Short-flow-time expansion of four-quark operators at NNLO QCD

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The heavy quark expansion

- Lifetimes are fundamental parameters of mesons
- Operator product expansion

$$\Gamma(B \to X) = \sum_i \Gamma_i \langle B|O_i|B\rangle$$

Contribution from $\Delta B = 0$ four quark operators

- Wilson coefficients Γ_i perturbative

$$\Gamma_i = \Gamma_i^{(0)} + \frac{\alpha_s}{4\pi}\Gamma_i^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \Gamma_i^{(2)} + \dots$$

- Matrix elements non-perturbative
- Only outdated lattice results

The gradient flow

- Operator product expansion

$$\sum_i \Gamma_i \langle B|O_i|B\rangle = \sum_i \tilde{\Gamma}_i \langle B|\tilde{O}_i|B\rangle$$

- Finite operators ideal for the lattice
- Procedure has been validated
  - Energy momentum tensor [Makino, Suzuki 2014] [Harlander, Kluth, Lange 2019]
  - Kaon bag parameters [Suzuki, Taniguchi, Suzuki, Kanaya 2020] [Black, Harlander, Lange, Rago, Shindler, Witzel 2023] [Black, Harlander, Lange, Rago, Shindler, Witzel 2024]
  - Parton distribution functions [Shindler 2023] [Francis, Fritzsch, Karur, Kim et al. 2024]

The short-flow-time expansion

- Short-flow-time expansion

$$\tilde{\mathcal{O}}_i \sim \sum_j \zeta_{ij} \mathcal{O}_j + \dots$$

$$\sum_i \Gamma_i \langle B|O_i|B\rangle = \sum_{i,j} \underbrace{\Gamma_j \zeta_{ji}^{-1}}_{\tilde{\Gamma}_i} \langle B|\tilde{O}_i|B\rangle$$

- Mixing shifted to the perturbative calculation

Perturbative calculation

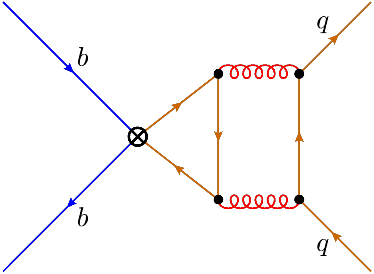
- $\Delta B = 0$ operators

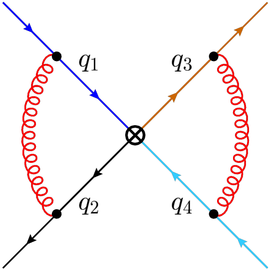
$$\begin{aligned}
\mathcal{Q}_1 &= \left(\bar{b}\gamma_\mu P_L q\right)\,\left(\bar{q}\gamma_\mu P_L b\right)\\
\mathcal{Q}_2 &= \left(\bar{b} P_L q\right)\,\left(\bar{q} P_L b\right)\\
\mathcal{T}_1 &= \left(\bar{b}\gamma_\mu P_L t^a q\right)\,\left(\bar{q}\gamma_\mu P_L t^a b\right)\\
\mathcal{T}_2 &= \left(\bar{b} P_L t^a q\right)\,\left(\bar{q} P_L t^a b\right)
\end{aligned}$$

- Not a closed set of operators under renormalization

Perturbative calculation

$$\left(\bar{b}\Gamma q\right)\ \left(\bar{q}\Gamma b\right) \qquad \left(\bar{q_1}\Gamma q_2\right)\ \left(\bar{q_3}\Gamma q_4\right)$$





Lifetime differences

- Mixing with penguin operators and lower dimensional operators drops out

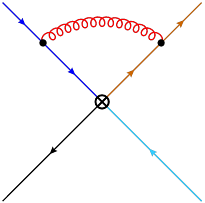
$$
\begin{aligned}
\mathcal{Q}_1 &= \left(\bar{q}_1 \gamma_\mu P_L q_2\right) \left(\bar{q}_3 \gamma_\mu P_L q_4\right) \\
\mathcal{Q}_2 &= \left(\bar{q}_1 P_L q_2\right) \left(\bar{q}_3 P_L q_4\right) \\
\mathcal{T}_1 &= \left(\bar{q}_1 \gamma_\mu P_L t^a q_2\right) \left(\bar{q}_3 \gamma_\mu P_L t^a q_4\right) \\
\mathcal{T}_2 &= \left(\bar{q}_1 P_L t^a q_2\right) \left(\bar{q}_3 P_L t^a q_4\right)
\end{aligned}
$$

