Quark mass effects in the gradient flow at higher orders in perturbation theory

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Mass corrections

Mass effects of $S(t) = \langle \bar{\chi}(t)\chi(t) \rangle$ and $R(t) = \langle \bar{\chi}(t) \overleftrightarrow{\not{D}} \chi(t) \rangle$ can be used to supplement lattice data in precision determination of quark masses. [Takaura, Harlander & Lange 2024].

Aim: Calculate these to processes to three loop level.

- Much machinery has been created to compute short flow time expansions perturbatively.
- Can only consider mass corrections of certain processes, such as VEVs.

Outline of calculations

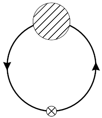
To perform these computations we can make use of the setup created for massless computations:

- Choose an operator and calculate its Feynman rule: `frules`
  [Harlander & Geuskens (unpublished)].
- Consider diagrams contributing to the process and calculate their expansion in terms of gradient flow vacuum bubbles: `qgraf`, `tapir`, `exp` and `form`.
  [Nogueira 1991; Gerlach, Herren, Lang 2022; Harlander, Seidensticker, Steinhauser 1998; Seidensticker 1999; Vermaseren 1989].
- Integration by parts can be performed with `Kira` + `firefly`.
  [Klappert, Lange 2019; Klappert, Klein, Lange 2020].
- Numerically evaluate the integrals for each flow time: `ftint`
  [Harlander, Nellopoulos, Olsson & Wesle 2025].

$S(t)$ and $R(t)$

Both processes are described by diagrams that are one-point functions with an operator insertion.

- $S(t) = \langle \bar{\chi}(t) \chi(t) \rangle$.
  Operator Feynman rule: δ_{ij}.



Diagrams contributing to $S(t)$ and $R(t)$.

- $R(t) = \langle \bar{\chi}(t) \overleftrightarrow{\not{D}} \chi(t) \rangle$.
  Operator Feynman rules:

$$-2i\delta_{ij}\not{k} \quad \text{and} \quad -2gT^a_{ij}\not{A}^a.$$

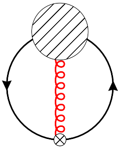


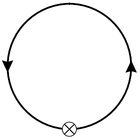
Diagram contributing to $R(t)$.

Diagrams drawn with FeynGame.

[Bündgen, Harlander, Klein, Schaaf 2025]

$S(t)$ at the one loop level

- At one loop, simple quark loop
- Factor out dimensionful scale and write in terms of dimensionless ratio



One loop contribution to $S(t)$.

$$
\begin{aligned}
S(t) &\approx -\left(\frac{3m}{4\pi^2 t}\right)(4\pi t)^{\frac{D}{2}}\, t^{-1} \int_k \frac{e^{-2tk^2}}{k^2+m^2} \\
&\approx -\left(\frac{3m}{4\pi^2 t}\right)\left(1 - 2m^2 t e^{-2m^2 t}\Gamma(0, 2m^2 t)\right).
\end{aligned}
$$

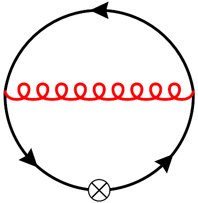
[Lüscher 2013; Harlander 2021].

Approximations to $S(t)$

No closed form solution is known at the two-loop level:

- The full mass expansion has also been considered numerically to this order.
- Small mass expansion is known to $\mathcal{O}(m^2 t)$ and $\mathcal{O}(\alpha_s)$.
- Large mass expansion is known to $\mathcal{O}\left(\frac{1}{(m^2 t)^2}\right)$

[H. Takaura, R.V. Harlander & F. Lange 2024].



Renormalization

Two loop level introduces divergences that must be renormalized.

- Both Green's functions are renormalized with the flowed quark renormalization constant

$$S(t) = Z_\chi S(t)_0 \quad \text{and} \quad R(t) = Z_\chi R(t)_0$$

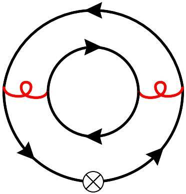
- Propagator mass also requires renormalization, according to

$$\frac{1}{p^2 + m_0^2} \;=\; \frac{1}{p^2 + m^2}\left(1 - (Z_m^2 - 1)\frac{m^2}{p^2 + m^2} + ...\right),$$

where $(Z_m^2 - 1) = \mathcal{O}(\alpha_s)$.

