Two-loop calculations toward quark mass determination using the gradient flow

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in collaboration with

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Applications of the gradient flow

The gradient flow plays a key role in recent lattice development:

- Scale setting
- Regularization independent expressions of physical observables
- Renormalization group flow

⋮

In this talk, we propose a new application, quark mass determination using the gradient flow.

-Quark $\overline{\mathrm{MS}}$ masses are fundamental parameters in QCD

-Precision is crucial in flavor and Higgs physics

Our proposal

We consider the ratio of bilinear operators of flowed quark:

$$S \equiv \langle \bar{\chi}(t,x)\chi(t,x)\rangle \;,\; R \equiv \langle \bar{\chi}(t,x)\overleftrightarrow{\not{D}}\chi(t,x)\rangle$$

Flow equations: 2010 Lüscher, 2011 Lüscher, Weisz, 2013 Lüscher

$$\begin{cases} \partial_t \chi(t,x) = (D_\mu D_\mu - \alpha_0 \partial_\mu B_\mu(t,x))\chi(t,x) \\ \partial_t \bar{\chi}(t,x) = \bar{\chi}(t,x)(\overleftarrow{D}_\mu \overleftarrow{D}_\mu + \alpha_0 \partial_\mu B_\mu(t,x)) \\ \partial_t B_\mu(t,x) = D_\nu G_{\nu\mu}(t,x) + \alpha_0 D_\mu \partial_\nu B_\nu(t,x) \end{cases}$$

with boundary conditions, $\chi(t=0,x) = \psi(x),\; \bar{\chi}(t=0,x) = \bar{\psi}(x),\; B_\mu(t=0,x) = A_\mu(x)$.

Our proposal

We consider the ratio of bilinear operators of flowed quark:

$$S \equiv \langle \bar{\chi}(t,x)\chi(t,x)\rangle \;,\; R \equiv \langle \bar{\chi}(t,x)\overleftrightarrow{\not{D}}\chi(t,x)\rangle$$

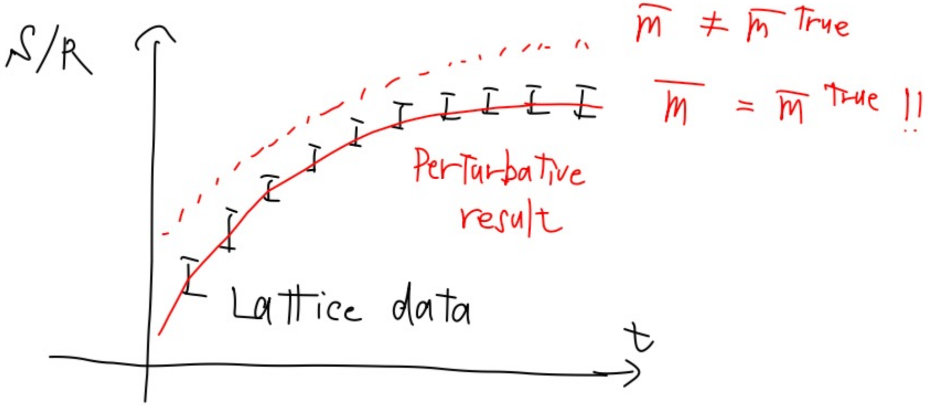
*The wave function renormalization is canceled in the ratio. 2013 Lüscher, 2014 Makino, Suzuki

To determine mass:

Calculate it both in continuum spacetime and on the lattice

Continuum: Perturbation theory gives a function of $t, \overline{m}, \alpha_s$

Lattice: Gives a physical result once the lattice quark mass is determined properly (to reproduce the mass of hadrons)



Existing lattice approaches

Quark masses have been determined using

- Regularization-independent (symmetric) momentum subtraction scheme [RI-(S)MOM]

  1995 Martinelli et al., 2007 Sturm et al.

- quark current (or current-like) correlator method (for heavy quark)

  2008 HPQCD collaboration

- minimal renormalon-subtracted (MSR) scheme

  2018 TUM collaboration

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RI-(S)MOM scheme considers the renormalization of the local bilinear operator $\bar{\psi}(x)\psi(x)$.
Since $m\bar{\psi}(x)\psi(x)$ is a finite bare operator, $Z_m Z_{\bar{\psi}\psi} = 1$, so $Z_m = Z_{\bar{\psi}\psi}^{-1}$.

Our proposal is similar to RI-(S)MOM in that bilinear operators are considered.
But distinct differences arise by applying the gradient flow!

Difference from RI-(S)MOM

RI-(S)MOM	Gradient-flow approach
$\langle \bar{\psi}(z_1)\bar{\psi}\psi(x)\psi(z_2)\rangle$	$S/R = \langle \bar{\chi}(t,x)\chi(t,x)\rangle / \langle \bar{\chi}(t,x)\overleftrightarrow{\not{D}}\chi(t,x)\rangle$
Three-point function	One-point functions
NOT gauge invariant (discussed within the Landau gauge)	gauge invariant
Dimension-two nonperturbative correction? $\langle A^a_\mu(x)A^a_\mu(x)\rangle$	Dimension-four nonperturbative corrections $\langle F^a_{\mu\nu}(x)F^a_{\mu\nu}(x)\rangle, \cdots$

Difference from RI-(S)MOM

Expected properties (to be confirmed)

RI-(S)MOM	Gradient-flow approach
$\langle \bar\psi(z_1)\bar\psi\psi(x)\psi(z_2)\rangle$	$S/R = \langle\bar\chi(t,x)\chi(t,x)\rangle / \langle\bar\chi(t,x)\overleftrightarrow{\not{D}}\chi(t,x)\rangle$ Noise suppression due to gradient flow
Three-point function	One-point functions $a \to 0$ is the only necessary extrapolation
NOT gauge invariant (discussed within the Landau gauge)	gauge invariant Solid foundation
Dimension-two nonperturbative correction? $\langle A_\mu^a(x)A_\mu^a(x)\rangle$	Dimension-four nonperturbative corrections $\langle F_{\mu\nu}^a(x)F_{\mu\nu}^a(x)\rangle, \cdots$ Better perturbative convergence?

