The short-flow-time expansion and its applications

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QUARK-GLUON TOMOGRAPHY COLLABORATION

Zürich Gradient Flow Workshop 2025

HadStruc

Overview

My charge:

“short-flow-time expansion and its applications”

First thought: I’ll just quickly list and review all its applications

- Energy density
- Quark bilinears
- Energy-momentum tensor
- Hadron vacuum polarisation
- Electroweak Hamiltonian
- EFTs for the Standard Model
- Quark masses
- Four-quark operators for flavour physics
- Nonlocal operators for parton distribution functions
- Local operators for moments of parton distribution functions
- …

See talks by:
R Harlander
J Borgulat
H Takaura
R Mason
A Shindler
…

And those are just the ones from yesterday…

Overview

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“~~short-flow-time expansion and its applications~~”

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Second thought: I’m radically underqualified to discuss this topic in this setting

“Some thoughts on the short-flow-time expansion and one application”

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Second thought: I’m radically underqualified to discuss this topic in this setting

“Some thoughts on the short-flow-time expansion and one application”

Third thought: proceed assuming I do not need to introduce the gradient flow

Narayanan & Neuberger, JHEP 03 (2006) 064
Luscher, CMP 293 (2010) 899
Luscher, JHEP 08 (2010) 071
Luscher & Weisz, JHEP (2011) 051
Luscher, JHEP 04 (2013) 123

Outline

Operator product expansions and the short flow-time expansion

An application of the short flow-time expansion

Systematic and quantitative error estimates

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Preliminary report on ongoing work by Alex Sturzu at William & Mary as part of the HadStruc Collaboration

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Operator product expansions

Expansion proposed by Wilson

Wilson PR 179 (1969) 1499

$$\mathcal{O}_1(x)\mathcal{O}_2(y) \sim \sum_{\mathcal{O}} c_{\mathcal{O}}(x-y,\mu)\,\mathcal{O}_R(0,\mu) + \mathcal{O}((x-y)^n)$$

where the sum runs over local renormalised composite operators, ordered by dimension, and subject to relevant symmetries (independent of spontaneous symmetry breaking)

Collins, Renormalization (1984) CUP
Weinberg, The Quantum Theory of Fields Vol. II (1995) CUP

Defines renormalised composite operators without involving any regularisation

Enables one to draw conclusions about behaviour of $\mathcal{O}_1(x)\mathcal{O}_2(y)$ in the limit $(x-y) \to 0$ for which the composite operator is singular

Wilson coefficients

In the operator product expansion

$$\mathcal{O}_1(x)\mathcal{O}_2(y) \sim \sum_{\mathcal{O}} c_{\mathcal{O}}(x-y,\mu)\,\mathcal{O}_R(0,\mu) + \mathcal{O}((x-y)^n)$$

Coefficient functions are

- Singular functions in the limit $(x-y) \to 0$ that behave, up to logarithms, as

  $$c_{\mathcal{O}}(x-y,\mu) \sim (x-y)^{d_{\mathcal{O}_R}-d_{\mathcal{O}_1}-d_{\mathcal{O}_2}}$$

- Independent of external states and momenta
  - Essential for determination of Wilson coefficients
- In asymptotically free theories, calculable in perturbation theory
  - Hard scale provided by operator separation
- Obey renormalisation group equations

Some technical comments

The validity of this expansion is nontrivial

- limit is nontrivial in Minkowski spacetime
- corresponding limit in momentum space nontrivial but physically interesting

For example $q^\mu \to \infty$ with q^2 finite is high-energy scattering

Radius of convergence

- Conformal field theories: at least as large as largest circle that excludes other operator insertions
- Converges in φ^4 theories
- QCD: unclear if any exists!

Holland et al., CMP 342 (2016) 385

Proof of existence to all orders in perturbation theory by Zimmerman

Short flow-time expansion

Expansion proposed by Lüscher and Weisz

Luscher & Weisz, JHEP (2011) 051

$$B_\mu(\tau) = c_B(\mu^2\tau)A_R(\mu) + \cdots$$

$$B_\mu(\tau) \sim \left(2b_0\overline{g}^2\left(\frac{1}{\sqrt{8\tau}}\right)\right)^{1/2-c_0/(2b_0)} R(g)A_R(\mu) + \cdots$$

For composite operators, such as

$$E = \frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu}$$

$$E_{\rm sub}(\tau) = c_E(\mu^2\tau)\frac{1}{4}\left[G^a_{\mu\nu}G^a_{\mu\nu}\right]_R(\mu) + \mathcal{O}(\tau)$$

See Robert Harlander's talk yesterday

Harlander & Neumann, JHEP 06 (2016) 161

Short flow-time expansion

In general

$$\mathcal{O}(\tau) \sim \sum_{\mathcal{O}} c_{\mathcal{O}}(\mu^2\tau)\, \mathcal{O}_R(\mu) + \mathcal{O}((\Lambda^2\tau)^n)$$

Suzuki, PTEP (2013) 083B03
"Smeared OPE" - CJM & Orginos, PRD 91 (2015) 074513
"Flowed OPE" - Harlander et al., JHEP 08 (2020) 109