Massless contributions

- At three loop level $S(t)$ introduces internal quark loops.
- These loops can still contribute if we have additional massless quarks.
- $R(t)$ also allows for massless contributions.



- **How should additional quarks be considered?**

Integrals

Up to the three-loop level for our processes we find integrals:

Integral Order	$\mathcal{O}(\alpha_s^0)$	$\mathcal{O}(\alpha_s)$	$\mathcal{O}(\alpha_s^2)$
No. Integrals	$\mathcal{O}(1)$	$\mathcal{O}(10)$	$\mathcal{O}(1000)$

Typically, two loop gradient flow calculations have the form

$$f(\mathbf{c},\mathbf{a},\mathbf{b}) = (4\pi t)^{-D}\left(t^{\sum_i b_i}\right)\int_0^1 \mathbf{u}^{\mathbf{c}}\,d\mathbf{u}\int_{p,k}\frac{\exp\left(-a_1p^2-a_2k^2-a_3(p-k)^2\right)}{(p^2+m_1^2)^{b_1}(k^2+m_2^2)^{b_2}((p-k)^2+m_3^2)^{b_3}}.$$

where $\{b_i, m_i\} \in \mathbf{b}$

ftint

ftint is a program for the numerical evaluation of one-point gradient flow integrals. [Harlander, Nellopoulos, Olsson & Wesle 2025].
Allows for massive flowtime integrals up to the three-loop level.

It is split into two stages:

- A sector decomposition step based on pySecDec:
  [Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk & Zirke 2018]

  ```
  ftint_pySecDec.py [-epsorder EPSORDER] in_ftint
  ```
- A numerical evaluation step

  ```
  ftint_integrate.py [-masses MASSES] run_directory
  ```

Sector decomposition

Approximate time taken for sector decomposition at different loop orders

Loop	Integral	Time (s)
1	One loop massless	$\mathcal{O}(1)$ sec
1	One loop massive	$\mathcal{O}(1)$ sec
2	Two loop massless	$\mathcal{O}(1)$ sec
2	Two loop massive	$\mathcal{O}(10)$ sec
3	Three loop massless	$\mathcal{O}(1)$ min
3	Three loop massive	$\mathcal{O}(10)$ min

- Integrals can take much less or more time depending on the propagator structure.
- Generally sector decomposition is more computationally expensive than numerical integration evaluation.

Numerical Results

One example for $t = 0.223\text{GeV}^{-2}$

$$\begin{aligned} S(t) &= -0.0581 - 0.778\alpha_s - 0.0584\alpha_s^2 \\ &\quad + \left(9.21 \times 10^{-16}\alpha_s + 1.47 \times 10^{-7}\alpha_s^2\right)\epsilon^{-1} \\ &\quad -3.09 \times 10^{-9}\alpha_s^2\epsilon^{-2}, \end{aligned}$$

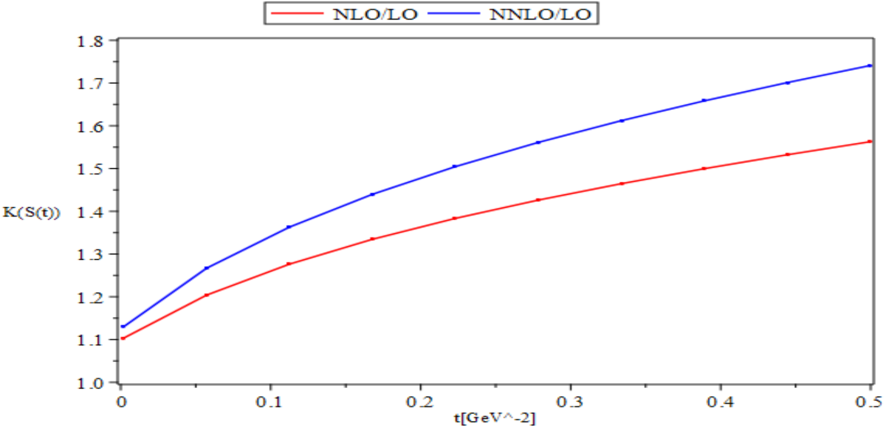
which has an approximate numerical accuracy