Gradient-flow approach could provide a cleaner and precise method.

Today's target

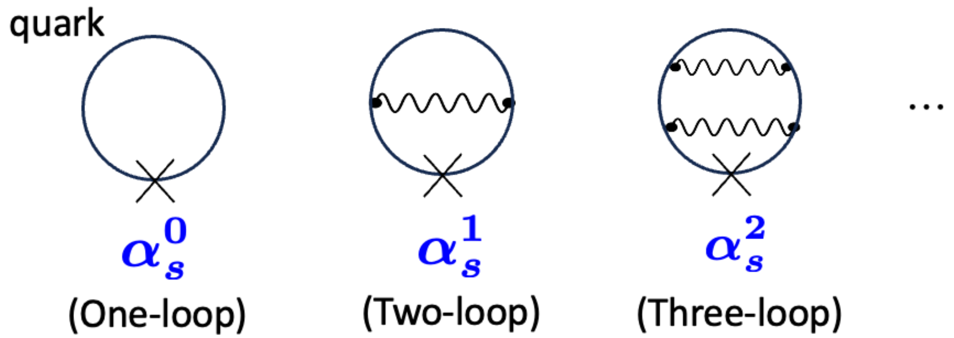
The perturbative result is available for an approximately *massless quark*

NLO: 2014 Makino, Suzuki
NNLO: 2019 Artz, Harlander, Lange, Neumann, and Prausa

$$\langle\bar{\chi}(t,x)\chi(t,x)\rangle/\langle\bar{\chi}(t,x)\overleftrightarrow{\not{D}}\chi(t,x)\rangle$$
$$=\overline{m}t[(\underline{c_0}+\underline{c_1}\alpha_s+\underline{c_2}\alpha_s^2+\underline{\mathcal{O}(\alpha_s^3)})+\underline{\mathcal{O}(\overline{m}^2t)}]$$

Known Missing

Together with our proposal, we provide the perturbative result at the first nontrivial order [O(α_s)^1] for a *massive quark*.



Contents

Two-loop integrals

Eleven scalar integrals contributing to *S.*

$$
\begin{aligned}
I_1 &= \int_0^t ds \int_{p,k} \frac{m}{k^2(p^2+m^2)} e^{-2tp^2-2sk^2}, \\
I_2 &= \int_0^t ds \int_0^s ds' \int_{p,k} \frac{mp^2}{k^2(p^2+m^2)} e^{-(2t-s+s')p^2-(s+s')k^2-(s-s')(k-p)^2}, \\
I_3 &= \int_0^t ds \int_{p,k} \frac{m}{k^2(p^2+m^2)} e^{-(2t-s)p^2-sk^2-s(k-p)^2}, \\
I_4 &= \int_0^t ds \int_0^t ds' \int_{p,k} \frac{m(p-k)^2}{k^2((p-k)^2+m^2)} e^{-(2t-s-s')p^2-(s+s')k^2-(s+s')(k-p)^2}, \\
I_5 &= \int_{p,k} \frac{m}{k^2(p^2+m^2)((p-k)^2+m^2)} e^{-tp^2-tk^2-t(k-p)^2}, \\
I_6 &= \int_0^t ds \int_{p,k} \frac{m}{k^2((p-k)^2+m^2)} e^{-(2t-s)p^2-sk^2-s(k-p)^2}, \\
I_7 &= \int_0^t ds \int_{p,k} \frac{m^3}{k^2(p^2+m^2)((p-k)^2+m^2)} e^{-(2t-s)p^2-sk^2-s(k-p)^2}, \\
I_8 &= \int_{p,k} \frac{m}{k^2(p^2+m^2)((p-k)^2+m^2)} e^{-2tp^2}, \\
I_9 &= \int_{p,k} \frac{m}{k^2(p^2+m^2)^2} e^{-2tp^2}, \\
I_{10} &= \int_{p,k} \frac{m}{(p^2+m^2)^2((p-k)^2+m^2)} e^{-2tp^2}, \\
I_{11} &= \int_{p,k} \frac{m^3}{k^2(p^2+m^2)^2((p-k)^2+m^2)} e^{-2tp^2}.
\end{aligned}
$$

Loop momenta *p,k*, quark (bare) mass *m*, flow time integrals

$$\int_{\boldsymbol{p}} \equiv \int \frac{\boldsymbol{d^d p}}{\boldsymbol{(2\pi)^d}} \quad \text{w/} \quad \boldsymbol{d = 4 - 2\epsilon}$$

Similarly, we have twenty integrals contributing to *R*.

How to evaluate them

We evaluate them using two approaches.

(i) Expand the integrals for small- or large-m^2t to give semianalytic expressions in these limits

(ii) Numerical evaluation with full mass dependence retained

In (i), we develop a new method based on the Laplace transform
rather than the standard technique called ``expansion by regions.” 1998 Beneke, Smirnov
2021 Harlander [application to gradient flow]

Method

Given a loop integral $I(m^2,t)$, we consider the Laplace transform:

$$\tilde{I}(v,t) \equiv \int_0^\infty \mathrm{d}(m^2)(m^2)^{-v-1} I(m^2,t)$$

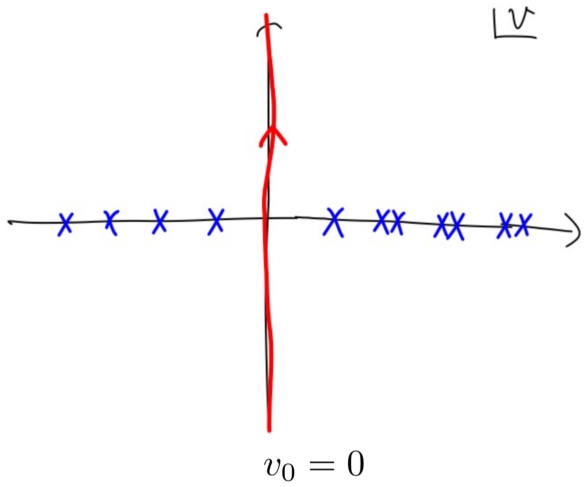
From dimensional analysis,
$\tilde{I}(v,t) \propto t^{v-\dim[I]/2}$

