Universality and cutoff effects of pure gauge theories from gradient flow scales

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  - Important but difficult

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- Universality of lattice actions
  - Are we in the scaling region?
    $a \in (0.05 - 0.1)\mathrm{fm}$
    Renormalization scheme/observable independence
  - Which S_{latt} shows smaller cutoff effects

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  - Extrapolation valid if a-dependence under control

- Understand scale setting
  - Matching an energy scale to an experiment
  - Required for physical predictions
  - t_0 or w_0 – very precise
  - Systematics of extrapolation

- Universality of lattice actions
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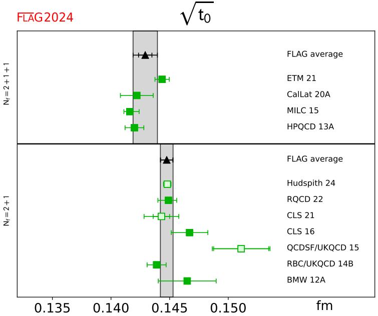

How to understand cutoff effects?

Symanzik effective theory

- Any lattice action S_{latt} can be described by an effective continuum action

$$S_{\text{latt}} \overset{a\to 0}{\sim} S_{\text{cont}} + a^2 S_2 + \ldots \qquad \longrightarrow \qquad \langle O \rangle_{\text{latt}} \overset{a\to 0}{\sim} \langle O \rangle + O_1 a^{2+\eta} + \ldots$$

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Gradient Flow Quantities

- Small systematic
- Known Symanzik expansion

Gradient flow scales

- t_0 scales [Lüscher 2010]

$$t^2 \langle E(t) \rangle \Big|_{t=t_c} = \begin{cases} 0.15, & t_2 \longrightarrow \text{short distance} \\ 0.3, & t_0 \longrightarrow \text{medium distance} \\ 0.5, & t_1 \longrightarrow \text{long distance} \end{cases}$$

- w_0 scales [Borsányi et al. 2012]

$$t \frac{\mathrm{d}}{\mathrm{d}t} t^2 \langle E(t) \rangle \Big|_{t=w_c^2} = c$$

  - (no time but similar conclusions)

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  - (Theory + Flow) as a 5D local field theory [Martin Luscher and Peter Weisz 2011]

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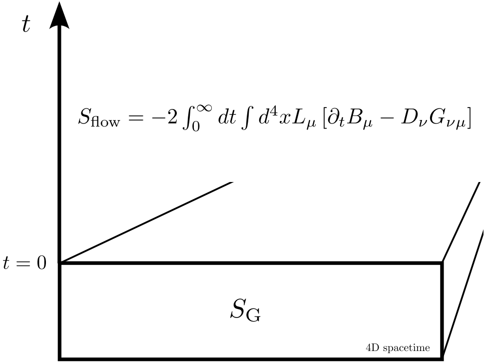

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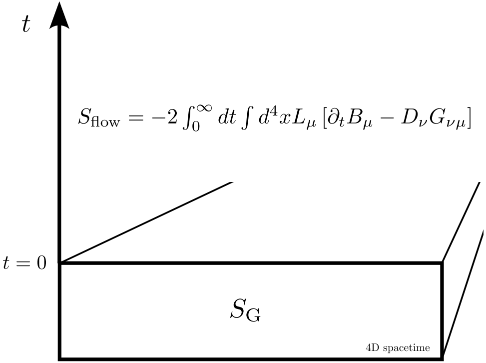

- Gauge action S_G
- Flow action S_{flow}
- L_μ enforces flow equation (Lagrangian multiplier)
- 'Classical' theory at $t > 0$

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Extra contributions [Ramos and Sint 2016]

$$S^{5D}_{\text{latt}} \overset{a\to 0}{\sim} S^{5D}_{\text{cont}} + a^2 S_{2,G} + a^2 S_{2,\text{flow}} + \ldots$$

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- Flow discretization
  - Use Zeuthen flow $S_{2,\text{flow}} = 0$

$$a^2 \frac{\mathrm{d}}{\mathrm{d}t} V_\mu(x,t) = -g_0^2 \left(1 + \frac{a^2}{12} D_\mu D_\mu^*\right) \frac{\delta S^{\text{LW}}[V]}{\delta V_\mu(x,t)} V_\mu(x,t)$$

