Progress on the infinite volume based gradient flow for high precision determination of the $\Lambda_{\overline{MS}}$ scale of QCD.

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On behalf of LatHC collaboration

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Gradient Flow 2025

Determination of $\alpha_S(m_Z)$

- Goal: High precision determination of the strong coupling $\alpha_S(m_Z)$ at the Z-pole in QCD with massless $N_f = 3$ fermions.
- The determination of $\alpha_S(m_Z)$, equivalently $\mu^{-1}\Lambda_{\overline{MS}}$, requires integration of the inverse β-function from perturbative regime to the scale of hadron physics at strong coupling.

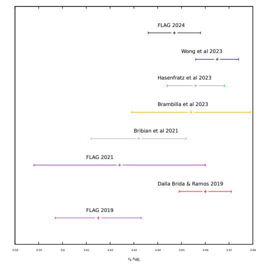
$$\mu^{-1} \cdot \Lambda_{\overline{MS}} = (b_0 \bar{g}^2)^{-b_1/2b_0^2} \cdot \exp(-1/2b_0\bar{g}^2) \cdot \exp\left(-\int_0^{\bar{g}} dx[1/\beta(x) + 1/b_0x^3 - b_1/b_0^2x]\right)$$

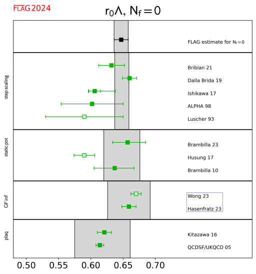
- The integration beyond the perturbative regime requires non-perturbative calculation of the β-function, which can be done on lattice
- On the lattice:
  - g^2 and β-function are not defined in $\overline{MS}$ scheme, but it is not an issue as long as the conversion to $\Lambda_{\overline{MS}}$ is known
  - μ^{-1} is typically r_0 or $\sqrt{8t_0}$ where $\bar{g}^2 = g^2(\mu)$ is known

Determination of $\alpha_S(m_Z)$

- Since 2017, we developed a variety of high-precision, gradient-flow based strategies utilizing β-function defined in infinite physical volume, applied in the sextet $N_f = 2$ model, fund $N_f = 10$ and fund $N_f = 12$ models.
- Our goal is to apply this method on $N_f = 3$ QCD. In Lattice 2022, we applied this to $N_f = 0$ and fund $N_f = 10$ as pilot studies.
- It turns out $N_f = 0$, supposedly the simplest case, is already challenging

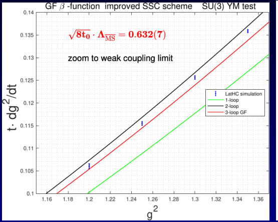
$N_f = 0$ model





- The FLAG 2019 world average summarizes studies prior to GF-based determinations
- The GF-based determination by Dalla Brida & Ramos 2019 significantly deviates from FLAG 2019 world average, creating tension with previous studies
- Our results, Wong et al 2023(PoS of Lattice 2022), agrees with Dalla Brida & Ramos 2019
- Hasenfratz et al 2023, using the same approach as us, got very consistent results with us and worsens the tension
- Other recent GF-based results also seem to have the same trend

$N_f = 0$ model



Collaboration	Ref.	N_f	publication status	renormalization scale	perturbative behaviour	continuum extrapolation	scale	$\sqrt{8t_0}\Lambda_{\overline{MS}}$	$r_0\Lambda_{\overline{MS}}$ *
Hasenfratz 23	[699]	0	A	★	★	★	$\sqrt{t_0}$	0.622(10)	0.659(11)
Wong 23	[700]	0	C	★	○	★	$\sqrt{t_0}$	0.632(7)	0.670(8)

and the 3-loop β-function coefficient b_2 is known. In the pure gauge theory is is given by

$$b_2^{\rm GF} = -1.90395(4)/(4\pi)^3, \quad (N_f = 0). \qquad (387)$$

This is almost three times larger in magnitude than in the $\overline{\rm MS}$ scheme and of opposite sign. One naturally worries about higher-order corrections being large, too. As a result, making contact with perturbation theory requires very small couplings. To quantify the