$\tilde{I}(v,t)$ develops singularities reflecting the series expansions of $I(m^2,t)$:

- If $I(m^2,t) = \mathcal{O}((m^2)^{a_s})$ for small m^2, the integral diverges at $v = a_s$.
- If $I(m^2,t) = \mathcal{O}((m^2)^{-a_l})$ for large m^2, the integral diverges at $v = -a_l$.

The inverse transform is given by

$$I(m^2,t) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty+v_0}^{\mathrm{i}\infty+v_0} \mathrm{d}v \tilde{I}(v,t)(m^2)^v$$



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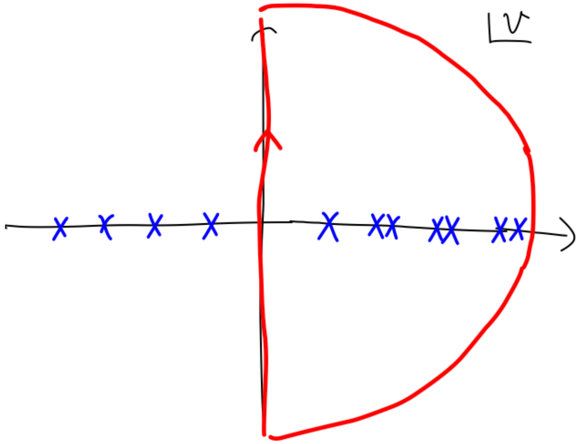
Small-$m^2 t$ expansion:

$$I(m^2,t) = -\sum_{v_{\text{sing}} > v_0} \mathrm{Res}[\tilde{I}(v,t)(m^2)^v]|_{v=v_{\text{sing}}}$$



$(m^2 t)^v \to 0$
as $v \to +\infty$

Method

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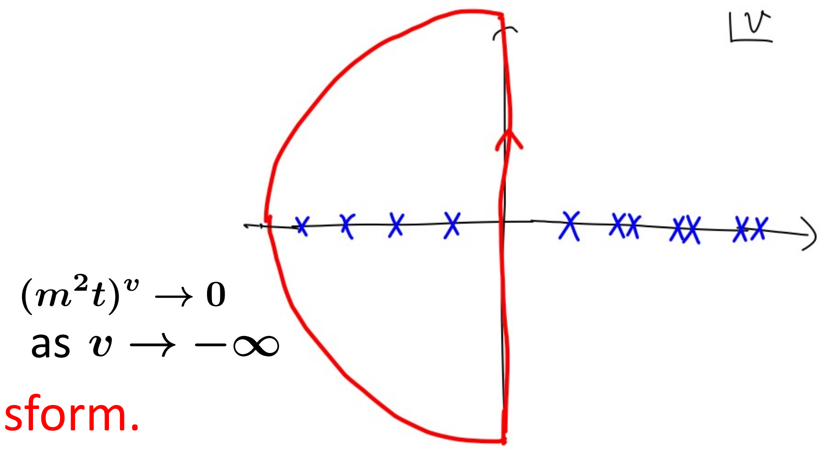
Large-m^2t expansion:

$$I(m^2,t) = \sum_{v_{\text{sing}} < v_0} \mathrm{Res}[\tilde{I}(v,t)(m^2)^v]|_{v=v_{\text{sing}}}$$



We study the singularities of the Laplace transform.
This single quantity can produce both expansions.

One-loop example

$$S|_{1-\text{loop}} = -4N_c \int_p \frac{m}{m^2+p^2} e^{-2tp^2}$$

The Laplace transform

$$\tilde{S}|_{1-\text{loop}} = -4N_c \int_p e^{-2tp^2} \int_0^\infty \mathrm{d}(m^2)(m^2)^{-v-1} \frac{m}{m^2+p^2}$$

$$= -4N_c \frac{\pi}{\cos(\pi v)} \int_p \left(\frac{1}{p^2}\right)^{v+\frac{1}{2}} e^{-2tp^2} = -4N_c \frac{\pi}{\cos(\pi v)} \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(3/2-\epsilon-v)}{\Gamma(2-\epsilon)} (2t)^{v-3/2+\epsilon}$$

The positive singularities: $v = 1/2, 3/2, 3/2-\epsilon, 5/2, 5/2-\epsilon, \ldots$

$$v = 1/2: \quad -\mathrm{Res}[\tilde{I}(v,t)(m^2)^v]|_{v=1/2} = -\frac{N_c m}{8\pi^2 t} + \mathcal{O}(\epsilon)$$

$$v = 3/2: \quad -\mathrm{Res}[\tilde{I}(v,t)(m^2)^v]|_{v=3/2} = -\frac{N_c m^3 t^\epsilon}{4\pi^2}\left(\frac{1}{\epsilon} + 1 + 3\log 2 + \log\pi\right) + \mathcal{O}(\epsilon)$$

$$v = 3/2-\epsilon: \quad -\mathrm{Res}[\tilde{I}(v,t)(m^2)^v]|_{v=3/2-\epsilon} = \frac{N_c m^{3-2\epsilon}}{4\pi^2}\left(\frac{1}{\epsilon} + 1 - \gamma_E + \log(4\pi)\right) + \mathcal{O}(\epsilon)$$