Given a complete set of operators, this expansion can be inverted

The inverse expansion is a more natural formulation from the lattice perspective

To extract perturbative coefficients requires carefully-chosen Greens functions

$$\langle p|\mathcal{O}(\tau)|p\rangle = \sum_{\mathcal{O}} c_{\mathcal{O}}(\mu^2\tau)\langle p|\mathcal{O}_R(\mu)|p\rangle + \mathcal{O}((\Lambda^2\tau)^n)$$

Solved order-by-order in the coupling constant

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Solved order-by-order in the coupling constant

Natural to associate the small flow-time expansion with perturbation theory, but from the lattice perspective, the coefficients can, and sometimes must, be determined nonperturbatively

Short flow-time expansion

Particularly powerful in the presence of mixing with lower-dimensional operators

Examples given yesterday

(1) CP-violating operators relevant to nucleon electric dipole moments
    Four-quark effective operators
(2) Twist-2 operators on the lattice

For power-divergent mixing in the continuum, short flow-time coefficient must be determined nonperturbatively

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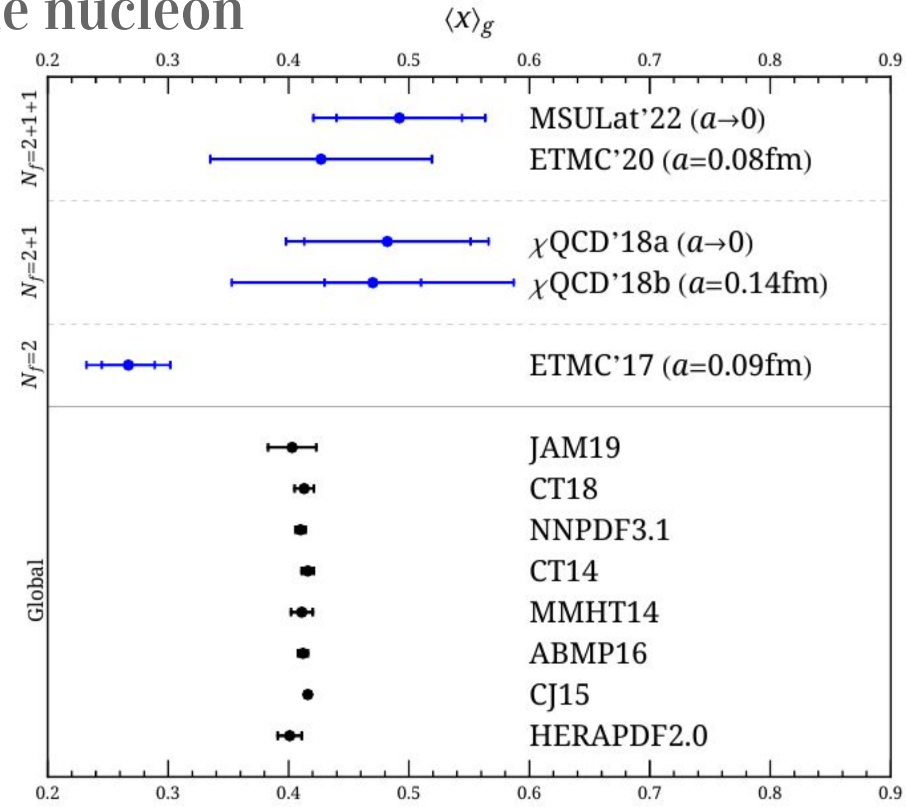
An application of the short flow-time expansion

Gluon momentum fraction of the nucleon

Systematic and quantitative error estimates

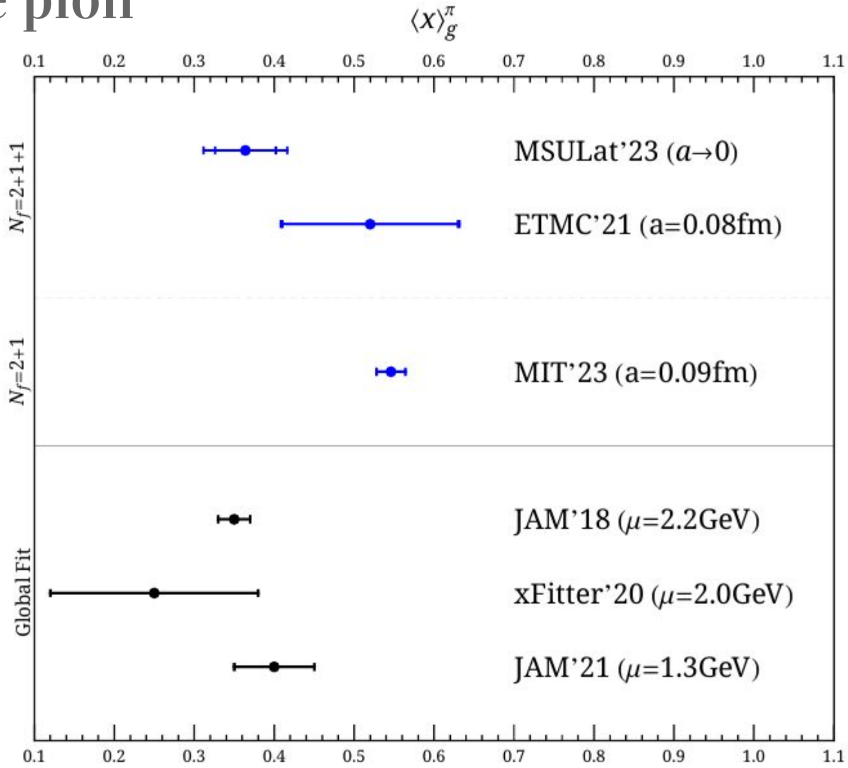
Gluon momentum fraction of the nucleon



Fan et al., PRD 107 (2023) 034505

Note contrast with A Shindler's talk, which referred to the precise calculations of quark PDF moments!