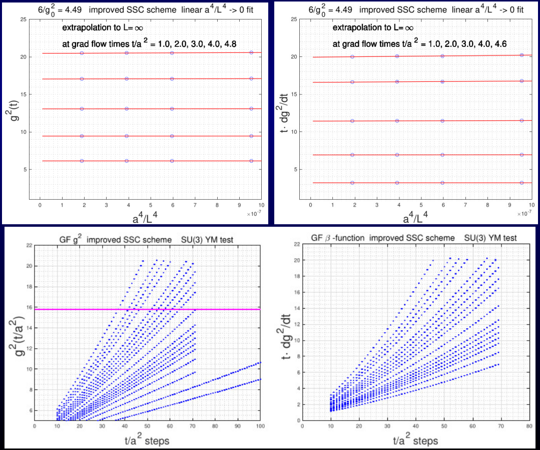
- In Lattice 2022, we made contact at $\alpha_S \approx 0.095$ (i.e. $g^2_{\rm PT} = 4\pi\alpha_S \approx 1.2$) and got an error $\approx 1.4\%$
- FLAG 2024 criticizes that this introduces an uncontrolled systematic error and hence gave a lower rating
- This motivates us to improve our dataset and analysis. We conclude that it is not a significant effect
- In this talk, we discuss the progress on such investigation

LATTICE 2022 Results

- Simulation details of $N_f = 0$:
  - Symanzik improved $SU(3)$ gauge action in periodic boundary conditions
  - Heat-bath + over relaxation ($N_{\text{overrelax}} = 3$)
  - Only improved SSC scheme (definitions in [2203.15847]) is discussed

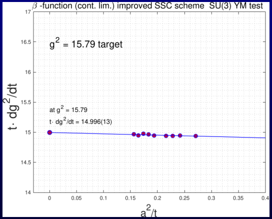
LATTICE 2022 Results

- Methodology:
  - $g^2(t)$ is defined in infinite physical volume wtih Gradient Flow schemes at flow time t
  - $\beta(g^2(t)) = t \cdot dg^2(t)/dt$ is obtained
  - Infinite volume limit $\Rightarrow$ Target g^2 $\Rightarrow$ Continuum limit
  - Scan g^2 values from weak to strong coupling regimes up to $g^2(t_0)$



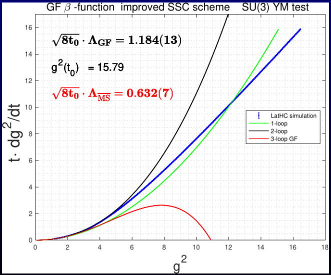
LATTICE 2022 Results

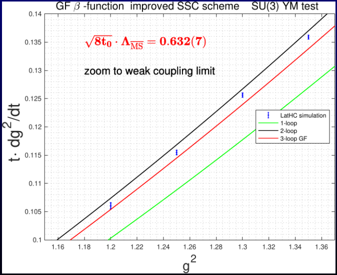
- Methodology:
  - $g^2(t)$ is defined in infinite physical volume wtih Gradient Flow schemes at flow time t
  - $\beta(g^2(t)) = t \cdot dg^2(t)/dt$ is obtained, approximated by 5-point stencil
  - Infinite volume limit $\Rightarrow$ Target g^2 $\Rightarrow$ Continuum limit
  - Scan g^2 values from weak to strong coupling regimes up to $g^2(t_0)$



