The static force with gradient flow from the lattice

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Motivation

- Static force relies on the static potential which is well known
- Static potential/force well known perturbatively and on the lattice
- Useful for testing new methodologies
  Here: Force from generalized Wilson loops, Gradient flow
  $\rightarrow$ preparation for similar objects needed in NREFTs

- Static potential/force is used for scale setting
- Static potential/force can be used to extract $\Lambda_0 = \Lambda_{\overline{\mathrm{MS}}}^{n_f=0}$
- Λ_0 got more attention recently with new methods

Motivation

- Lattice: we use stochastic methods to solve the Euclidean path integral non-perturbatively:

$$\langle O(t)O(0)\rangle = \frac{1}{Z}\int \mathcal{D}[U]e^{-S_E[U]}O(t)O(0) \approx \frac{1}{N}\sum_{p(U)\propto e^{-S_E}} O(t)O(0)$$

  on a discretized space-time grid (lattice), with lattice spacing/lattice regulator a
- We are interested in long-t correlation to obtain spectra
- Comparing of non-perturbative lattice results and PT expressions → extract α_S
- Several issues:
  - large t → larger statistical error/bad signal-to-noise ratio
  - discretization errors → continuum limit
    But: Not trivial in our case

➡ We use gradient flow to tackle both problems

Overview

- Perturbative/Continuum:
  QCD Static Force in Gradient Flow
  Nora Brambilla, Hee Sok Chung, Antonio Vairo, and Xiang-Peng Wang
  JHEP01 (2022) 184, arXiv:2111.07811
- Lattice:
  Static force from generalized Wilson loops on the lattice using the gradient flow
  Nora Brambilla, Viljami Leino, Julian Mayer-Steudte, Antonio Vairo
  PRD109 (2024) 11,114517 arXiv:2312.17231

Physical Background: Static Potential

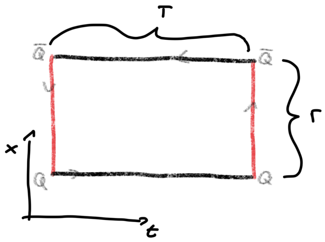
- Static potential $V(r)$ encoded in the spectrum of Wilson loops at fixed r:

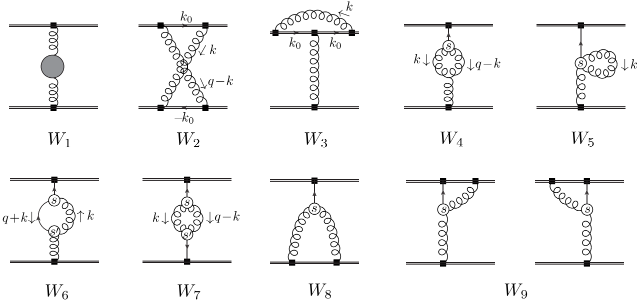
$$\langle W_{r\times T}\rangle \overset{\text{large}\,T}{\propto} e^{-aV(r)}$$

-

$$V(r) = -\lim_{T\to\infty}\frac{1}{T}\log\langle W_{r\times T}\rangle = -\lim_{T\to\infty}\frac{1}{a}\log\frac{\langle W_{r\times T+a}\rangle}{\langle W_{r\times T}\rangle}$$

- In continuum PT: renormalon ambiguity
- In continuum PT: at $t_F = 0$ known up to N^{3}LL
- On the lattice: $1/a$ - divergence (r-independent)
- Lattice and PT should agree for $r\Lambda_{\mathrm{QCD}} \ll 1$
- Derivative: $\partial_r V(r) \equiv F(r)$





- 1-loop diagrams in Feynman gauge, momentum space, neglecting $\mathbf{q} = 0$ contributions
-

$$V(r) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} \tilde{V}(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{r}} \Rightarrow \partial_r V(r)$$

Perturbative Static Force

- Full expression of the 1-loop static force at finite flow time:

$$F(r,t_F) = \frac{\alpha_S(\mu) C_F}{r^2} \left[\left(1 + \frac{\alpha_S}{4\pi} a_1\right) \mathcal{F}_0(r,t_F) + \frac{\alpha_S}{4\pi} \beta_0 \mathcal{F}^L_{\mathrm{NLO}}(r,t_F,\mu) + \frac{\alpha_S C_A}{4\pi} \mathcal{F}^F_{\mathrm{NLO}}(r,t_F) \right] + \mathcal{O}(\alpha_S^3)$$