Gluon momentum fraction of the pion



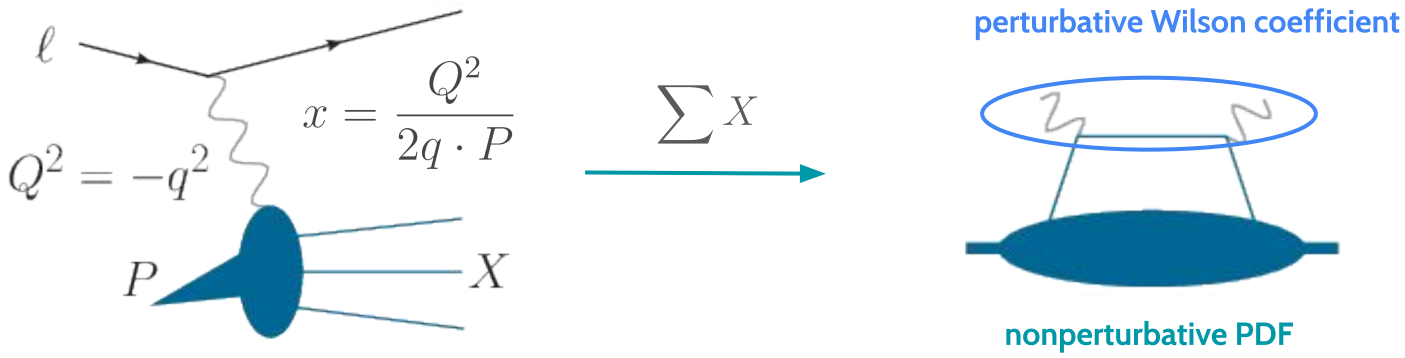
Good et al., PRD 109 (2024) 114509

Parton distribution functions

Parton distribution functions (PDFs) encode longitudinal partonic structure of hadrons

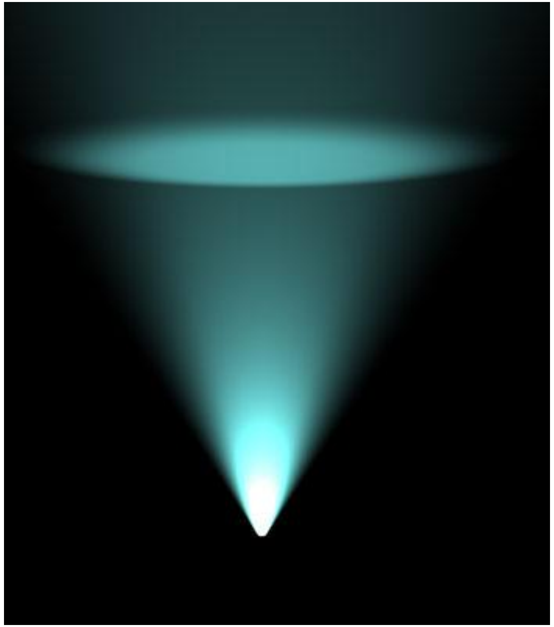
- Directly connect the standard model to nuclear physics
- Important source of systematic uncertainties at hadron colliders

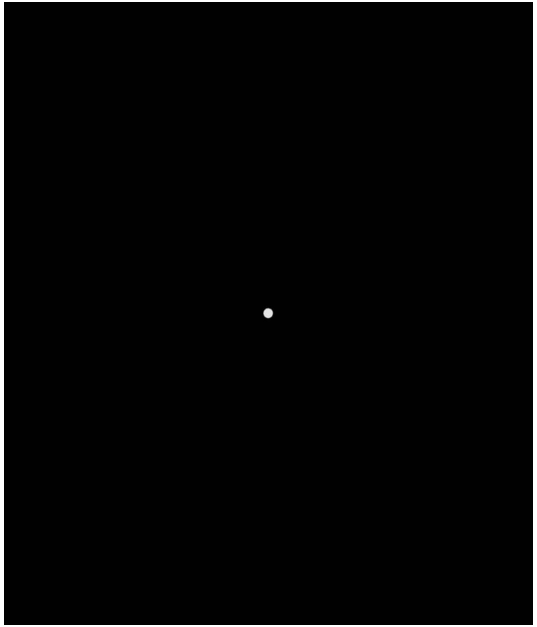
Deep inelastic scattering is the primary theoretical and experimental probe of PDFs



In QFT, PDFs are defined through matrix elements of fields at light-like separations