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    Improvement with $\left\langle E(t + c_b(g_0)a^2) \right\rangle$? similar to τ-shift [Cheng et al. 2014]
  - Lüscher-Weisz TL imp. action: $c_b = 0 + \mathcal{O}\left(g_0^2\right)$
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S_G, $t = 0$ improvement

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  - Zeuthen Flow – $S_{2,\mathrm{flow}} = 0$
  - Improved Observable – $\langle O_2 \rangle = 0$

Understanding common scaling tests

Observable discretization – $t_0^{\mathrm{pl}}/t_0^{\mathrm{cl}}$

- Ideal quantity (?)
  - Very precise quantity
  - Known limit: $\lim_{a\to 0} t_0^{\mathrm{pl}}/t_0^{\mathrm{cl}} = 1$

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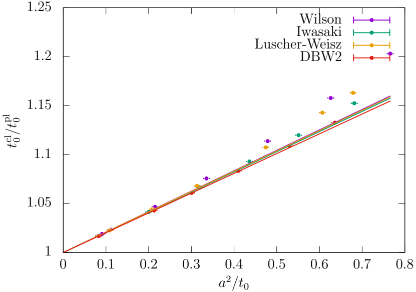

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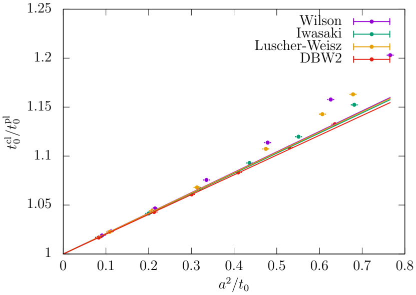
- Action effects drop out $S_{2,\text{G}}$
- Sensitive only to O_2
  - Improved observable $E_2^{\text{imp}} = 0$



τ-shift – $t_0(c_b)/t_0$

- Compute t_0 scales with a shift

$$t_0 \longrightarrow \langle E(t) \rangle$$
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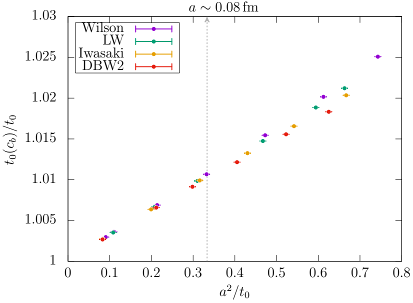

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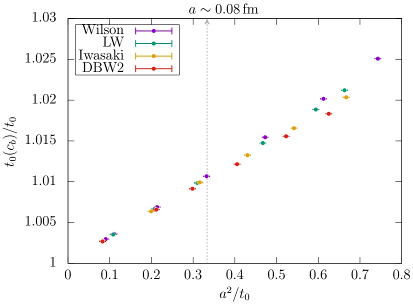
$$\frac{t_0(c_b)}{t_0} \overset{a\to 0}{\sim} 1 - \frac{a^2}{D_0} \left\{ c_b t_0^2 \frac{\mathrm{d}}{\mathrm{d}t} \langle E(t_0) \rangle \Big|_{t_0} \right\}$$

- Action effects drop out $S_{2,\mathrm{G}}$
- Sensitive only to c_b
  - Removed to TL by LW action



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Some results for t_0-like scales

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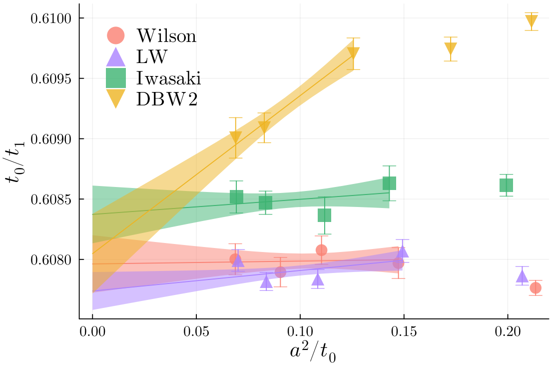
$$\frac{t_0^{\text{imp}}}{t_1^{\text{imp}}} \overset{a\to 0}{\sim} \frac{t_0}{t_1} - a^2 t_0 \left\{ \frac{t_0}{t_1}\frac{G_0}{D_0} - \frac{G_1}{D_1} \right\}, \qquad G_0 = \langle E(t_0) S_{2,\text{G}} \rangle - c_b \frac{\mathrm{d}}{\mathrm{d}t} \langle E(t) \rangle \Big|_{t_0}$$