$\vdots$

Canceled

One-loop example

$$S|_{1-\text{loop}} = -4N_c \int_p \frac{m}{m^2+p^2} e^{-2tp^2}$$

The Laplace transform

$$\tilde{S}|_{1-\text{loop}} = -4N_c \int_p e^{-2tp^2} \int_0^\infty \mathrm{d}(m^2)(m^2)^{-v-1} \frac{m}{m^2+p^2}$$

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The positive singularities: $v = 1/2, 3/2, 3/2-\epsilon, 5/2, 5/2-\epsilon, \ldots$

$$S|_{1-\text{loop}} = -\frac{N_c}{8\pi^2} \frac{m}{t} \left[1 + 2m^2 t(\gamma_E + \log 2 + \log(m^2 t)) + 4(m^2 t)^2(-1 + \gamma_E + \log 2 + \log(m^2 t)) + \mathcal{O}((m^2 t)^3)\right] + \mathcal{O}(\epsilon)$$

Small-m^2t expansion

One-loop example

$$S|_{1-\mathrm{loop}} = -4N_c \int_p \frac{m}{m^2+p^2} e^{-2tp^2}$$

The Laplace transform

$$\tilde{S}|_{1-\mathrm{loop}} = -4N_c \int_p e^{-2tp^2} \int_0^\infty \mathrm{d}(m^2)(m^2)^{-v-1} \frac{m}{m^2+p^2}$$

$$= -4N_c \frac{\pi}{\cos(\pi v)} \int_p \left(\frac{1}{p^2}\right)^{v+\frac{1}{2}} e^{-2tp^2} = -4N_c \frac{\pi}{\cos(\pi v)} \frac{1}{(4\pi)^{2-\epsilon}} \frac{\Gamma(3/2-\epsilon-v)}{\Gamma(2-\epsilon)} (2t)^{v-3/2+\epsilon}$$

The nagative singularities: $v = -1/2, -3/2, -5/2, \ldots$

$$S|_{1-\mathrm{loop}} = \frac{N_c}{16\pi^2} \frac{1}{mt^2} \left[1 - \frac{1}{m^2 t} + \frac{3}{2} \frac{1}{(m^2 t)^2} + \mathcal{O}((m^2 t)^{-3})\right] + \mathcal{O}(\epsilon)$$

Large-m^2t expansion

Comments

- Singularities originate from series expansion of the loop *integrand* and *divergences of the loop integral* of the Laplace transformed quantity.

- Two-loop computation can be completed by (mainly) using this method.

- The small-m^2t expansion is given with numerical coefficients, whereas the large-m^2t expansion is given with analytic coefficients.

- We make use of differential equations for evaluating more complicated integrals.

The expansions of S

Small-m²t expansion

$$\overline{S} = \frac{\overline{m}}{t}\sum_{n,k=0}^{\infty} C_{n,k}^{S,\ll 1}(\overline{m}^2 t,\mu^2)(\overline{m}^2 t)^k \alpha_s^n(\mu^2)$$

where

$$C_{1,0}^{S,\ll 1} = -0.00962021 N_c C_F$$

$$C_{1,1}^{S,\ll 1} = N_c C_F\Big(-0.0295528 - 0.0282201\log(\overline{m}^2 t) + 0.0274585\log\left(\frac{\overline{m}^2}{\mu^2}\right) + 0.0120943\log(\overline{m}^2 t)\log\left(\frac{\overline{m}^2}{\mu^2}\right)\Big),$$

$$C_{1,2}^{S,\ll 1} = N_c C_F\Big(-0.020508 - 0.0680192\log(\overline{m}^2 t) - 0.0120943\log(\overline{m}^2 t)^2 + 0.0372681\log\left(\frac{\overline{m}^2}{\mu^2}\right) + 0.0483773\log(\overline{m}^2 t)\log\left(\frac{\overline{m}^2}{\mu^2}\right)\Big),$$

$$C_{1,3}^{S,\ll 1} = N_c C_F\Big(0.00853359 - 0.0694318\log(\overline{m}^2 t) - 0.0282201\log(\overline{m}^2 t)^2 + 0.00752481\log\left(\frac{\overline{m}^2}{\mu^2}\right) + 0.072566\log(\overline{m}^2 t)\log\left(\frac{\overline{m}^2}{\mu^2}\right)\Big),$$

$$C_{1,4}^{S,\ll 1} = N_c C_F\Big(0.0230986 - 0.0388166\log(\overline{m}^2 t) - 0.0342673\log(\overline{m}^2 t)^2 - 0.0201876\log\left(\frac{\overline{m}^2}{\mu^2}\right) + 0.0645031\log(\overline{m}^2 t)\log\left(\frac{\overline{m}^2}{\mu^2}\right)\Big),$$

$$C_{1,5}^{S,\ll 1} = N_c C_F\Big(0.0186958 - 0.00890339\log(\overline{m}^2 t) - 0.029228\log(\overline{m}^2 t)^2 - 0.0247115\log\left(\frac{\overline{m}^2}{\mu^2}\right) + 0.0403144\log(\overline{m}^2 t)\log\left(\frac{\overline{m}^2}{\mu^2}\right)\Big).$$

Large-m²t expansion

$$\overline{S} = \frac{1}{\overline{m}t^2}\sum_{n,k=0}^{\infty} C_{n,k}^{S,\gg 1}(\overline{m}^2 t,\mu^2)(\overline{m}^2 t)^k \alpha_s^n(\mu^2)$$

where

$$C_{1,0}^{S,\gg 1} = \frac{N_c C_F}{64\pi^3}\left(2 + 3\gamma_E + 9\log 2 - 9\log 3 + 3\log(\overline{m}^2 t) - 6\log\left(\frac{\overline{m}^2}{\mu^2}\right)\right),$$

$$C_{1,1}^{S,\gg 1} = -\frac{N_c C_F}{256\pi^3}\left(22 + 21\gamma_E + 63\log 2 - 51\log 3 + 21\log(\overline{m}^2 t) - 48\log\left(\frac{\overline{m}^2}{\mu^2}\right)\right),$$

