Gradient flow and the (Wilsonian) renormalization group

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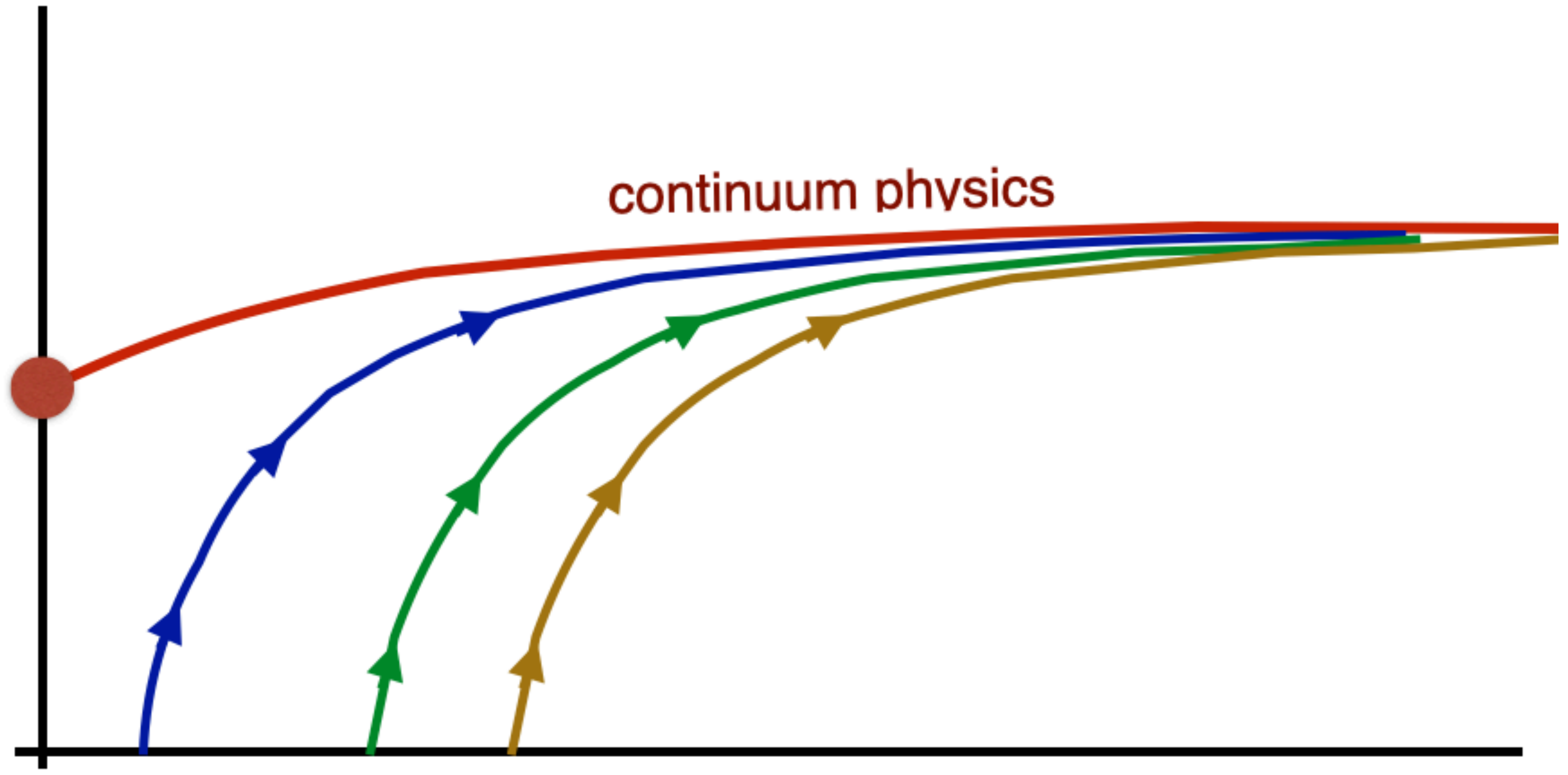
University of Colorado Boulder