LATTICE 2022 Results

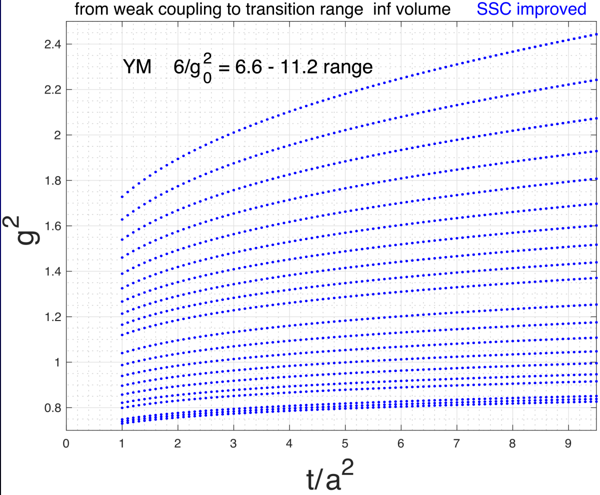
- Numerical data is not available for $g^2 < g^2_{\rm PT} = 1.2$
- Assume that β-function is well-described by the known 3-loop perturbative GF β-function($\beta^{\text{3-loop}}_{\rm GF}$)[JHEP06(2016)161] at $g^2 < g^2_{\rm PT}$
- $\sqrt{8t_0}\,\Lambda_{\rm GF}$ is obtained by :
  - Integrating $\beta^{\text{3-loop}}_{\rm GF}$ from 0 to $g^2_{\rm PT}$
  - Integrate a spline-fit using data between $g^2_{\rm PT}$ and $g^2(t_0)$
- Convert $\Lambda_{\rm GF}$ into $\Lambda_{\rm MS}$ using the well known 1-loop calculation
- Convert $\sqrt{8t_0}$ into r_0 with $\sqrt{8t_0}/r_0 = 0.948(7)$[JHEP08(2010)071]
- $r_0\,\Lambda_{\overline{MS}} = 0.665(9)$ is obtained





Improvements

- In this talk, we focus on the weak coupling regime $g^2 \leq 1.8$
- We have improvements in both dataset and analysis
- The dataset extended by ≈ 20 more ensembles aiming at the weak coupling regime



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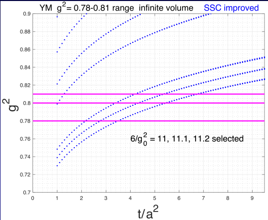
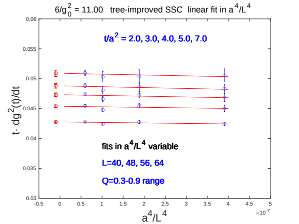
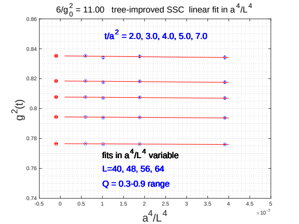
Improvements

- In the LAT 22 analysis, there are very few intersections between targeted g^2 and available data unless $6/g_0^2$ is sampled densely and carefully chosen in the right ranges



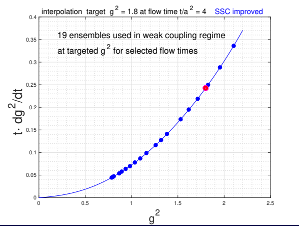
- $\Rightarrow$ We modify our analysis as follows:
  - At each t/a^2 and $6/g_0^2$, take infinite volume limit of g^2 and β

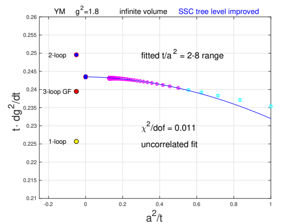




Improvements

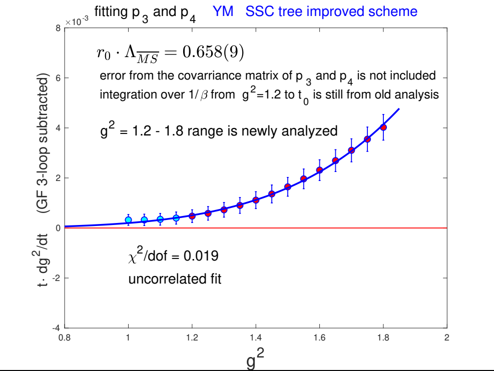
- At each t/a^2 and $6/g_0^2$, take infinite volume limit of g^2 and β
- At each t/a^2, for a set of target g^2 values, interpolate g^2 and β among $6/g_0^2$ values in infinite volume limit
- Take continuum limit of β at each targeted g^2 values





