U_2(x\!+\!e_1)U_1^\dagger(x\!+\!e_2)U_2^\dagger(x)$

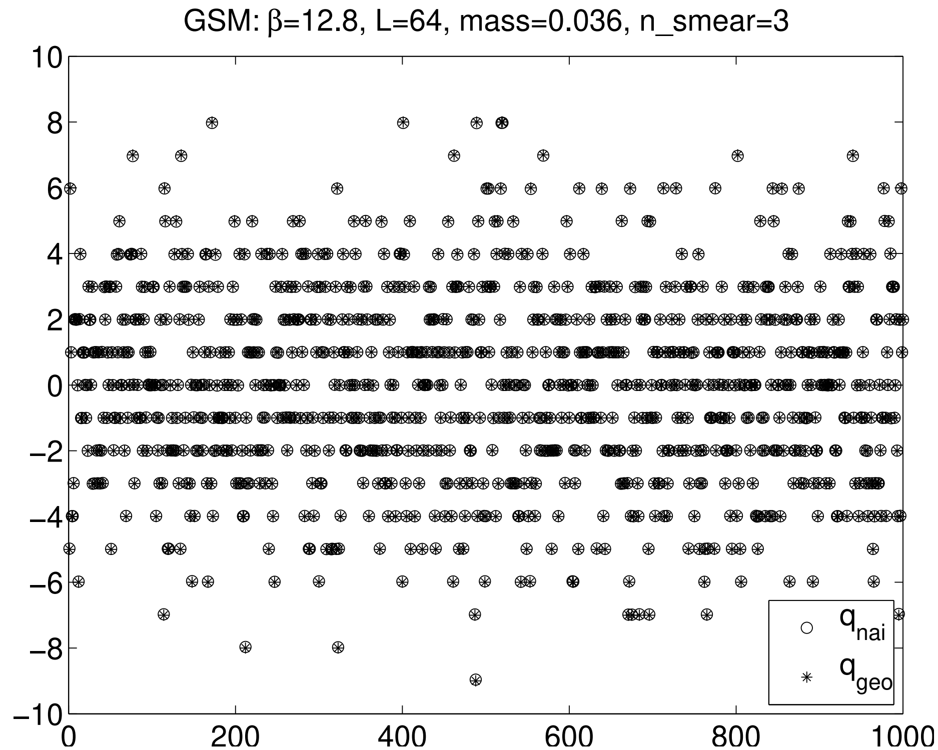
$U(x) = \exp(\mathrm{i}\theta(x))$

Two gluonic topological charges:

$q_{\mathrm{raw}}^{(n)} = \sum\sin(\theta^{(n)}(x))/(2\pi) \in \mathbf{R}$ ("fth/Z")

$q_{\mathrm{geo}}^{(n)} = \sum\theta^{(n)}(x)/(2\pi) \in \mathbf{Z}$ ("geometric")

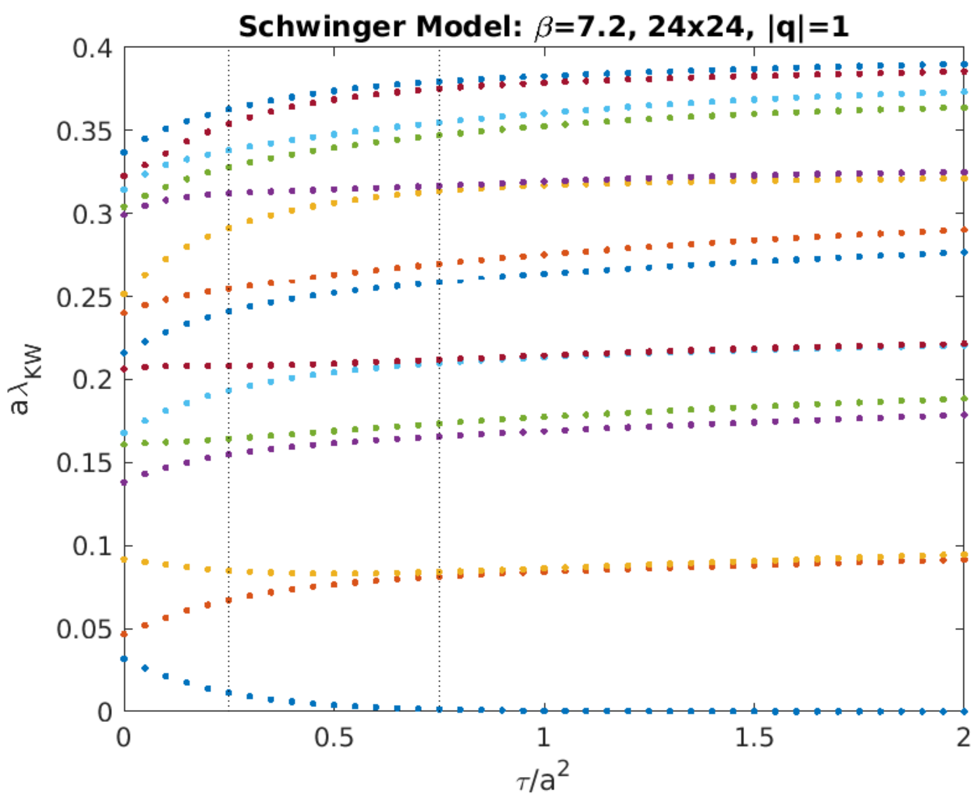
$\theta^{(n)}$ plaquette angle after n smearings