Gradient Flow Workshop, Zurich
Feb 14, 2025

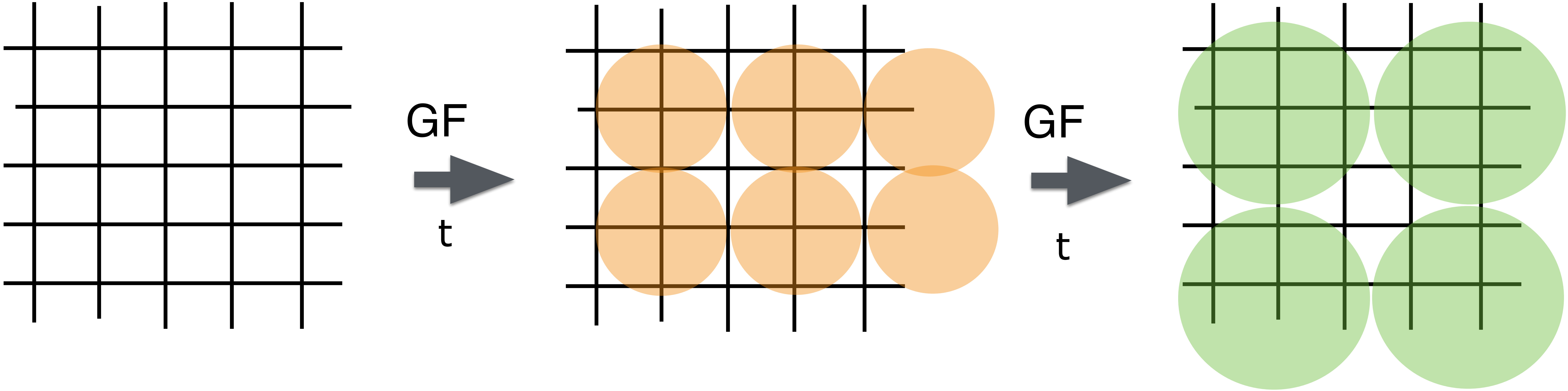


Gradient flow: non-perturbative interpretation

GF is a continuous smearing transformation of radius $\rho \propto \sqrt{8t}$:

➡ defines "block spins" or "block links"

A. Carosso, AH, E. Neil,
PRL 121,201601 (2018)
Sonoda, H., Suzuki,
H. PTEP,023B05 (2021)



GF does not have coarse graining, not an RG transformation;
BUT it can be *interpreted* as Wilsonian RG with $\mu \propto 1/\sqrt{8t}$ for *local* operators