$$C_{1,2}^{S,\gg 1} = \frac{3N_c C_F}{1024\pi^3}\left(68 + 57\gamma_E + 171\log 2 - 127\log 3 + 57\log(\overline{m}^2 t) - 144\log\left(\frac{\overline{m}^2}{\mu^2}\right)\right),$$

$$C_{1,3}^{S,\gg 1} = -\frac{N_c C_F}{12288\pi^3}\left(6634 + 5049\gamma_E + 15147\log 2 - 10719\log 3 + 5049\log(\overline{m}^2 t) - 13824\log\left(\frac{\overline{m}^2}{\mu^2}\right)\right),$$

$$C_{1,4}^{S,\gg 1} = \frac{N_c C_F}{147456\pi^3}\Big(253556 + 176985\gamma_E + 530955\log 2 - 364095\log 3 + 176985\log(\overline{m}^2 t) - 518400\log\left(\frac{\overline{m}^2}{\mu^2}\right)\Big),$$

$$C_{1,5}^{S,\gg 1} = -\frac{5N_c C_F}{589824\pi^3}\Big(746314 + 480897\gamma_E + 1442691\log 2 - 967383\log 3 + 480897\log(\overline{m}^2 t) - 1492992\log\left(\frac{\overline{m}^2}{\mu^2}\right)\Big).$$

The expansions of *R*

Small-m²t expansion

$$\overline{R} = \frac{1}{t^2}\sum_{n,k=0}^{\infty} C_{n,k}^{R,\ll 1}(\overline{m}^2 t, \mu^2)(\overline{m}^2 t)^k \alpha_s^n(\mu^2)$$

where

$$C_{1,0}^{R,\ll 1} = N_c C_F\left(-0.00227508 + 0.00302358\log(\overline{m}^2 t) - 0.00302358\log\left(\frac{\overline{m}^2}{\mu^2}\right)\right),$$

$$C_{1,1}^{R,\ll 1} = N_c C_F\left(0.0273809 + 0.00604716\log(\overline{m}^2 t) - 0.00604716\log\left(\frac{\overline{m}^2}{\mu^2}\right)\right),$$

$$C_{1,2}^{R,\ll 1} = N_c C_F\Big(0.0837551 + 0.0972472\log(\overline{m}^2 t) + 0.0120943\log(\overline{m}^2 t)^2 - 0.0702812\log\left(\frac{\overline{m}^2}{\mu^2}\right) - 0.036283\log(\overline{m}^2 t)\log\left(\frac{\overline{m}^2}{\mu^2}\right)\Big),$$

$$C_{1,3}^{R,\ll 1} = N_c C_F\Big(0.0559953 + 0.21159\log(\overline{m}^2 t) + 0.0645031\log(\overline{m}^2 t)^2 - 0.0810759\log\left(\frac{\overline{m}^2}{\mu^2}\right) - 0.120943\log(\overline{m}^2 t)\log\left(\frac{\overline{m}^2}{\mu^2}\right)\Big),$$

$$C_{1,4}^{R,\ll 1} = N_c C_F\Big(-0.0250186 + 0.216651\log(\overline{m}^2 t) + 0.131022\log(\overline{m}^2 t)^2 - 0.00949501\log\left(\frac{\overline{m}^2}{\mu^2}\right) - 0.169321\log(\overline{m}^2 t)\log\left(\frac{\overline{m}^2}{\mu^2}\right)\Big),$$

$$C_{1,5}^{R,\ll 1} = N_c C_F\Big(-0.0685145 + 0.125177\log(\overline{m}^2 t) + 0.169321\log(\overline{m}^2 t)^2 + 0.0494534\log\left(\frac{\overline{m}^2}{\mu^2}\right) - 0.145132\log(\overline{m}^2 t)\log\left(\frac{\overline{m}^2}{\mu^2}\right)\Big).$$

Large-m²t expansion

$$\overline{R} = \frac{1}{\overline{m}^2 t^3}\sum_{n,k=0}^{\infty} C_{n,k}^{R,\gg 1}(\overline{m}^2 t, \mu^2)(\overline{m}^2 t)^{-k} \alpha_s^n(\mu^2)$$

where

$$C_{1,0}^{R,\gg 1} = \frac{N_c C_F}{128\pi^3}\left(4 + 18\gamma_E + 54\log 2 - 45\log 3 + 18\log(\overline{m}^2 t) - 36\log\left(\frac{\overline{m}^2}{\mu^2}\right)\right),$$

$$C_{1,1}^{R,\gg 1} = -\frac{N_c C_F}{512\pi^3}\left(94 + 156\gamma_E + 468\log 2 - 351\log 3 + 156\log(\overline{m}^2 t) - 360\log\left(\frac{\overline{m}^2}{\mu^2}\right)\right),$$

$$C_{1,2}^{R,\gg 1} = \frac{N_c C_F}{18432\pi^3}\Big(11686 + 14202\gamma_E + 42606\log 2 - 30267\log 3 + 14202\log(\overline{m}^2 t) - 36288\log\left(\frac{\overline{m}^2}{\mu^2}\right)\Big),$$

$$C_{1,3}^{R,\gg 1} = -\frac{N_c C_F}{73728\pi^3}\Big(168494 + 168480\gamma_E + 505440\log 2 - 347085\log 3 + 168480\log(\overline{m}^2 t) - 466560\log\left(\frac{\overline{m}^2}{\mu^2}\right)\Big),$$

$$C_{1,4}^{R,\gg 1} = \frac{N_c C_F}{98304\pi^3}\Big(891478 + 770310\gamma_E + 2310930\log 2 - 1549935\log 3 + 770310\log(\overline{m}^2 t) - 2280960\log\left(\frac{\overline{m}^2}{\mu^2}\right)\Big).$$