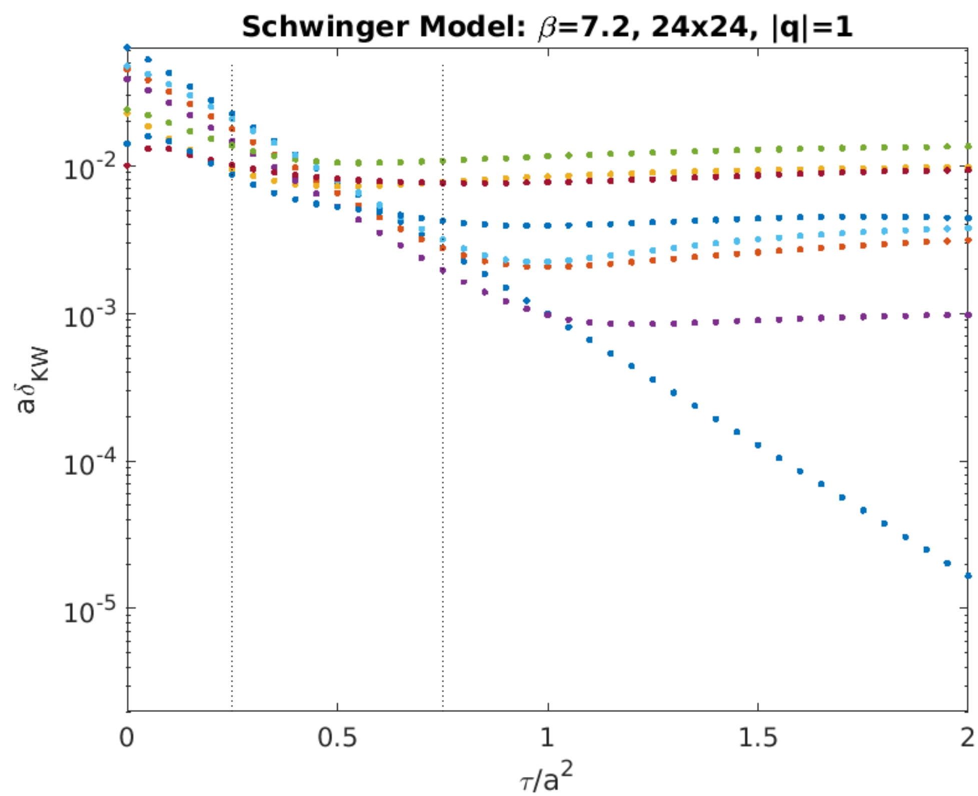


Operators use $n\!=\!0,1,3$ steps of $\rho\!=\!0.25$ stout-smearing [Morningstar Peardon 2003].
Flowtime $\tau/a^2=0.75$ reached by $(n,\rho)=(3,0.25)$ or $(5,0.15)$ or $(15,0.05)$ or ...

Taste splittings: $a\delta_{\mathrm{KW}}$ under gradient flow

Eigenvalues $\pm i\lambda_1, ..., \pm i\lambda_{15}$ of D_{KW} on a $|q|=1$ configuration at $(\beta, L/a) = (7.2, 24)$ versus gradient flow time τ (everything in lattice units).
Note that λ_1 pairs with $-\lambda_1$ for $|q| = 1$, while $\lambda_2 \simeq \lambda_3$ and $\lambda_4 \simeq \lambda_5$ pair, and so on.
Splittings defined with proper pairing: $\delta_1 = 2\lambda_1$, $\delta_2 = \lambda_3 - \lambda_2$ and so on for $|q| = 1$.