The challenge





One “traditional” approach

See A Shindlers talk

Determine the Mellin moments of PDFs

$$\langle x^n \rangle_{f_q}(\mu) = \int_0^1 \mathrm{d}x\, x^n \left[f_q(x,\mu) + (-1)^{n+1} f_{\bar{q}}(x,\mu) \right]$$

Through an operator product expansion, moments are related to local twist-2 operators

$$2\langle x^n \rangle_{f_q}(\mu) \left[P_{\mu_1} \cdots P_{\mu_n} - \text{traces} \right] = \frac{1}{2} \sum_{s=1}^{2} \langle P, s | \mathcal{O}^q_{\{\mu_1 \cdots \mu_{n+1}\}} | P, s \rangle$$

$$\mathcal{O}^q_{\{\mu_1 \cdots \mu_n\}} = \bar{q} \gamma_{\{\mu} i \overleftrightarrow{D}_{\mu_2} \cdots i \overleftrightarrow{D}_{\mu_n\}} q$$

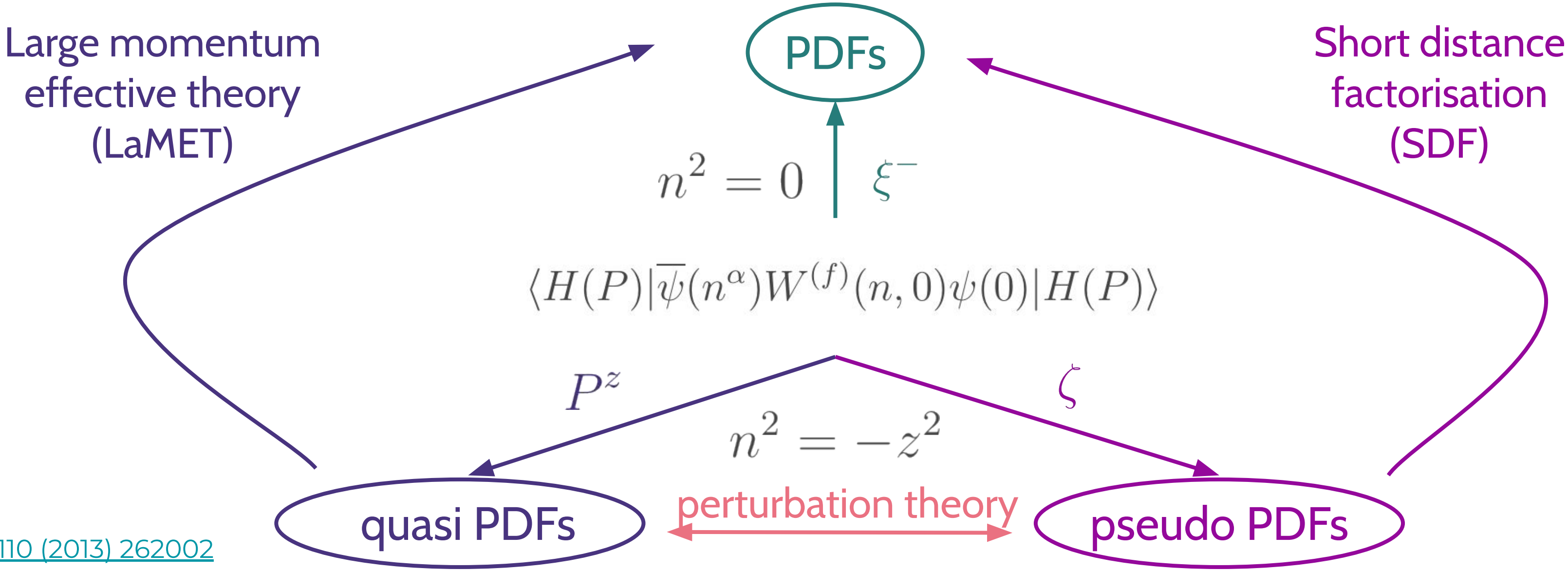
Given a sufficient number of moments, one can reconstruct the original function

$$f_q(x,\mu) + (-1)^{n+1} f_{\bar{q}}(x,\mu) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}n}{x^{n+1}} \langle x^n \rangle_{f_q}(\mu)$$

Gradient flow applied in: Monahan & Orginos JHEP 03 (2017) 116
Monahan, PRD 97 (2018) 054507
Shindler, PRD 110 (2024) L051503
Shindler, arXiv:2410.00198

x-dependent parton distributions

$$f^{(0)}_{q/H}(x) = \int_{-1}^{1} \frac{\mathrm{d}\xi^-}{4\pi} e^{-i\xi^- x P^+} \langle H(P)|\overline{\psi}(n^\alpha) W^{(f)}(n,0)\psi(0)|H(P)\rangle$$