Long-distance ratio t_0/t_1



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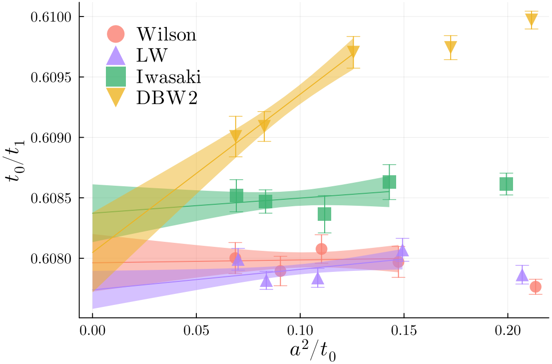

Fit quality

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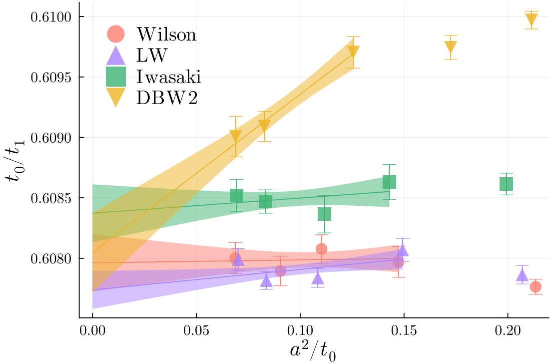

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  $\sqrt{8t_1}/L \sim 0.168$

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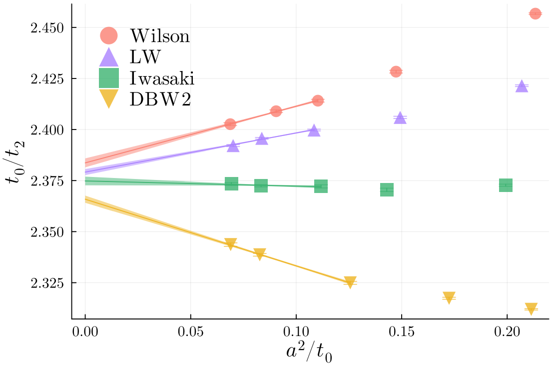

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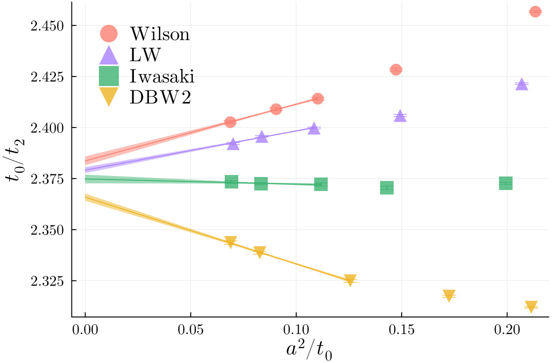

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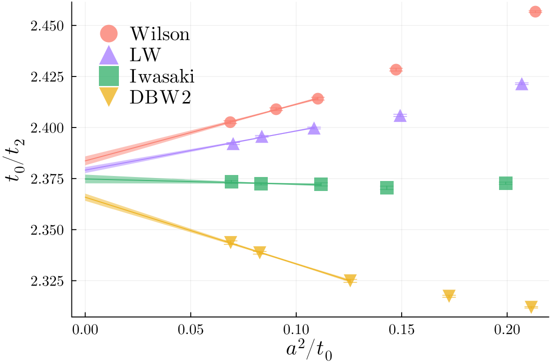