- Basis of operators smaller
- Mixing matrix of $\mathcal{Q}_1$ and $\mathcal{T}_1$ known [Harlander, Lange 2023]

Evanescent operators

- Consider 1 loop diagrams of

$$\mathcal{Q}_1 \sim \gamma_\mu P_L \otimes \gamma^\mu P_L$$



$$\sim \gamma_{\mu\nu\rho} P_L \otimes \gamma^{\mu\nu\rho} P_L$$

- In $D = 4$

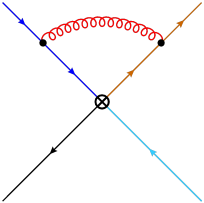
$$E_Q^1 = \gamma_{\mu\nu\rho} P_L \otimes \gamma^{\mu\nu\rho} P_L - 16 \gamma_\mu P_L \otimes \gamma^\mu P_L = 0$$

- Where $\gamma_{\mu\ldots\nu} = \gamma_\mu \ldots \gamma_\nu$

Evanescent operators

- Consider 1 loop diagrams of

$$\mathcal{Q}_1 \sim \gamma_\mu P_L \otimes \gamma^\mu P_L$$



$$\sim \gamma_{\mu\nu\rho} P_L \otimes \gamma^{\mu\nu\rho} P_L$$

- In $D = 4 - 2\varepsilon$

$$E_Q^1 = \gamma_{\mu\nu\rho} P_L \otimes \gamma^{\mu\nu\rho} P_L - 16\gamma_\mu P_L \otimes \gamma^\mu P_L = \mathcal{O}(\varepsilon)$$

- Operator mixing with E_Q^1
- Where $\gamma_{\mu\ldots\nu} = \gamma_\mu \ldots \gamma_\nu$

Evanescent operators

$$E \overset{D\to 4}{=} 0$$

- Mix into the physical operators

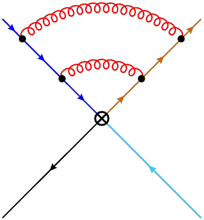
$$\begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}_R = \begin{pmatrix} Z_{\mathcal{O}\mathcal{O}} & Z_{\mathcal{O}E} \\ Z_{E\mathcal{O}} & Z_{EE} \end{pmatrix} \begin{pmatrix} \mathcal{O} \\ E \end{pmatrix}_B$$

- Part of operator basis
- $Z_{E\mathcal{O}}$ includes finite terms chosen such that

$$E_R = \mathcal{O}(\varepsilon)$$

Evanescent operators

- Same principle at two loops



$$\sim \gamma_{\mu\nu\rho\sigma\tau} P_L \otimes \gamma^{\mu\nu\rho\sigma\tau} P_L$$

- In $D = 4 - 2\varepsilon$

$$E_Q^2 = \gamma_{\mu\nu\rho\sigma\tau} P_L \otimes \gamma^{\mu\nu\rho\sigma\tau} P_L - 256 \gamma_\mu P_L \otimes \gamma^\mu P_L = \mathcal{O}(\varepsilon)$$

- 4 physical operators, 8 evanescent operators

Evanescent Operators scheme

- Evanescent operators not uniquely defined

$$E_i' = E_i + \varepsilon a_i^{(1)} + \varepsilon^2 a_i^{(2)} + \dots$$

- Values of $a_i^{(n)}$ define scheme of evanescent operators
- Our calculation in general scheme
- Scheme of ζ_{ji}^{-1} must be the same as scheme of wilson coefficient Γ_j