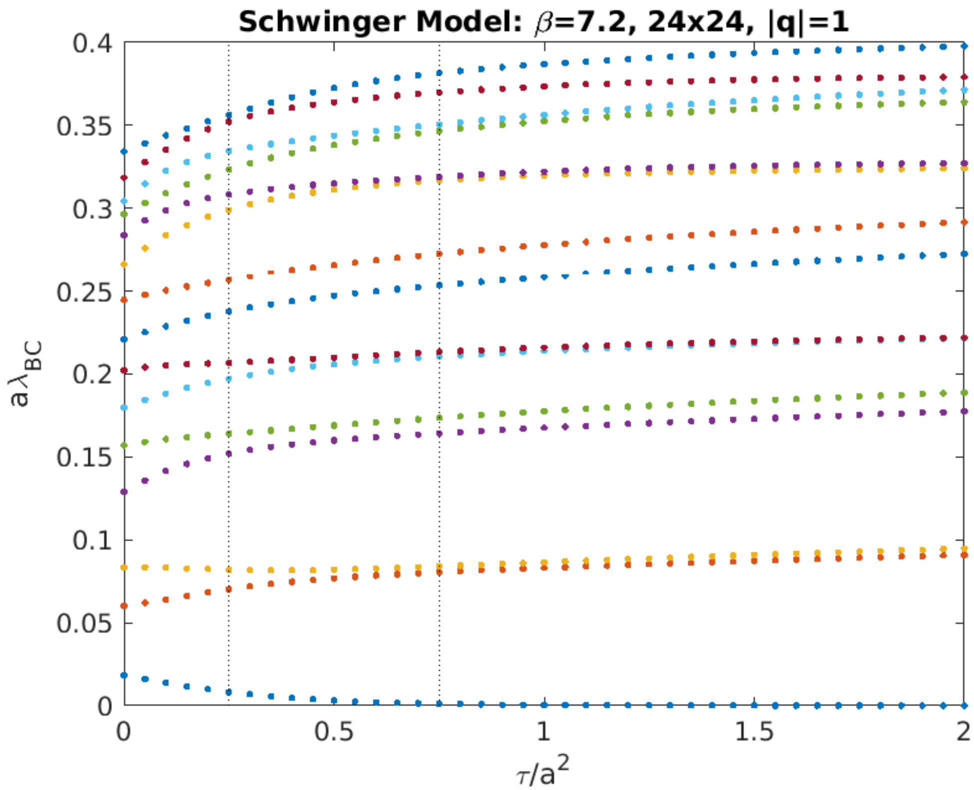


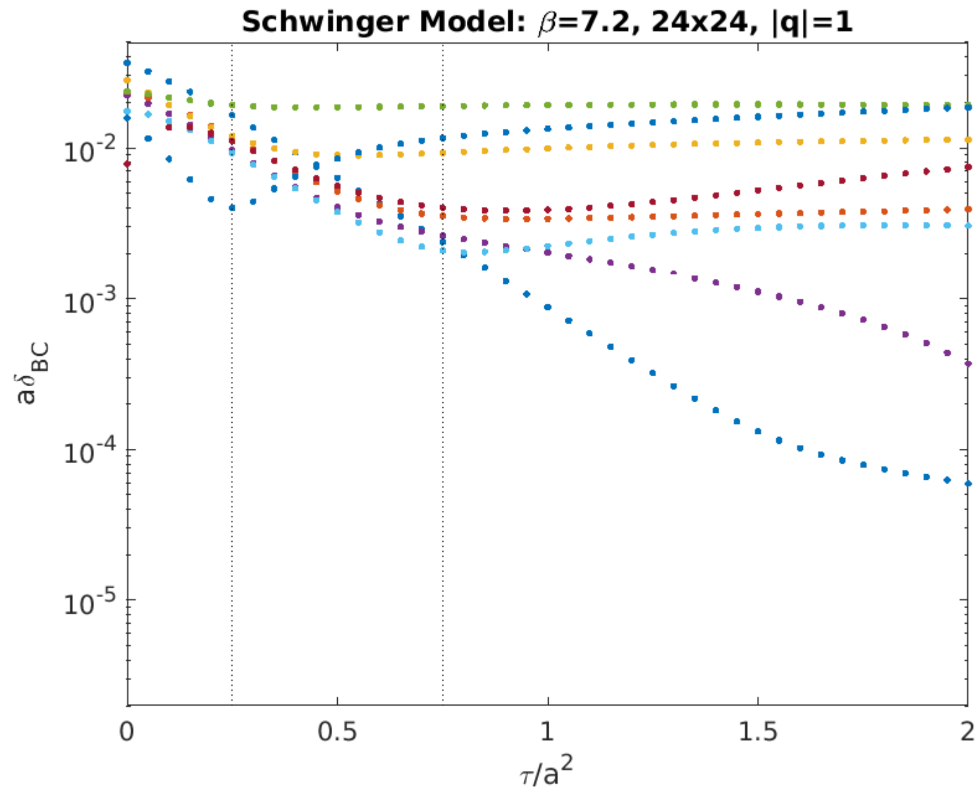


$\longrightarrow$ for D_{KW} some splittings stop decreasing after some gradient flow time

Taste splittings: $a\delta_{\mathrm{BC}}$ under gradient flow

Eigenvalues $\pm\mathrm{i}\lambda_1, ..., \pm\mathrm{i}\lambda_{15}$ of D_{BC} on a $|q|\!=\!1$ configuration at $(\beta, L/a) = (7.2, 24)$ versus gradient flow time τ (everything in lattice units).
Note that λ_1 pairs with $-\lambda_1$ for $|q| = 1$, while $\lambda_2 \simeq \lambda_3$ and $\lambda_4 \simeq \lambda_5$ pair, and so on.
Splittings defined with proper pairing: $\delta_1 = 2\lambda_1$, $\delta_2 = \lambda_3 - \lambda_2$ and so on for $|q| = 1$.





$\longrightarrow$ for D_{BC} decrease of splittings with gradient flow time seems more chaotic