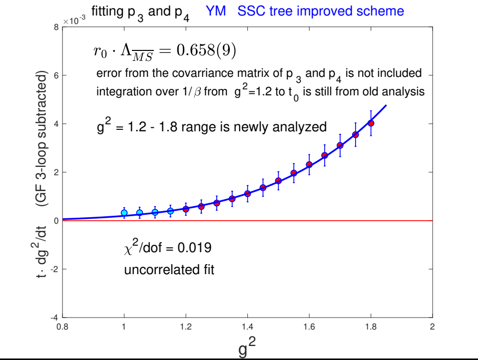
Improvements

- At $g^2 = 1.2$, the difference between 2-loop and 3-loop perturbative β-functions is large and 4-loop is not available
- $\Rightarrow$ Replacement with 3-loop perturbative β-function below $g^2_{\rm PT}$ may introduce systematic errors coming from unknown higher-order contributions
- The effect can be put under control by fitting β a polynomial with p_3 and p_4: $\beta(g^2) = g^4(b_0 + b_1\, g^2 + b_2\, g^4 + p_3\, g^6 + p_4\, g^8)$, then integrate using the fitted polynomial



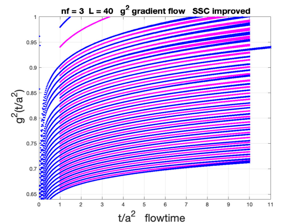
Results

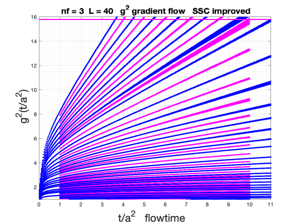
- Result of replacing the three-loop β function with fitted polynomial within $g^2 = 0$ to 1.2 : $r_0 \cdot \Lambda_{\overline{MS}} = 0.658(9)$
- LATTICE 2022: $0.665(9)$ $\Leftarrow 1-\sigma$ difference
- The data at different g^2 are correlated. Correlation fits are required
- The consistency among different schemes, different fit ansatz and different orders of limits remains to be checked
- We will extend the modified analysis into the range $g^2 > 1.2$
- The limit $g^2_{\rm PT} \to 0$ needs to be taken



Updates on $N_f = 3$ QCD

- We have generated much more ensembles with focus on the perturbative matching and on improved resolution at strong coupling
- Interpolation of g^2 at each L as a function of $6/g_0^2$ will increase the resolution
- Spline based χ^2 interpolation minimizes the model dependence of the interpolation
- Blue: data Magenta: interpolated
- Symanzik-improved gauge action with 4-step $\rho = 0.12$ stout-smeared staggered fermions in periodic boundary conditions for gauge links and anti-periodic boundary conditions for $N_f = 3$ massless fermions.





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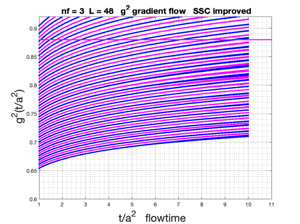

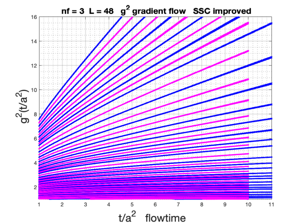

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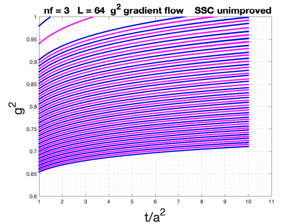
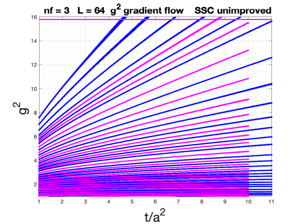
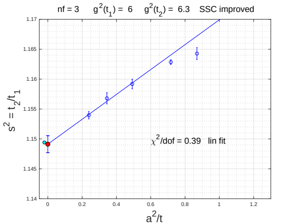
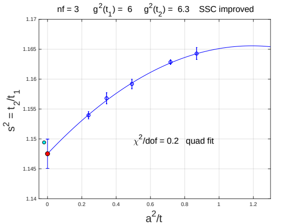




Updates on $N_f = 3$ QCD

- New analysis with improvements (ongoing):
  Take the continuum limit of $\ln(t_2/t_1) \equiv \int_{g^2(t_1)}^{g^2(t_2)} dg^2/\beta(g^2)$, where for each chosen $g^2(t_1)$ and $g^2(t_2)$:
  - At each L, within the range of corresponding t values, no systematic volume dependence is detectable below the noise level
  - $\Rightarrow$ At each L, extrapolate $s^2 \equiv t_2/t_1$ to infinite volume limit with ansatz's : "s^2 =constant" and "s^2 linear in a^4/L^4" with Akaike analysis





(cyan dot: 1-loop)