$$\tilde{\Gamma}_i = \Gamma_j \zeta_{ji}^{-1}$$

- Scheme dependence cancels

Preliminary results

$$\mathcal{O}_i = \zeta_{ij}^{-1} \tilde{\mathcal{O}}_j + \ldots$$

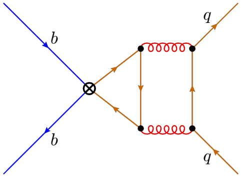
$$\begin{aligned}
\zeta_{22}^{-1} = 1 &+ \left(\frac{\alpha_s}{\pi}\right) \left(\frac{4}{3} + 4L_{\mu t}\right) \\
&+ \left(\frac{\alpha_s}{\pi}\right)^2 \left[\frac{2291}{72} + \frac{7}{4}\zeta_2 + \frac{397}{9}\ln(2) - \frac{89}{2}\ln(3) - \frac{26}{9}\ln^2(2)\right. \\
&- \frac{47}{2}\mathrm{Li}_2\left(\frac{1}{4}\right) - n_{\mathrm{f}}\left(\frac{10}{9} + \frac{1}{3}\zeta_2\right) + L_{\mu t}\left(\frac{425}{12} - \frac{52}{9}\ln(2) - \frac{10}{9}n_{\mathrm{f}}\right) \\
&\left. + L_{\mu t}^2\left(\frac{27}{2} - \frac{1}{3}n_{\mathrm{f}}\right)\right] + \mathcal{O}\left(\alpha_s^3\right)
\end{aligned}$$

where $L_{\mu t} = \ln(2\mu^2 t) + \gamma_E$

Penguin operators

- Closed fermion loops introduce penguin operators

$$\mathcal{Q} = \left(\bar{b}\Gamma q\right)\,\left(\bar{q}\Gamma b\right) \qquad \mathcal{P}_Q = \left(\bar{b}\Gamma b\right)\,\sum_q \left(\bar{q}\Gamma q\right)$$



- Lower dimensional operators
- Extends the operator basis

Setup for the calculation

- `qgraf` [Nogueira 1991]
- `tapir` [Gerlach, Herren, Lang 2022]
- `exp` [Harlander, Seidensticker, Steinhauser 1998, Seidensticker 1999]
- `FORM` [Vermaseren 1989]
- `Kira` [Maierhöfer, Usovitsch, Uwer 2017; Klappert, Lange, Maierhöfer, Usovitsch 2020]
- `FireFly` [Klappert, Lange 2019], [Klappert, Klein, Lange 2020]
- `ftint` [Harlander, Nellopoulos, Olsson, Wesle 2024]
  - `pySecDec` [Borowka, Heinrich, Jahn, Jones, Kerner, Schlenk, Zirke 2017]

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Automization

- Automatical calculation of Feynman rules with

`frules`

[Harlander, Geuskens (unpublished)]

- Standard Feynman rules already implemented

- Tool for the implementation of the Feynman rules of the operators

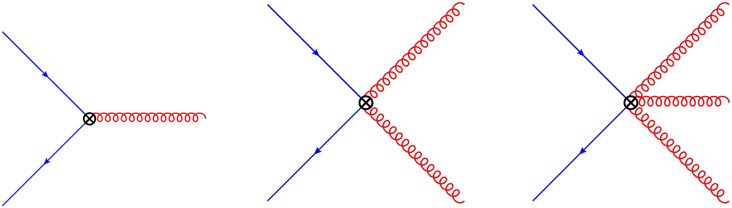
`prepsetup`

Prepsetup example

- Operator related to penguins by EOM

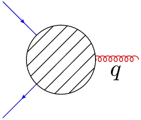
$$\mathcal{N}_1 = \bar{b}\gamma_\mu P_L t^a b \; D_\nu G^a_{\mu\nu}$$

- Multiple Feynman rules



- Projector

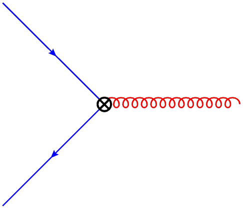
$$P_{\mathcal{N}_1} \sim \left(\frac{\partial}{\partial q}\right)^2$$



Prepsetup example

qgraf

```
[N1, g, Qb, qb]
[N1, b, Qb, qb]
[N1, g, Sb, qb]
[N1, b, Sb, rb]
```



tapir

```
{N1, g, Qb, qb: *VN1gqq(<lorentz_index_2>,<momentum_2>,...)|
                *CN1gqq(<color_index_2>,...)||}
```

form

```
id VN1gqq(mu1,q1, ...) = Vec(mu1,q1)*...;
```

Conclusion

- Lifetimes of mesons with an heavy quark calculated with HQE
- Gradient flow allows lattice calculation of non-perturbative $\Delta B = 0$ matrix elements
- Perturbative matching necessary
- Automization for large operator basis