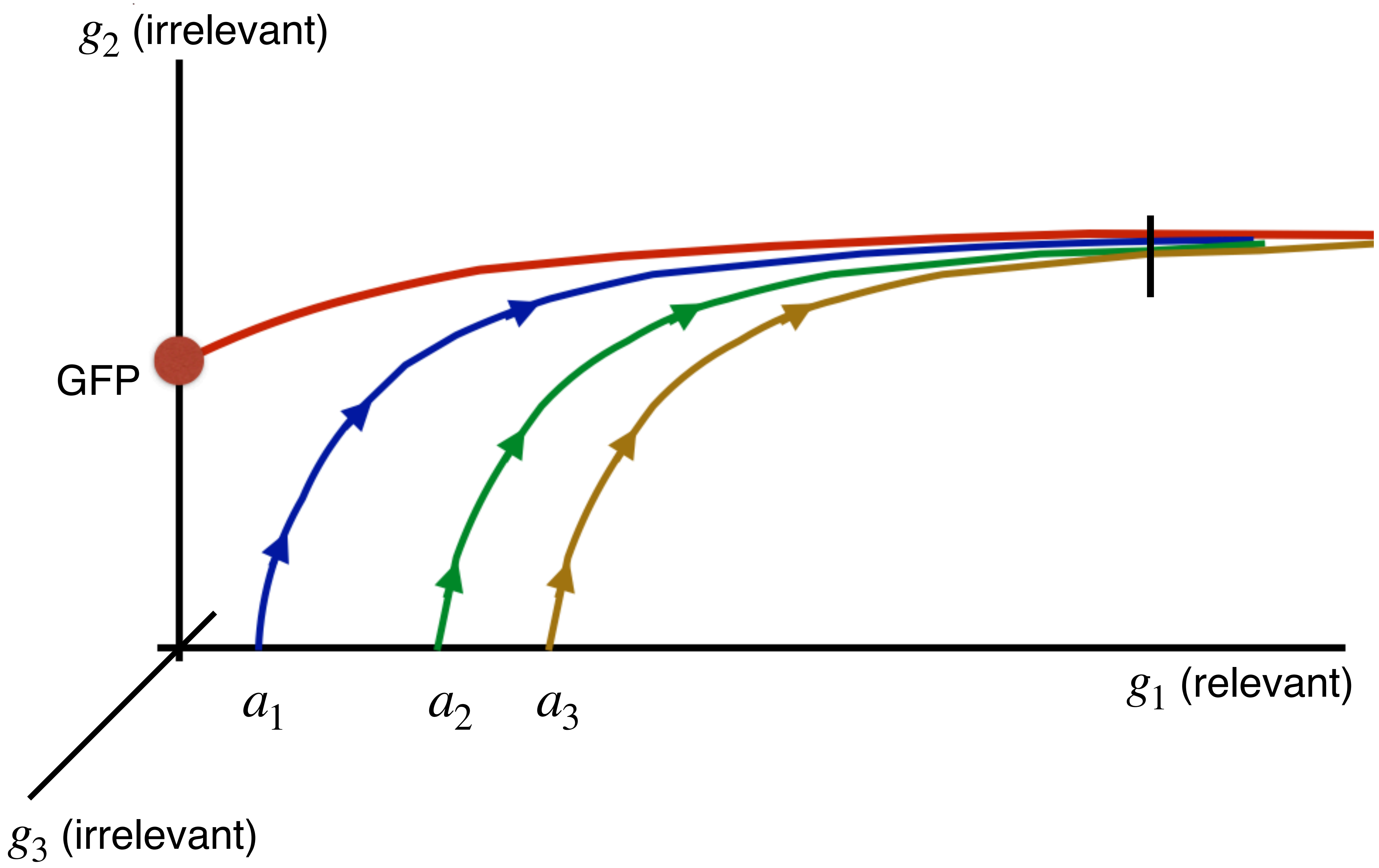
GF from lattice to continuum

Bare action parameter space:

GF/RG: $t=0:\ S(g_i, m_i; U) \longrightarrow t/a^2:\ S(g_i(t), m_i(t); U(t))$

$\mathcal{O}(g_i) \equiv \mathcal{O}(U) \qquad \mathcal{O}(g_i(t)) \equiv \mathcal{O}(U(t))$

Urs Wegner's talk



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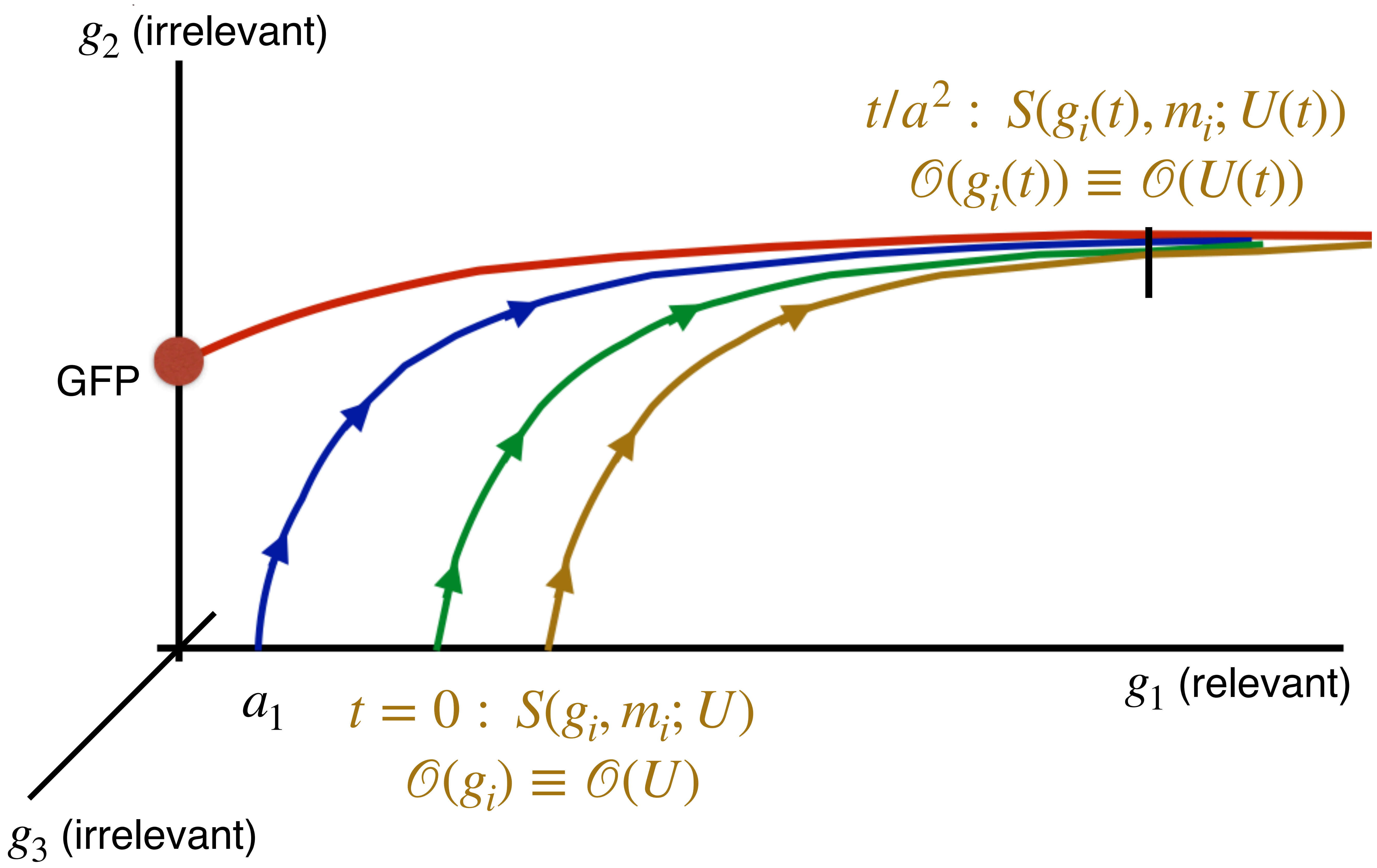