Ji, PRL 110 (2013) 262002

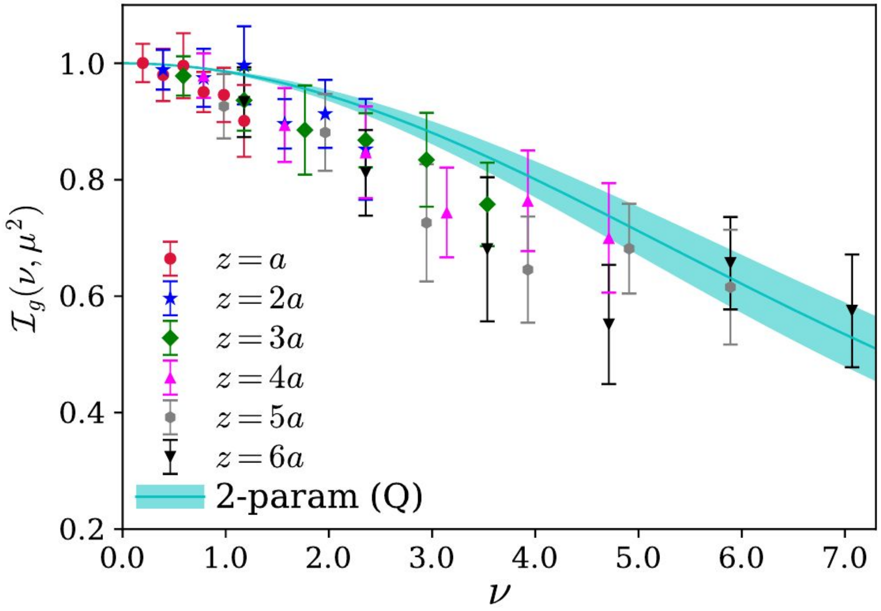
$$\widetilde{f}^{(0)}_{j/H}(\xi, P^z) = \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{4\pi} e^{i\xi P_z z} \langle H(P)\left|\overline{\psi}(0,z,\mathbf{0}_\mathrm{T}) W(z,0)\Gamma_j \psi(0)\right| H(P)\rangle$$

$$\widetilde{p}^{(0)}_{j/H}(\xi, z^2) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta}{4\pi} e^{i\xi\zeta} \langle H(P)\left|\overline{\psi}(0,z,\mathbf{0}_\mathrm{T}) W(z,0)\Gamma_j \psi(0)\right| H(P)\rangle$$

Radyushkin, PRD 96 (2017) 034025

HadStruc calculations

$\nu = P \cdot n$



Khan et al., PRD 104 (2021) 094516

HadStruc calculations

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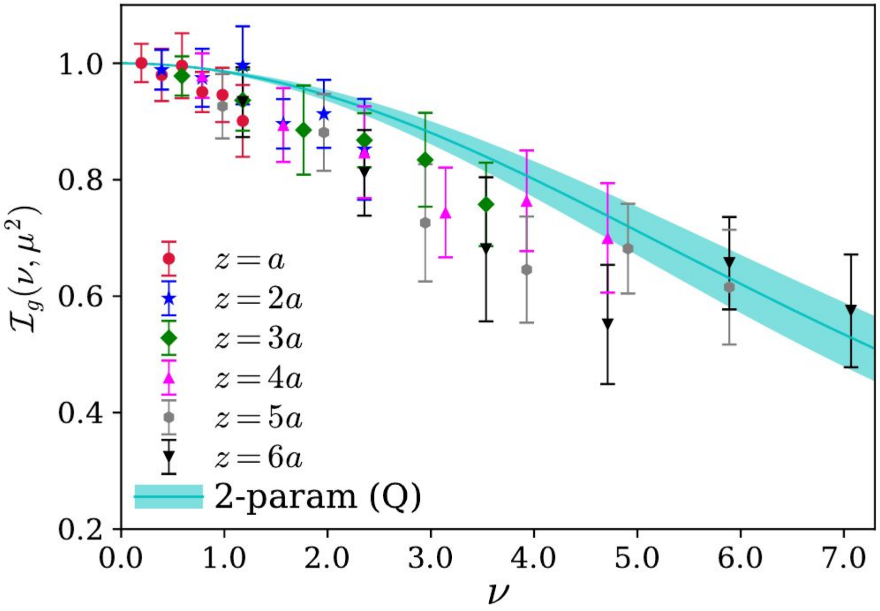
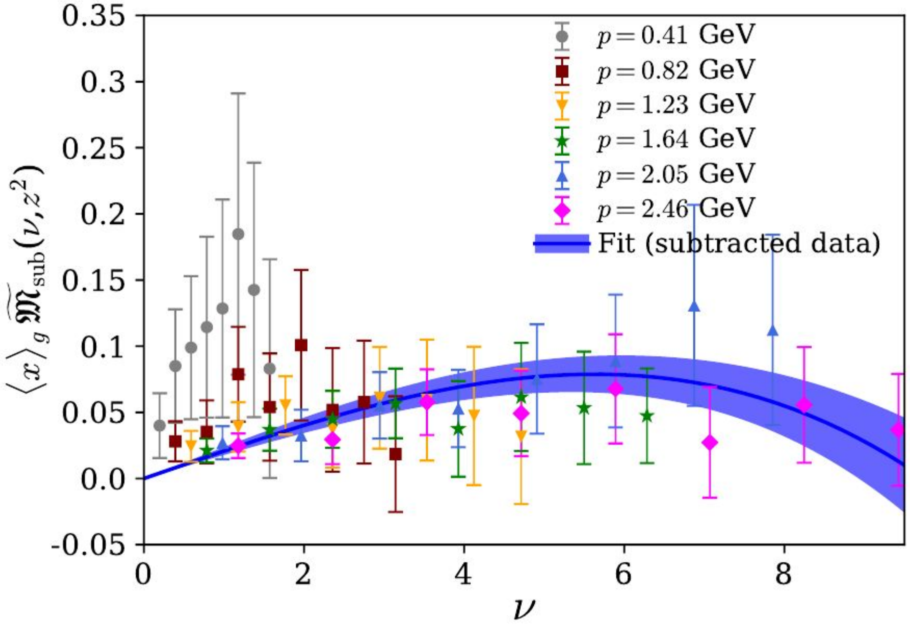