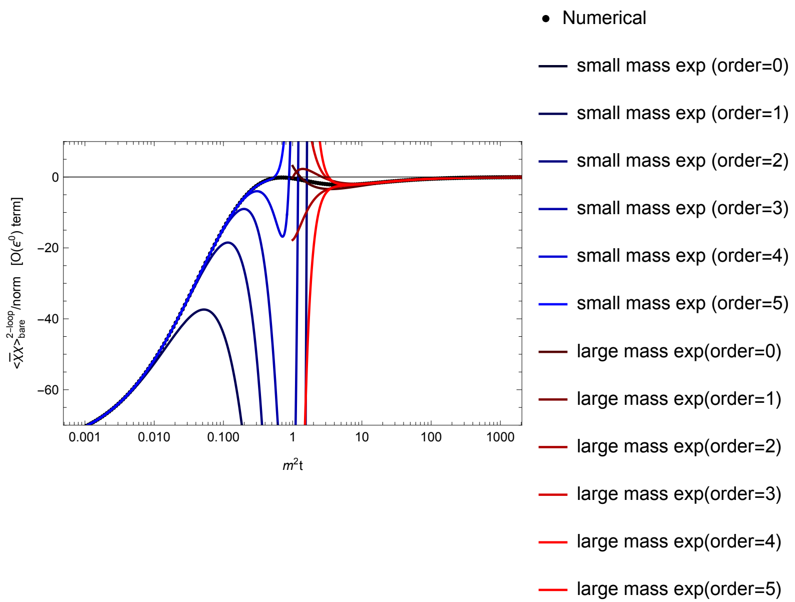
Numerical evaluation

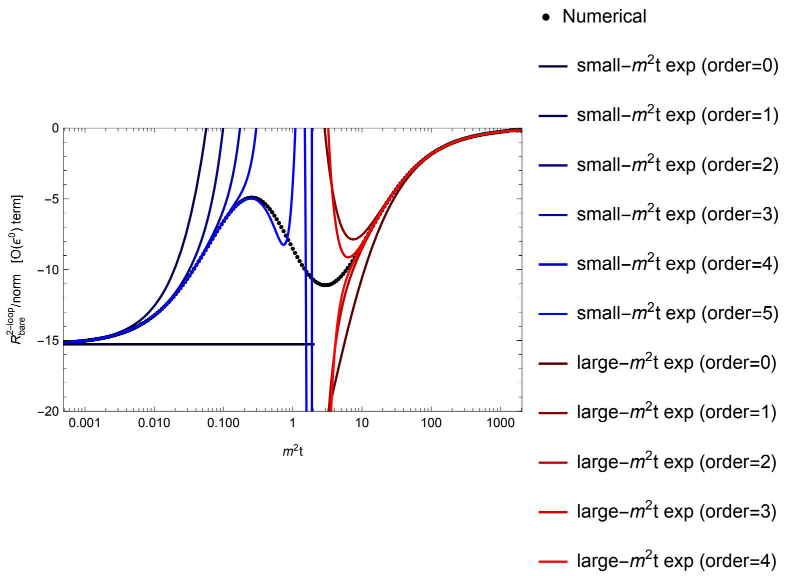
We also provide numerical evaluation of the loop integrals with full mass dependence retained.

Using “ftint” [2024 Harlander, Nellopoulos, Olsson, Wesle], we evaluate them for a wide range,

$$2^{-11} \leq \overline{m}^2 t \leq 2^{11}$$

We verified the agreement between our expansions and our numerical results.





Contents

Lattice observables

We consider

(i) $S/R = \bar{S}(\overline{m})/\bar{R}(\overline{m})$

(ii) $R/R_{m=0} = \bar{R}(\overline{m})/\bar{R}(0)$

These are finite quantities due to the cancellation of the wavefunction renormalization factor and can be measured on the lattice.
Note that flavors for the numerator and the denominator do not have to be the same.

Range of the flow time

Lattice simulation: $a^2 \ll 8t \ll L^2$

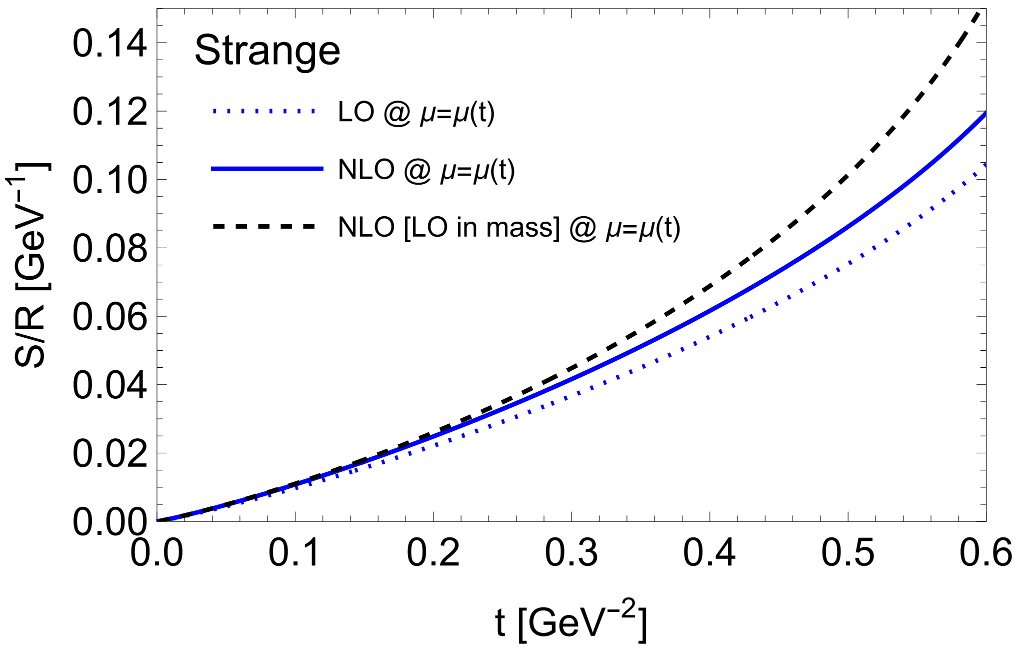
Perturbativity: $8t \ll \Lambda_{\overline{\mathrm{MS}}}^{-2}$