Evanescent operators 1 loop

$$\mathcal{E}^{(1)}_{Q_1} = \left(\bar{q}_1 \gamma_{\mu\nu\rho} P_L q_2\right) \left(\bar{q}_3 \gamma_{\mu\nu\rho} P_L q_4\right) - 16\mathcal{Q}_1$$
$$\mathcal{E}^{(1)}_{Q_2} = \left(\bar{q}_1 \gamma_{\mu\nu} P_L q_2\right) \left(\bar{q}_3 \gamma_{\mu\nu} P_L q_4\right) - 4\mathcal{Q}_2$$
$$\mathcal{E}^{(1)}_{T_1} = \left(\bar{q}_1 \gamma_{\mu\nu\rho} P_L t^a q_2\right) \left(\bar{q}_3 \gamma_{\mu\nu\rho} P_L t^a q_4\right) - 16\mathcal{T}_1$$
$$\mathcal{E}^{(1)}_{T_2} = \left(\bar{q}_1 \gamma_{\mu\nu} P_L t^a q_2\right) \left(\bar{q}_3 \gamma_{\mu\nu} P_L t^a q_4\right) - 4\mathcal{T}_2$$

Evanescent operators 2 loop

$$\mathcal{E}^{(2)}_{Q_1} = \left(\bar{q}_1 \gamma_{\mu\nu\rho\sigma\tau} P_L q_2\right) \left(\bar{q}_3 \gamma_{\mu\nu\rho\sigma\tau} P_L q_4\right) - 256 \mathcal{Q}_1$$

$$\mathcal{E}^{(1)}_{Q_2} = \left(\bar{q}_1 \gamma_{\mu\nu\rho\sigma} P_L q_2\right) \left(\bar{q}_3 \gamma_{\mu\nu\rho\sigma} P_L q_4\right) - 16 \mathcal{Q}_2$$

$$\mathcal{E}^{(1)}_{T_1} = \left(\bar{q}_1 \gamma_{\mu\nu\rho\sigma\tau} P_L t^a q_2\right) \left(\bar{q}_3 \gamma_{\mu\nu\rho\sigma\tau} P_L t^a q_4\right) - 256 \mathcal{T}_1$$

$$\mathcal{E}^{(1)}_{T_2} = \left(\bar{q}_1 \gamma_{\mu\nu\rho\sigma} P_L t^a q_2\right) \left(\bar{q}_3 \gamma_{\mu\nu\rho\sigma} P_L t^a q_4\right) - 16 \mathcal{T}_2$$

Calculating the mixing matrix

- It is possible to construct projectors $P_n[X]$ so that

  [Gorishny, Larin, Tkachov 1983; Gorishny, Larin 1986]

$$P_n[\mathcal{O}_m] = \delta_{nm}$$

  holds to all orders in perturbation theory

- Using these projectors on the flowed operators

  [Harlander, Kluth, Lange 2019]

$$\tilde{\mathcal{O}}_n(t) \approx \sum_m \xi^B_{nm} \mathcal{O}^B_m + \ldots$$

  leads to

$$\boxed{P_n[\tilde{\mathcal{O}}_m] = \xi^B_{mn}}$$

Form of the Projectors

$$P_n[\mathcal{O}_m] = \delta_{nm}$$

- The projectors have the general form

$$P_n[X] = \sum_k \Pi_k(\partial_p, \partial_m) \langle f_k | X | i_k \rangle \Big|_{p=m=0}$$

- Because all scales are set to zero this only has to hold at tree level

Renormalization of composite operators

- First all of the physical operators have to be found

$$\mathcal{O}^R = Z\mathcal{O}^B$$

- The off shell projectors have to be orthogonalized with respect to operators vanishing due to EOM

$$P[\mathcal{O}_{EOM}] = 0$$

Flowed quark fields

- The flow equations for the flowed quark field χ are

$$\partial_t \chi = \triangle \chi - \kappa \partial_\mu B_\mu^a T^a \chi, \tag{1}$$

$$\partial_t \overline{\chi} = \overline{\chi} \overleftarrow{\triangle} + \kappa \overline{\chi} \partial_\mu B_\mu^a T^a, \tag{2}$$

$$\chi^i(t = 0, x) = \psi^i(x) \tag{3}$$