Khan et al., PRD 104 (2021) 094516

Egerer et al., PRD 106 (2022) 094511

Gluon momentum fraction via the SFTE

Gradient flow provides controlled smearing method for improved signal-to-noise ratio

Properties of the flow also advantageous for renormalisation

- Gauge invariant: avoids mixing with gauge noninvariant operators
- Does not require additional nonperturbative calculation
- NNLO perturbative matching already available

Harlander et al., EPJC 78 (2018) 944

In the future

- Nonperturbative running to perturbative scales a possibility
- NNLO likely to become available

See A Hasenfratz's talk

Extracting the gluon momentum fraction

Gluon momentum fraction related to the energy-momentum tensor

$$T_g^{\{\mu\nu\}} = \frac{1}{4} g^{\mu\nu} G_{\alpha\beta} G^{\alpha\beta} - G^{\mu\alpha} G^{\nu}{}_{\alpha}$$

In the following we neglect mixing with isoscalar quark contributions

Extracted via two possible operators

$$\mathcal{O}_{Ai}(x) = 2\,\mathrm{Tr}\left[G_{i\alpha} G_{4\alpha}\right]$$

$$\mathcal{O}_{B}(x) = 2\,\mathrm{Tr}\left[G_{4\alpha} G_{4\alpha} - \frac{1}{3} G_{j\alpha} G_{j\alpha}\right]$$

with hadronic matrix elements

$$\langle P|\mathcal{O}_{Ai}|P\rangle = 4 i E_N P_i \langle x \rangle_g$$

$$\langle P|\mathcal{O}_{B}|P\rangle = -\left(E_N^2 + \frac{2}{3}\mathbf{P}^2\right) \langle x \rangle_g$$

Extracting the gluon momentum fraction

Finite gluon momentum fraction extracted as a function of flow time

Related to the result in the MS-bar scheme via the (inverse) short flow-time expansion

$$\mathcal{O}_B(\mu) = c(\mu^2\tau)\,\mathcal{O}_B(\tau) + \cdots$$

whose coefficients are known to NNLO in perturbation theory Harlander et al., EPJC 78 (2018) 944