$$\begin{aligned} \Delta S(t) &\approx +10^{-17} + 10^{-8}\alpha_s + 10^{-6}\alpha_s^2 \\ &\quad + \left(10^{-15}\alpha_s + 10^{-7}\alpha_s^2\right)\epsilon^{-1}. \\ &\quad +10^{-9}\alpha_s^2\epsilon^{-2}. \end{aligned} \tag{1}$$

$$\alpha_s(\mu_t) \approx 0.377 \quad \text{where} \quad \mu_t = 1.18\text{GeV}.$$

where $\mu_t = \frac{1}{\sqrt{2te^{\gamma_E}}}$.

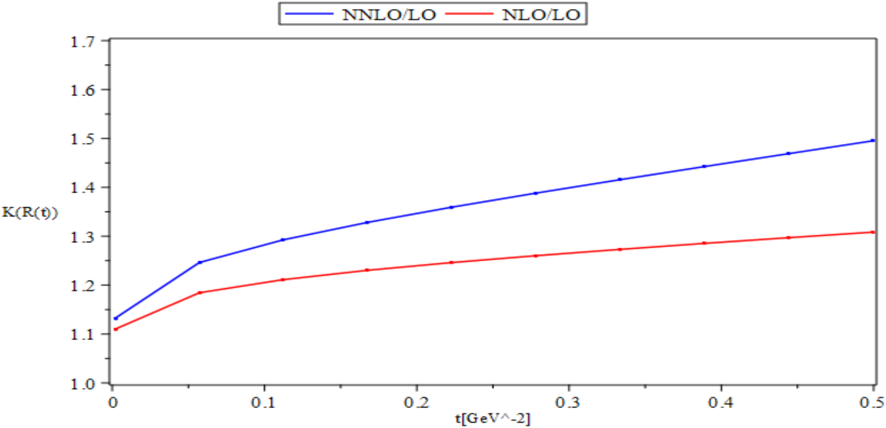
$\langle\bar{\chi}(t)\chi(t)\rangle$



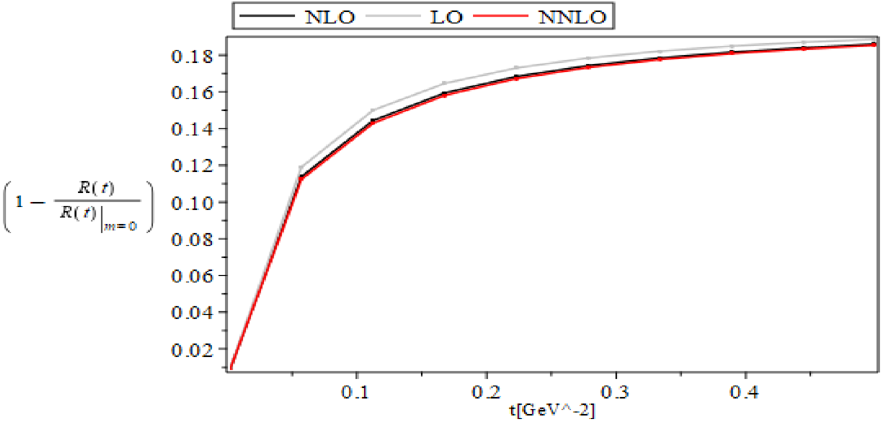
Running is taken into account with RunDec using:
[K.G. Chetyrkin, J.H. Kühn, M. Steinhauser 2020]

$$m_b(m_b) = 4.18\text{GeV}, \quad M_Z = 91.188\text{GeV} \quad \text{and} \quad \alpha_s(M_Z) = 0.1179.$$

$\langle \bar{\chi}(t) \overleftrightarrow{\not{D}} \chi(t) \rangle$



$\langle \bar{\chi}(t) \overleftrightarrow{\not{D}} \chi(t) \rangle$



Conclusion

- Can calculate various VEVs that do not require the method of projectors.
- These expressions are computed numerically and therefore only known for certain values of $m^2 t$.
- Can calculate these to the three loop level.
- These computations could be used in precision physics.

Conclusion

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- These expressions are computed numerically and therefore only known for certain values of $m^2 t$.
- Can calculate these to the three loop level.
- These computations could be used in precision physics.

Any questions?