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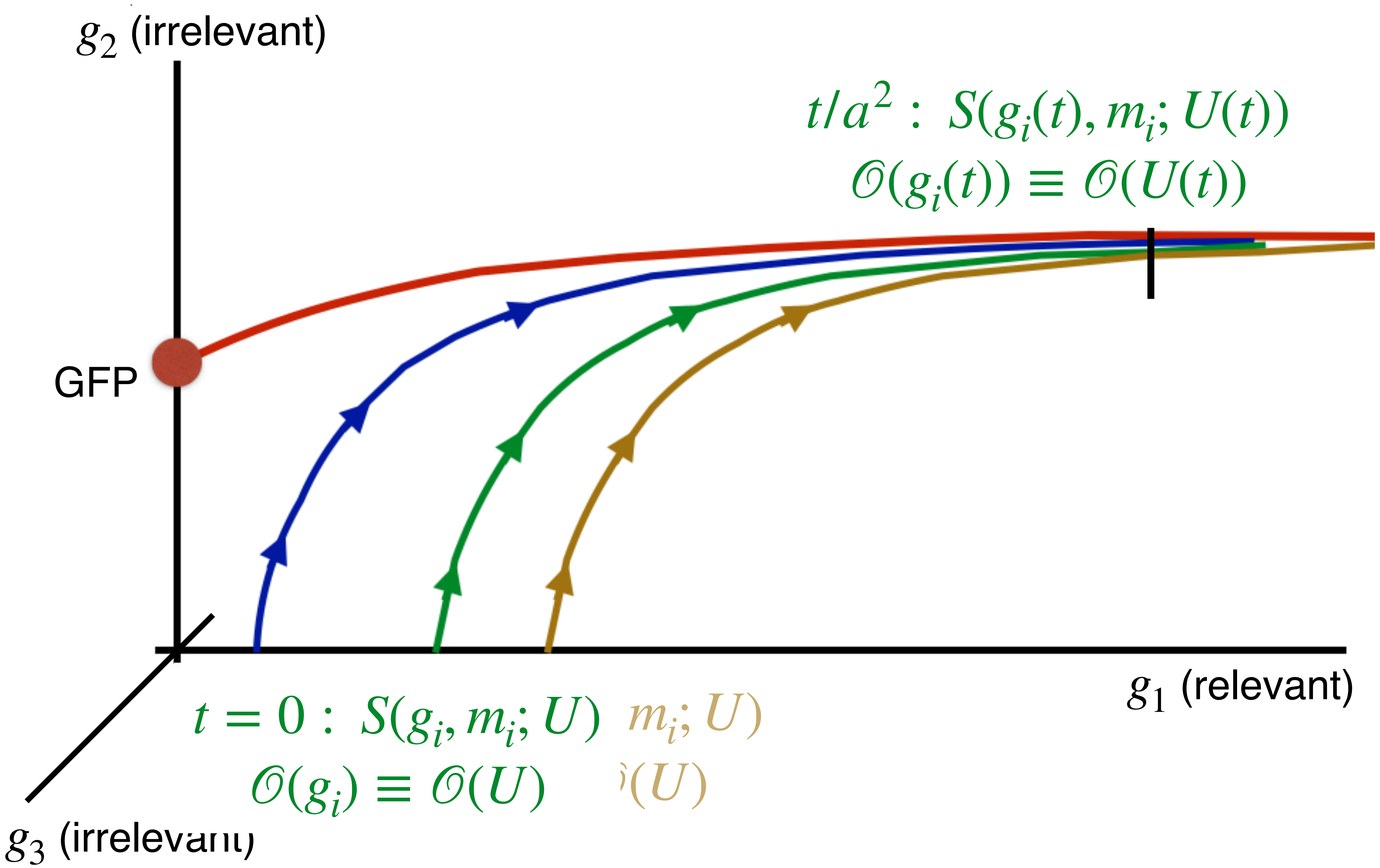