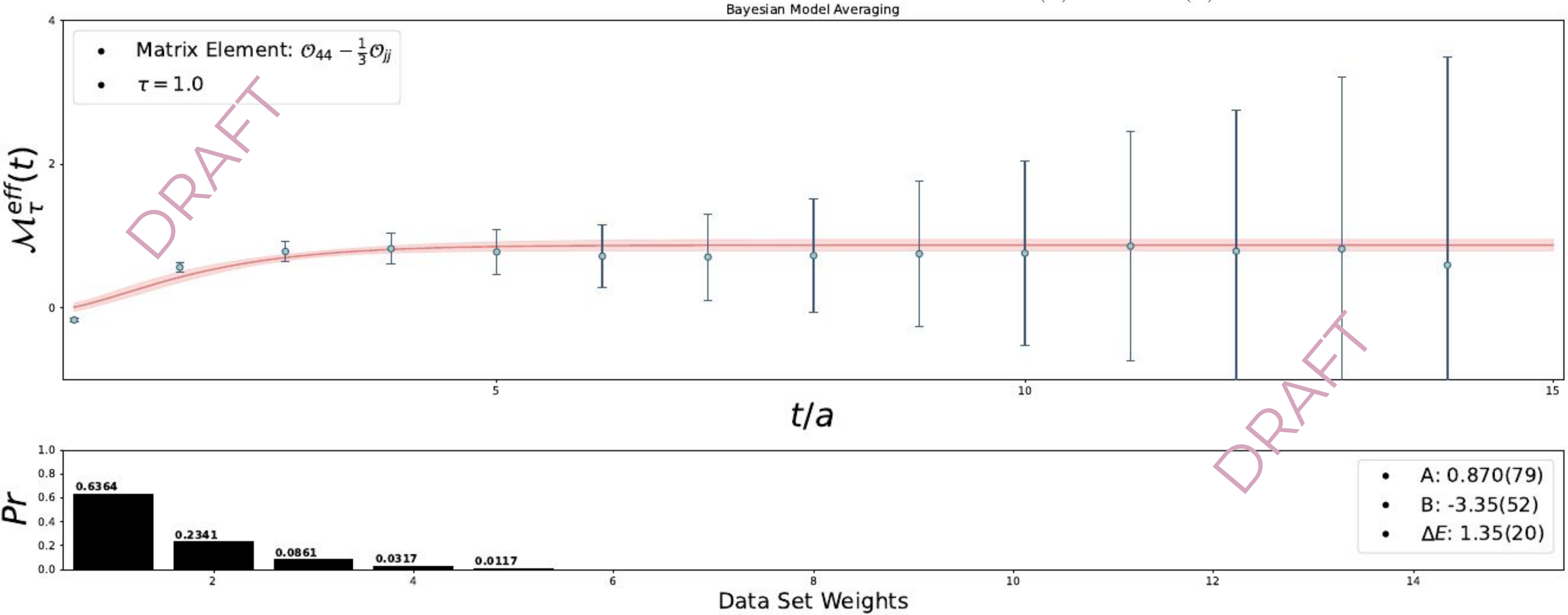
Product of scales set by

$$\mu^2\tau = \frac{e^{-\gamma_E}}{2}$$

$$T_g^{\{\mu\nu\}}(\mu) = \frac{1}{g^2}\left[\mathcal{O}_1^{\mu\nu}(\mu) - \frac{1}{4}\mathcal{O}_2^{\mu\nu}(\mu)\right]$$
$$\mathcal{O}_1^{\mu\nu}(\mu) = G_{\mu\rho}G_{\nu\rho}$$
$$\mathcal{O}_2^{\mu\nu}(\mu) = g_{\mu\nu}G_{\alpha\beta}G_{\alpha\beta}$$