- $C_A = N_c,\ C_F = (N_c^2-1)/(2N_c),\ \beta_0 = \frac{11}{3} C_A - \frac{2}{3} n_f,\ a_1 = \frac{31}{9} C_A - \frac{10}{9} n_f$
- Flow time part and logarithmic part:

$$\mathcal{F}_0(r,t_F) = \mathrm{erf}\left(\frac{r}{\sqrt{8t_F}}\right) - \frac{r}{\sqrt{2\pi t_F}} \exp\left(-\frac{r^2}{8t_F}\right)$$

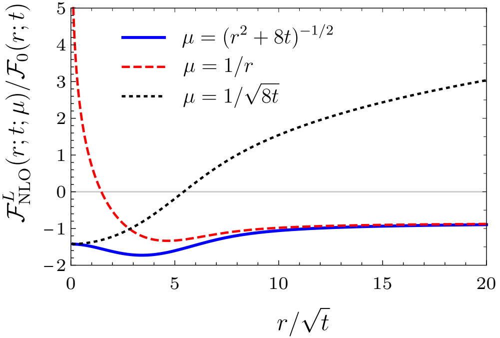
$$\mathcal{F}^L_{\mathrm{NLO}}(r,t_F,\mu) = \log(\mu^2 r^2) \mathcal{F}_0(r,t_F) + \log\left(\frac{8t_F}{r^2} e^{\gamma_E}\right) \mathcal{F}_0(r,t_F) - \frac{r}{\sqrt{2\pi t_F}} \left[... \right]$$

Perturbative Static Force: The scale(s)

- Logarithmic part:

$$\mathcal{F}^L_{\mathrm{NLO}}(r, t_F, \mu) = \log(\mu^2 r^2)\mathcal{F}_0(r, t_F) + \log\left(\frac{8t_F}{r^2} e^{\gamma_E}\right)\mathcal{F}_0(r, t_F) - \frac{r}{\sqrt{2\pi t_F}}\left[\ldots\right]$$

- Logarithmic behavior suppressed by choice $\mu = 1/\sqrt{r^2 + 8t_F}$
- Natural scale: $\mu = 1/r$
- On the lattice: $\mu = 1/\sqrt{r^2 + 8bt_F}$

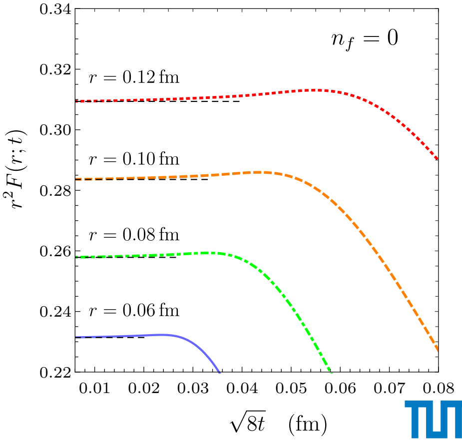


Perturbative Static Force: Expanding in small t_F

$$r^2 F(r, t_F) \approx r^2 F(r, t_F = 0) + \frac{\alpha_S^2 C_F}{4\pi}[-12\beta_0 + 44 C_A]\frac{t_F}{r^2}$$

$$F(r, t_F = 0) = \frac{\alpha_S(\mu) C_F}{r^2}\left\{1 + \frac{\alpha_S}{4\pi}[a_1 + 2\beta_0 \log(\mu r e^{\gamma_E - 1})]\right\} + \mathcal{O}(\alpha_S^3)$$

$$8 n_f = -12\beta_0 + 44 C_A$$



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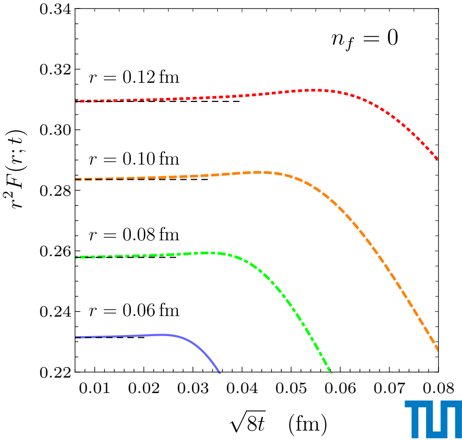
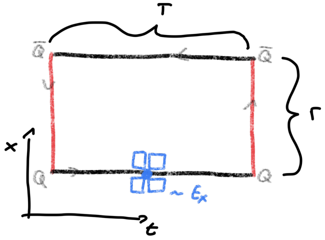
➡ $F(r,t_F)$ is constant for $n_f = 0$