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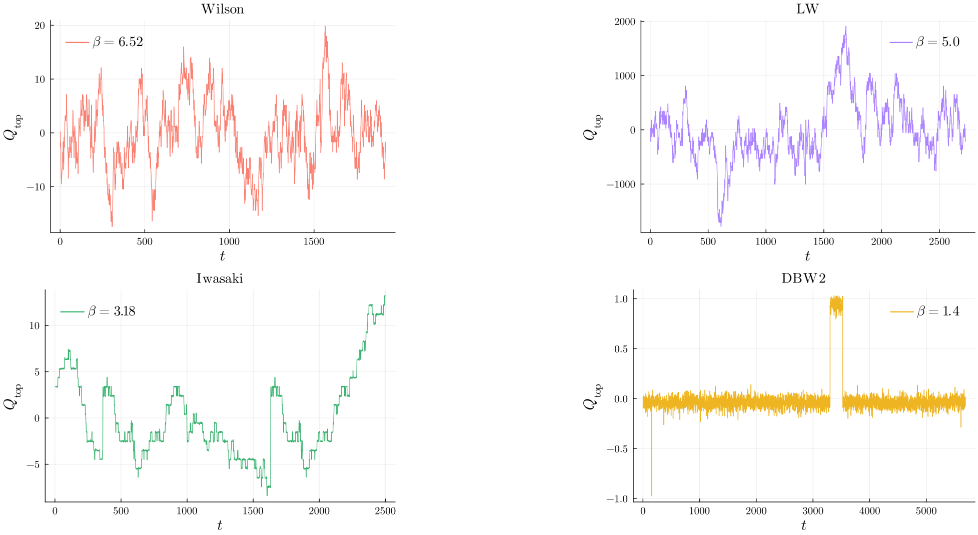
- Asymptotic prediction [Husung, Marquard, and Sommer 2020]

$$\langle O \rangle^{\text{latt}} = \langle O \rangle + a^2 b^{\text{latt}} \alpha(1/a)^{\gamma}, \qquad b^{\text{DBW2}}/b^{\text{PL}} = 16$$

  - c_b spoils the use of this condition as a scaling test

Topological charge



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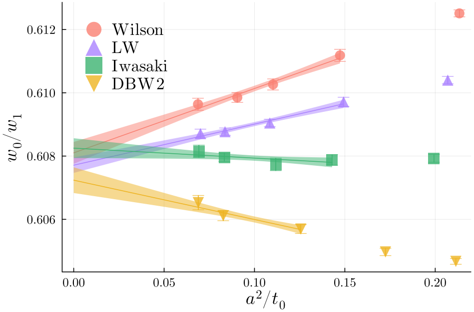
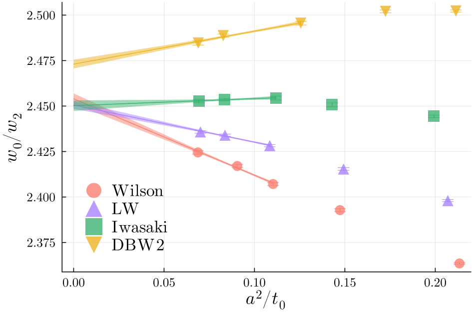
  - t_0/t_i using short & long distance scales
  - w_0/w_i less sensitive to c_b-term

w_0 scales

- w_0 scales [Borsányi et al. 2012]
- unnusual definition

$$t \frac{\mathrm{d}}{\mathrm{d}t} t^2 \langle E(t) \rangle \Big|_{t=w_c^2} = \begin{cases} 0.095, & w_C \longrightarrow \text{short distance} \\ 0.285, & w_A \longrightarrow \text{medium distance} \\ 0.550, & w_B \longrightarrow \text{long distance} \end{cases}$$





Ensembles

Wilson [Wilson 1975]					
$\tau_{\text{int}}^{\hat{t}_0}$	1.1185	1.9820	1.1224	1.3711	0.8414
N_{ind}	313	270	459	709	1142
TL-imp. LW [M. Luscher and P. Weisz 1985]					
$\tau_{\text{int}}^{\hat{t}_0}$	0.5006	0.5003	0.5001	0.5000	0.5000
N_{ind}	450	720	3132	5448	10904
Iwasaki [Itoh et al. 1984]					
$\tau_{\text{int}}^{\hat{t}_0}$	1.6590	4.9233	1.7093	1.9361	1.0827
N_{ind}	283	150	376	1531	1155
DBW2 [Forcrand et al. 1997]					
$\tau_{\text{int}}^{\hat{t}_0}$	0.7414	1.0516	2.4447	1.9192	3.6399
N_{ind}	620	361	334	888	782