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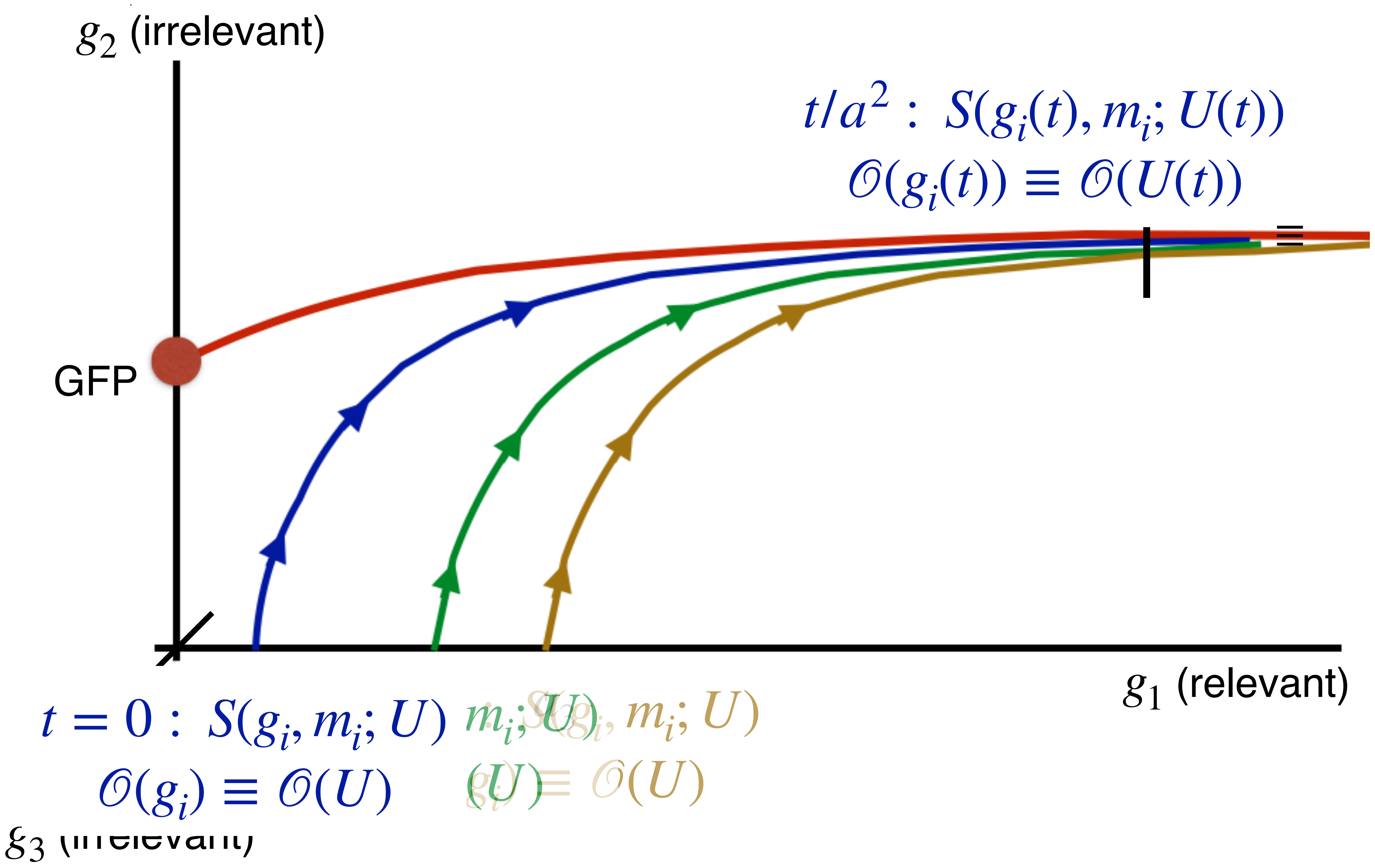