On the lattice: Static Force

- $F_E(r) = \lim_{T\to 0} \frac{\langle WE_{r\times T}\rangle}{\langle W_{r\times T}\rangle}$
  (Vairo, EPJ Web Conf. 126, 02031 (2016)) (Vairo, Mod.Phys.Lett.A 31, 1630039 (2016)) (Brambilla, Phys. Rev. D 63, 014023 (2001))
- Chromo E-field insertion in one of the temporal Wilson lines
- It is the same quantity as $\partial_r V(r)$
- Discretization of E causes a non-trivial behavior to the continuum



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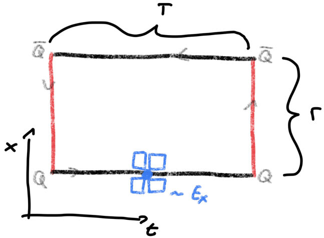
- It is the same quantity as $\partial_r V(r)$
- Discretization of E causes a non-trivial behavior to the continuum
- may be absorbed into a renormalization constant:

$$Z_E F_E(r) = \partial_r V(r) \qquad Z_E \to 1 \text{ for } a \to 0$$

- F_E and $\partial_r V(r)$ are clearly defined on the lattice



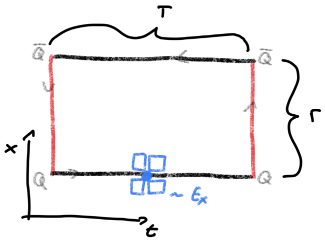
➡ Determine Z_E non-perturbatively on the lattice

Static Force w/o gradient flow

- $Z_E = \frac{\partial_r V(r)}{F_E(r)}$ with little r-dependence
- Brambilla, Phys.Rev.D105, 054514 (2022)):

a (fm)	Z_E
0.060	1.4068(63)
0.048	1.3853(30)
0.040	1.348(11)

- $Z_E \neq 1$
- Unable to perform the continuum limit



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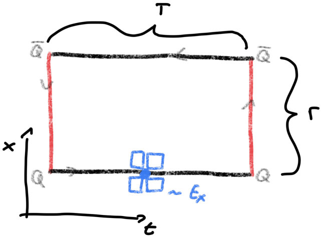
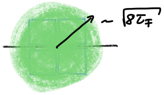
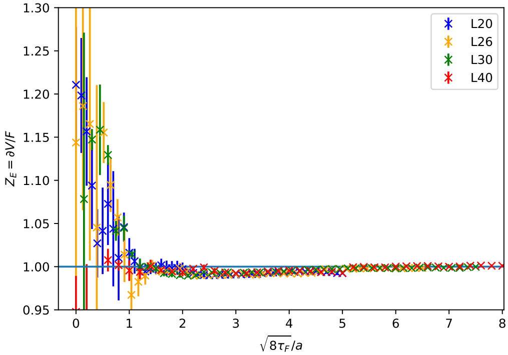
➡ Use gradient flow





Renormalization effect of gradient flow

- $Z_E(t_F) = \frac{\partial_r V(t_F)}{F_E(t_F)}$
- Z_E approaches 1 for $\sqrt{8t_F} > a$, for all lattice sizes (flow time scale dominates over lattice regulator scale)
  $\rightarrow$ renormalizes E-field insertion



Perform continuum limit at fixed t_F for $\sqrt{8t_F} > a$

Lattice & Scaling details

Simulation parameters:

N_S	N_T	β	a [fm]	t_0/a^2	N_{conf}	Label
20^3	40	6.284	0.060	7.868(8)	6000	L20
26^3	52	6.481	0.046	13.62(3)	6000	L26
30^3	60	6.594	0.040	18.10(5)	6000	L30
40^3	80	6.816	0.030	32.45(7)	3300	L40