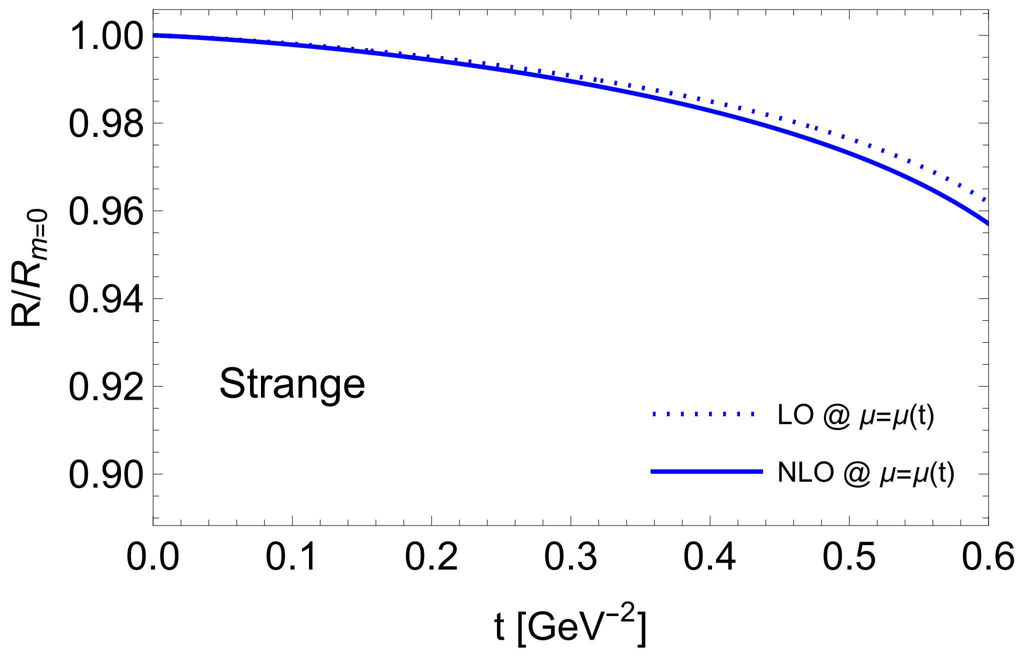
Strange

Because of $\overline{m}_s^2 t \ll 1$ in the figure below, we use the small-m^{2}t expansion.

We set the renormalization scale to $\mu(t) \equiv \frac{e^{-\gamma_E/2}}{\sqrt{2t}}$. 2019 Artz, Harlander, Lange, Neumann, Prausa

Preliminary





cf.

$$\frac{1}{\sqrt{8t}} = 5 \text{ GeV} \ @ \ t = 0.005 \text{ GeV}^{-2}$$

$$\frac{1}{\sqrt{8t}} = 0.5 \text{ GeV} \ @ \ t = 0.5 \text{ GeV}^{-2}$$

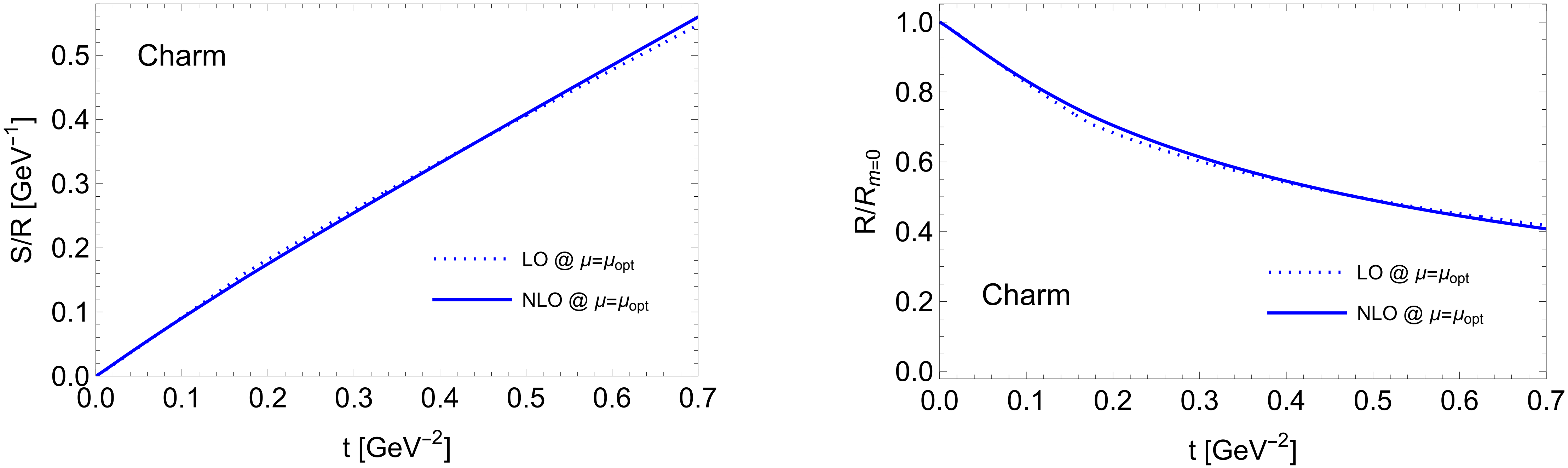
Charm

Because $\overline{m}_c^2 t$ can be $\overline{m}_c^2 t \ll 1$ or $\overline{m}_c^2 t \sim 1$, we use the numerical results.

$\mu(t) \equiv \frac{e^{-\gamma_E/2}}{\sqrt{2t}}$

To keep the expansion parameter small, we take the renormalization scale to $\mu_{\text{opt}} \equiv \max[\mu(t), \overline{m}_c(\overline{m}_c)]$.