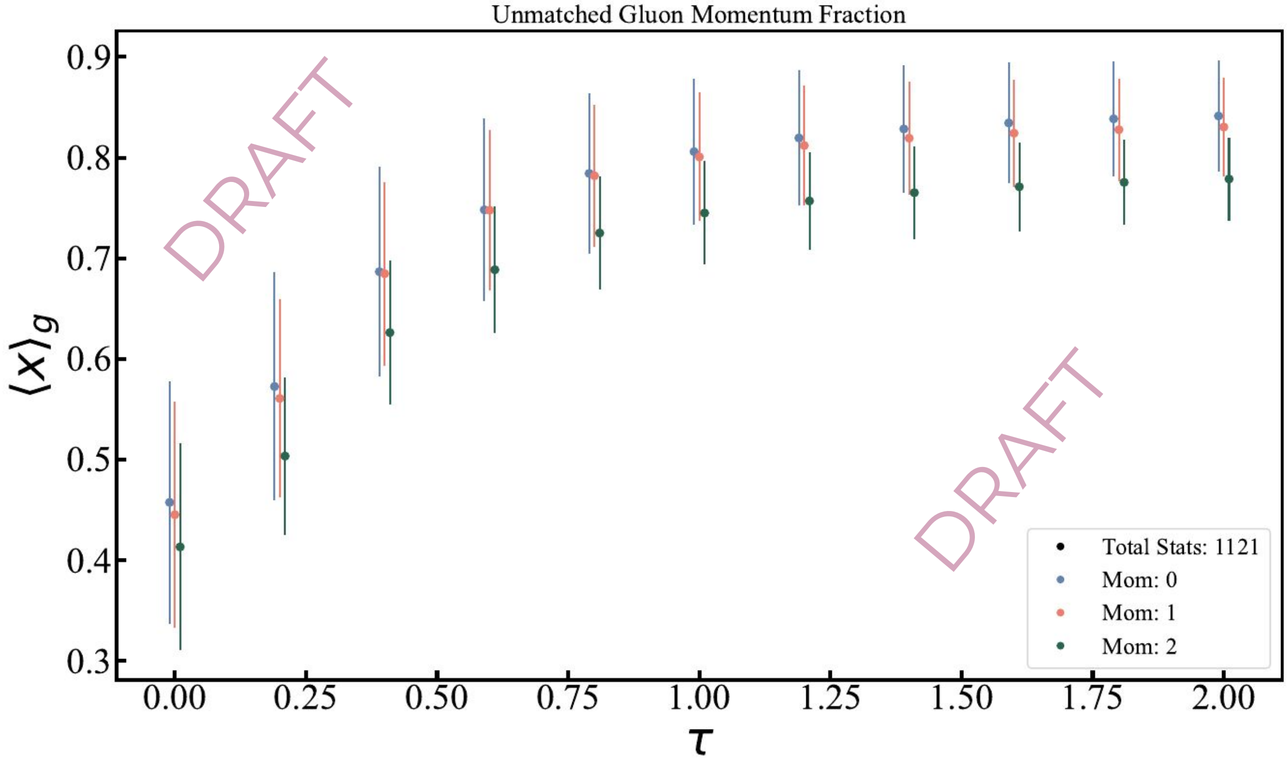
Extracting the gluon momentum fraction

a	m_π	$L/a \times T/a$	$\mathrm{N_{src}}$
0.094(1) fm	358(3) MeV	32×64	64

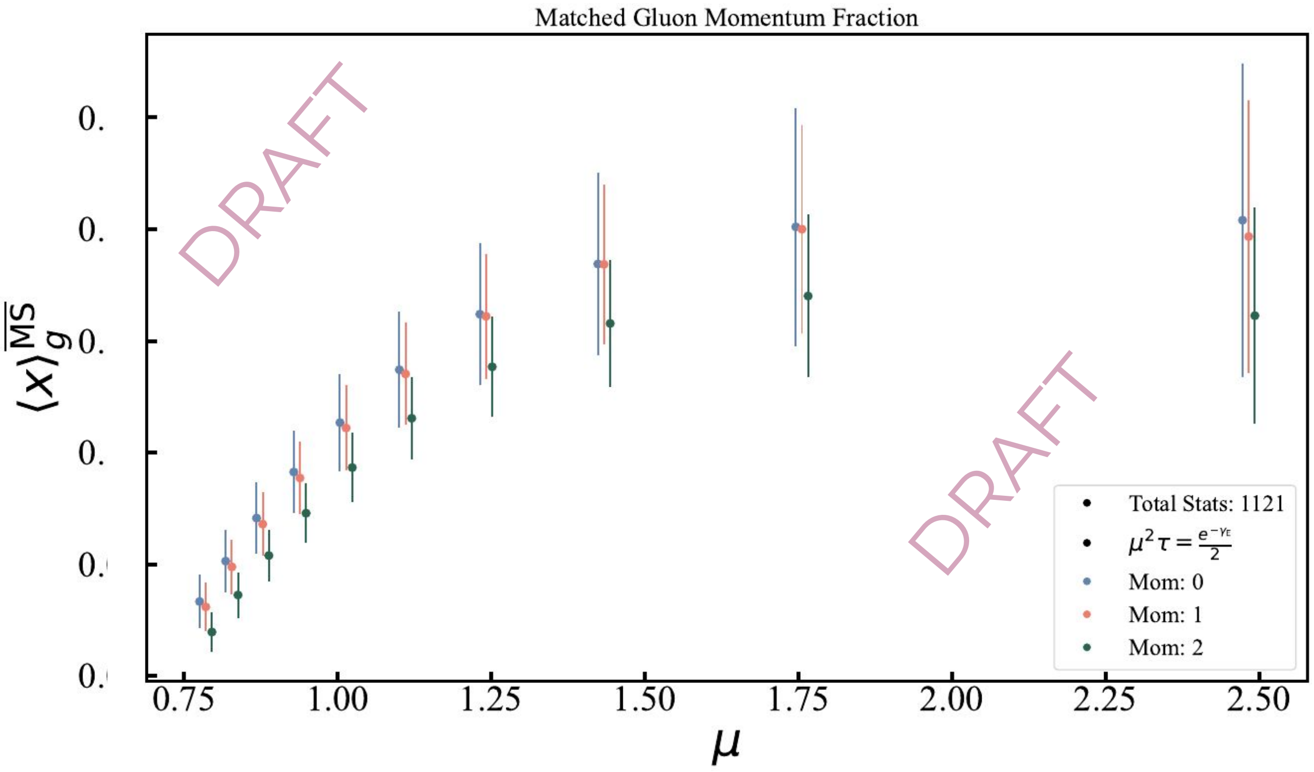


Jay & Neil, PRD 103 (2021) 114502

Gluon momentum fraction: flow-time dependence



Gluon momentum fraction: scale dependence



Outline

Operator product expansions and the short flow-time expansion

Applications of the short flow-time expansion

Systematic and quantitative error estimates?

Systematic tests of the short flow-time expansion

Flow-time dependence logarithmic at one-loop and log^n beyond that

- Deviations at large flow-time expected to be polynomial in the flow-time
- In principle, flow-time must be smaller than all other physical scales
- On the lattice, discretisation effects relevant at very short flow times
- Leads to a “window problem”

Short flow-time expansion coefficients calculable in perturbation theory

- Should be independent of external states

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- Should be independent of external states

In principle, provides a test of validity of the SFTE
For example: a comparison of weak coupling lattice simulations via hadronic states with perturbative calculations using external partonic states

Systematic tests of the short flow-time expansion

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Short flow-time expansion coefficients calculable in perturbation theory

- Should be independent of external states
- Detailed perturbative analysis of small flow-time limit feasible
    - Enables comparison with high-precision data

Suzuki & Takaura, PTEP (2021) 073B02

Summary

Short flow-time expansion

- Relates operators at finite flow time to renormalised operators at zero flow time
- Operator product expansion in the flow time
- Coefficients calculable in perturbation theory or nonperturbatively

Valid at the operator level in the short flow-time limit

- Short flow-time defined by all physical scales of relevant processes

Powerful tool for extracting renormalised operators from lattice calculations

- Particularly powerful for power-divergent operators

Precision lattice calculations require precision estimates of systematic uncertainties

Short flow-time expansion: open questions*

*These may simply reflect my ignorance

What is known of the properties to all orders in perturbation theory?

Can the radius of convergence be shown to be non-zero?

To what extent is this really (or really analogous to) an operator product expansion?

Does the validity of the expansion in bare operators hold beyond perturbation theory?

Can one analyse the window problem other than through numerical experiment?

What defines small flow time and can practical lattice calculations overcome the window problem in high-precision scenarios (at the 0.2 to 0.5% level)?

Thank you!

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