- find reference scale t_0/a^2 for continuum limit
- find r_0/a, r_1/a
- common to set $r_0 = 0.5\,\mathrm{fm}$ for pure gauge
- t_0-scale is the natural scale for gradient flow studies

- Potential scale:

$$r^2 F(r)|_{r=r_0} = 1.65$$

$$r^2 F(r)|_{r=r_1} = 1$$

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- Gradient flow scale:

$$t^2 \langle E \rangle|_{t=t_0} = 0.3$$

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$$\frac{\sqrt{8t_0}}{r_0} = 0.9569(66)$$

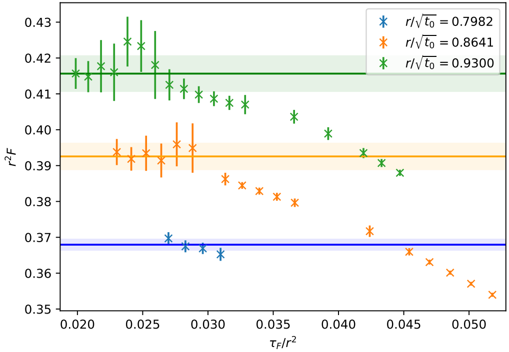
$$\frac{\sqrt{8t_0}}{r_1} = 1.325(13)$$

$$\frac{r_0}{r_1} = 1.380(14)$$

in continuum and after $t_F \to 0$

$t_F \to 0$ limit: $F(t_F \to 0)$, Λ_0 from small r

- 1-loop constant at small t_F
- constant $t_F \to 0$ limit
- obtain $F(t_F = 0)$



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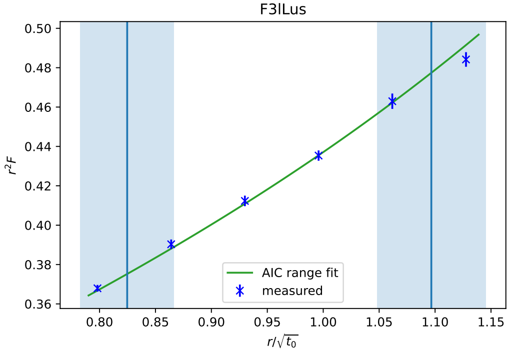
- 1-loop constant at small t_F
- constant $t_F \to 0$ limit
- obtain $F(t_F = 0)$
- Fit NLO, N^2LO, N^2LL, $N^3LO(+u.s.)$ where Λ_0 is the fit parameter

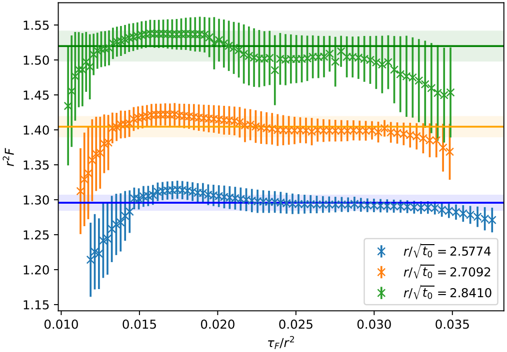
➡ Λ_0 extraction works, but only at larger r



$t_F \to 0$ limit: $F(t_F \to 0)$, large r

- Perform constant zero flow time limit at large r
- Perform Cornell fit:

$$r^2 F(r) = A + \sigma r^2$$



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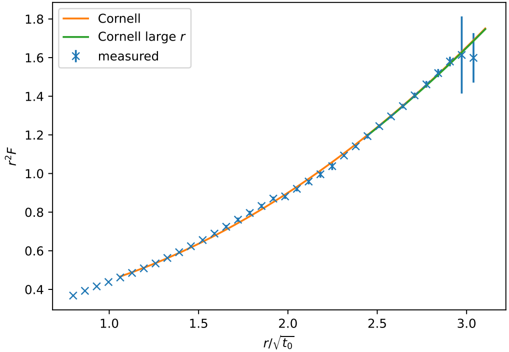
$$r^2 F(r) = A + \sigma r^2$$

- Fit result:

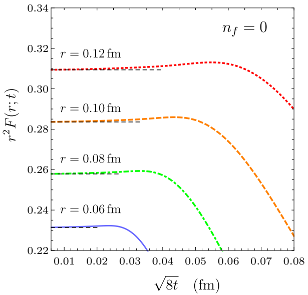
$$A = 0.268(33)$$
$$\sigma t_0 = 0.154(6)$$



$t_F \to 0$ limit: Λ_0 from $F(t_F)$

- Fit perturbative $F(t_F)$ directly to $F(t_F)$ obtained from the lattice
- PT $F(t_F)$ determined by Λ_0
- Higher order crucial for reliable Λ_0 extraction but only up to NLO is known at finite t_F
- Combined model function:

$$F^{\text{model}} = \begin{cases} F(r) \text{ at any order} & t_F = 0 \\ \text{flowtime part from NLO} & t_F \neq 0 \end{cases}$$



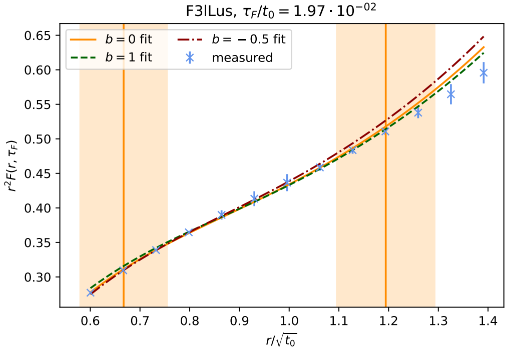
$t_F \to 0$ limit: Λ_0 from $F(t_F)$, r scale

- Fit at fixed t_F, along r
- scale: $\mu = \frac{1}{\sqrt{r^2+8bt_F}}$, $-0.5 \leq b \leq 1$

Obtainings:

- fit range depends on b, slope of fitted function is stable

different Λ_0 at different fixed t_F, but should not depend too much on that



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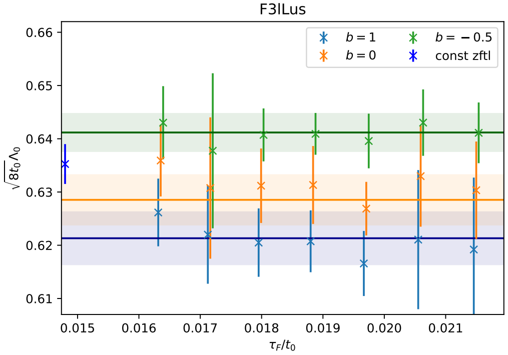

→ works at smaller r and along the physical relevant scale

Uncertainties of Λ_0

- Statistical error: Jackknife sampling
- Fit errors for different fit ranges: Akaike information criterion
- Perturbation errors:
  - $\mu = \frac{1}{\sqrt{r^2+8bt_F}}$ not unique
  - at finite t_F: variation $-0.5 \leq b \leq 1$
  - at $t_F = 0$: $\mu = \frac{s}{r}$ with $\frac{1}{\sqrt{2}} \leq s \leq \sqrt{2}$
- Errors are independent of each other
  $\rightarrow$ sum in quadrature

➡ Final result includes statistical and perturbative uncertainties

Results

- All methods agree within their errors
- We state fit at fixed t_F along r at N^3LO+u.s. as final result:

$$\sqrt{8t_0}\Lambda_0 = 0.629^{+22}_{-26}$$

$$\delta(\sqrt{8t_0}\Lambda_0) = (4)^{\mathrm{lattice}}(^{+18}_{-25})^{\mathrm{s-scale}}(^{+13}_{-7})^{\mathrm{b-scale}}$$

$$r_0\Lambda_0 = 0.657^{+23}_{-28}$$

$$\Lambda_0 = 259^{+9}_{-11}\,\mathrm{MeV},\ r_0 = 0.5\,\mathrm{fm}$$

- Cornell fit:

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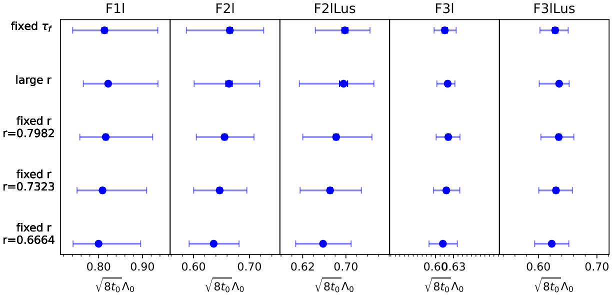
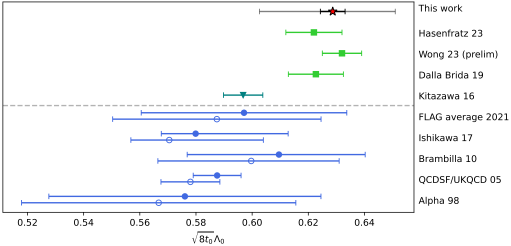


Summary

- With GF direct force measurement possible
- GF renormalizes E-field insertions
  $\rightarrow$ useful in other NREFT applications
- Λ_0 extraction in several ways
- Λ_0 compatible to recent GF studies
- Λ_0 with GF is systematically larger even in our study
- GF applicable with fermions



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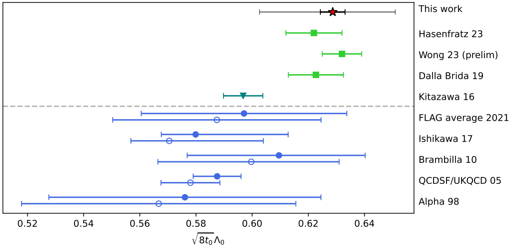


Thank you for your attention!

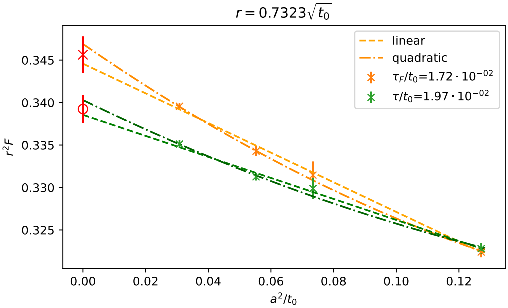
- Polynomial interpolations
- Tree level improvement at any fixed r and t_F:

$$F^{\text{impr latt}} = \frac{F^{\text{tree cont}}}{F^{\text{tree latt}}} F^{\text{latt}}$$

- Trivial continuum limit:

$$F^{\text{impr latt}} = \text{Polynomial}(a^2) = F^{\text{cont}} + \mathcal{O}(a^2)$$

where $\sqrt{8t_F} > a$ for the coarsest lattice

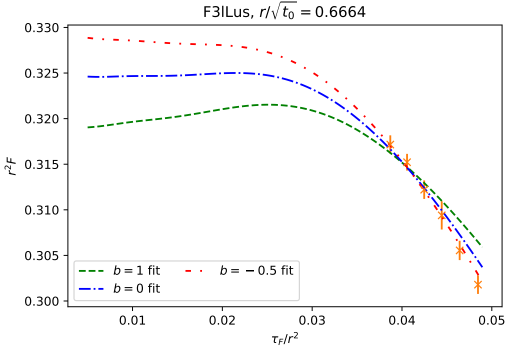


Backup: $t_F \to 0$ limit: Λ_0 from $F(t_F)$, t_F scale

- Fit at fixed r, along t_F
- scale: $\mu = \frac{1}{\sqrt{r^2+8bt_F}}$, $-0.5 \leq b \leq 1$

Obtainings:

- slope of fitted function highly depends on b
- works less good at larger r



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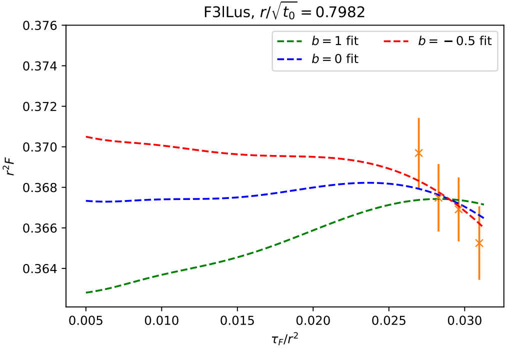

➡ Fit works, but t_F is not the physical scale

Sample frame title

This is some text in a sample frame. Don't waste your time and stay focused to the talks.

→ Knock Knock!! Who's there!?