Preliminary



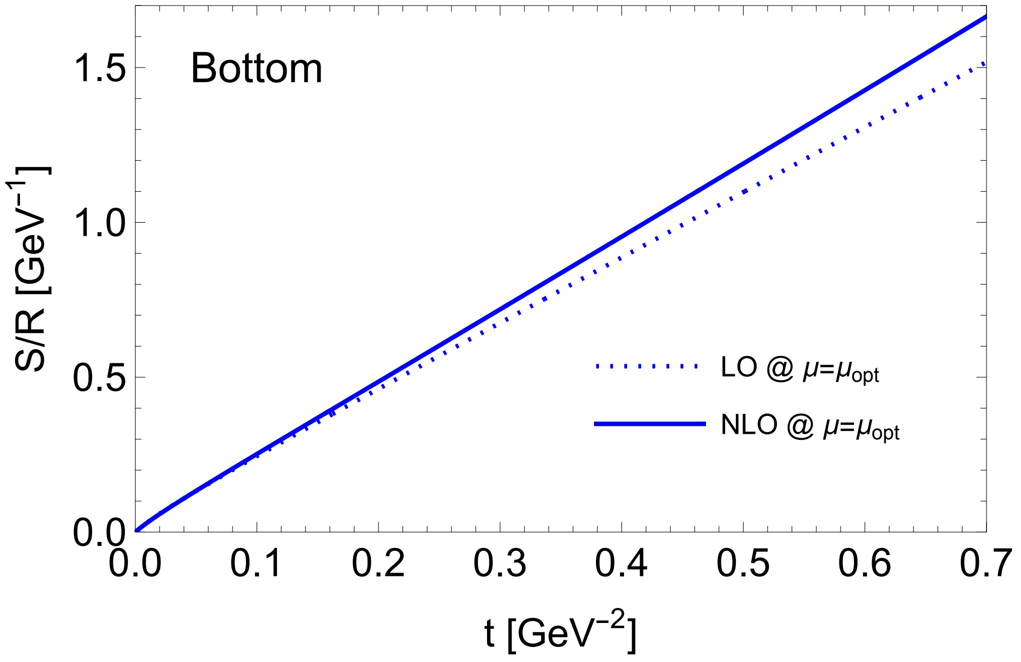
cf. $\mu(t) = \overline{m}_c(\overline{m}_c)$ @ $t = 0.173\ \text{GeV}^{-2}$

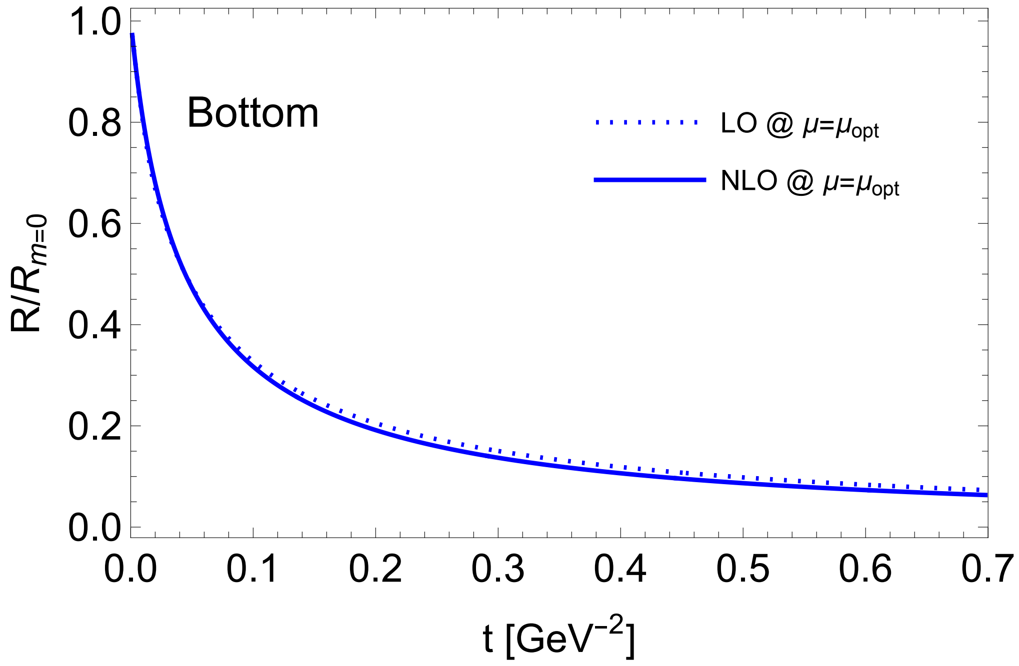
Bottom

Because $\overline{m}_b^2 t$ can be $\overline{m}_b^2 t \ll 1$ or $\overline{m}_b^2 t \gg 1$, we use the numerical results.

To keep the expansion parameter small, we take the renormalization scale to $\mu_{\rm opt} \equiv \max[\mu(t), \overline{m}_b(\overline{m}_b)]$.

Preliminary





cf. $\mu(t) = \overline{m}_b(\overline{m}_b)$ @ $t = 0.016\ \mathrm{GeV}^{-2}$

Summary

- *We proposed a new method for quark mass determination using the gradient flow*.

  → Ratios of bilinear operators of flowed quark

- This method can be regarded as a *gauge invariant* extension of the RI-(S)MOM schemes, and has potential to provide a cleaner and precise method.

- We provided two-loop results for these operators including mass effects.

  → *Significant for strange, charm, and bottom quarks*

- *We developed a new method to expand loop integrals using the Laplace transform*.

- Lattice observables were studied at NLO, where the NLO corrections were about 10 %.

Outlook

- Crucial for precision to calculate higher orders in perturbation theory. → Next talk by Mason
- Small flow time expansion/OPE should be studied for deeper theoretical understanding.
- Precision of lattice measurement?