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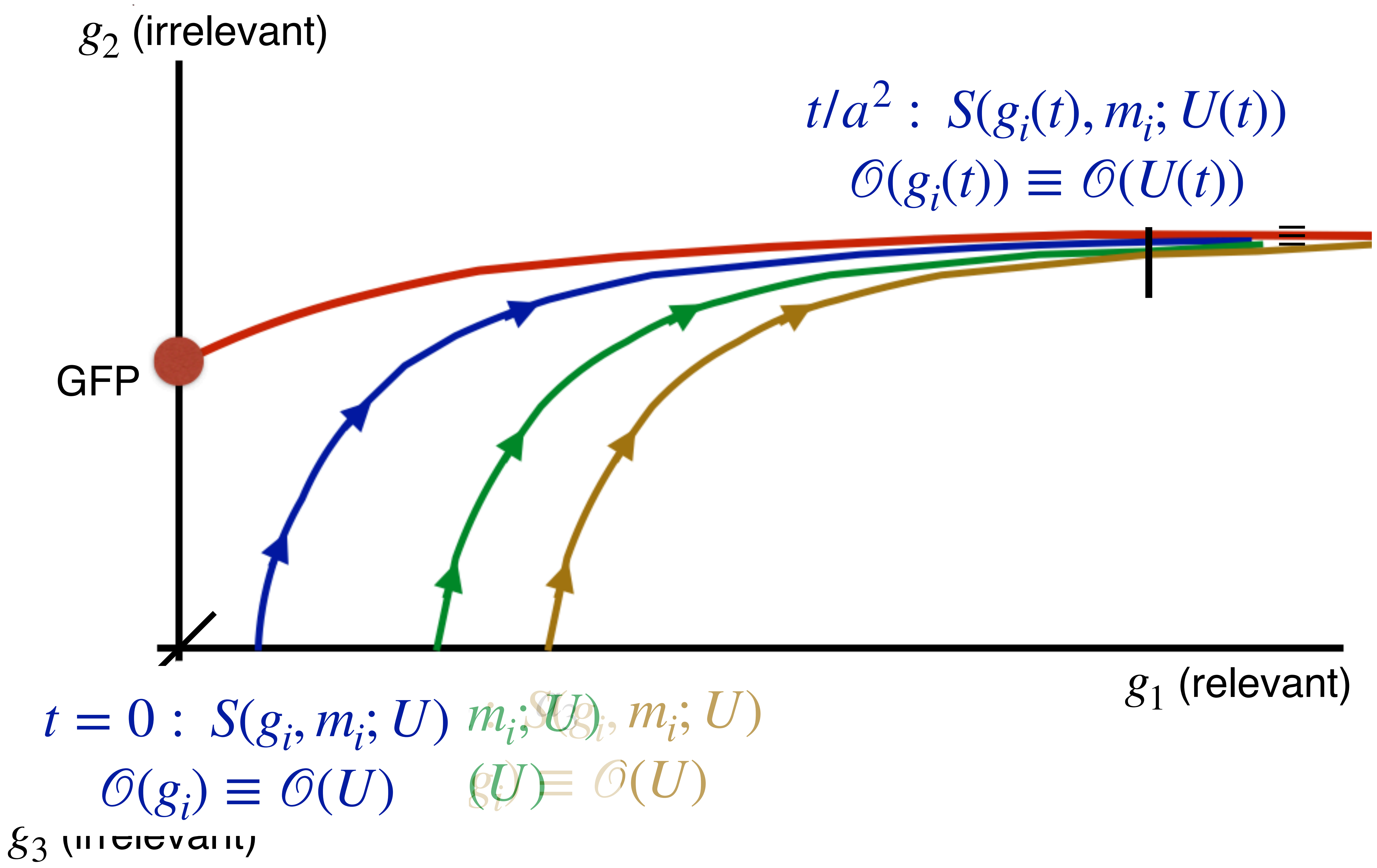

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Flowed actions, operators are (nearly) identical: renormalized

$t/a^2 : \; S(g_i(t), m_i; U(t))$

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|||

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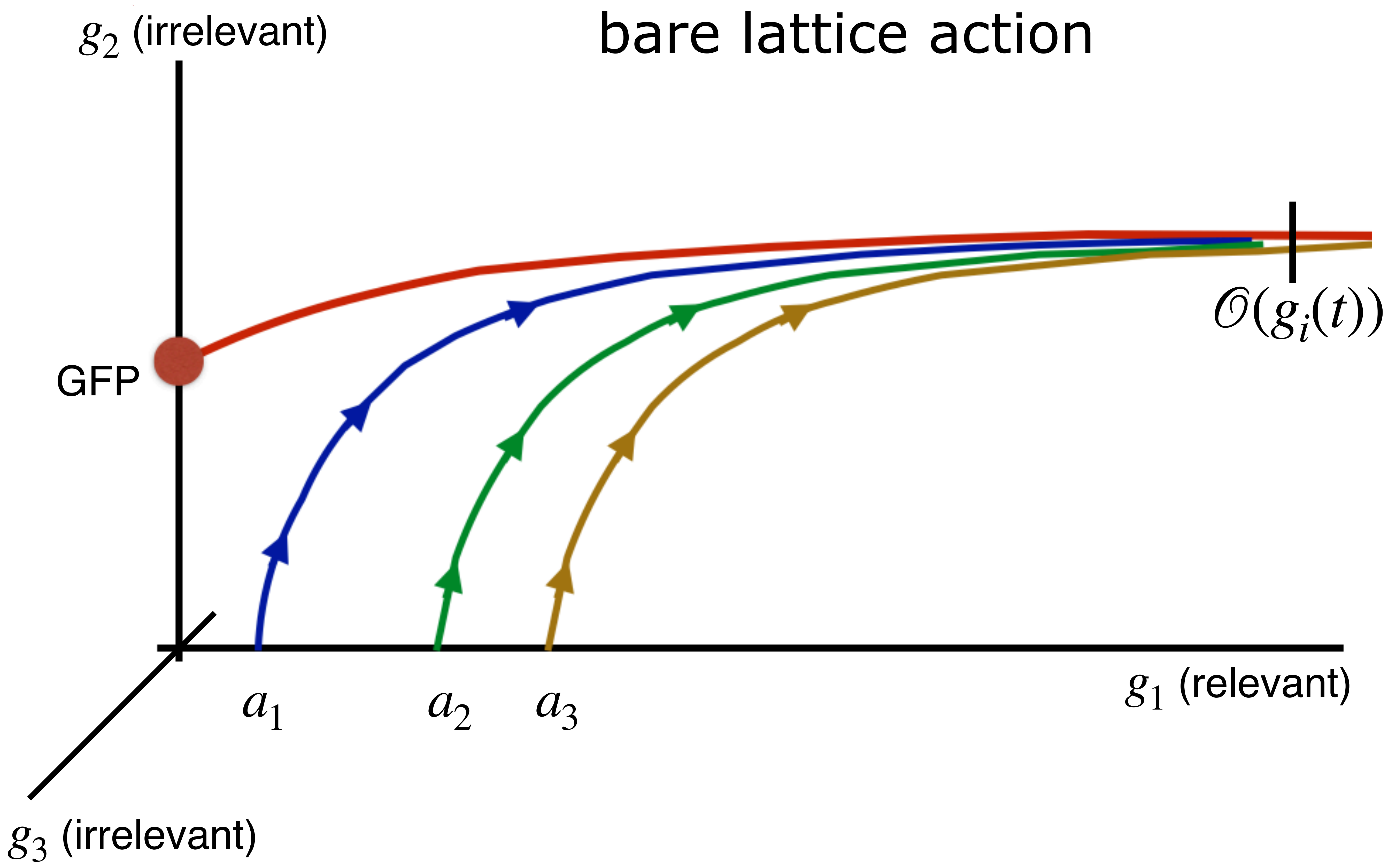
bare lattice action



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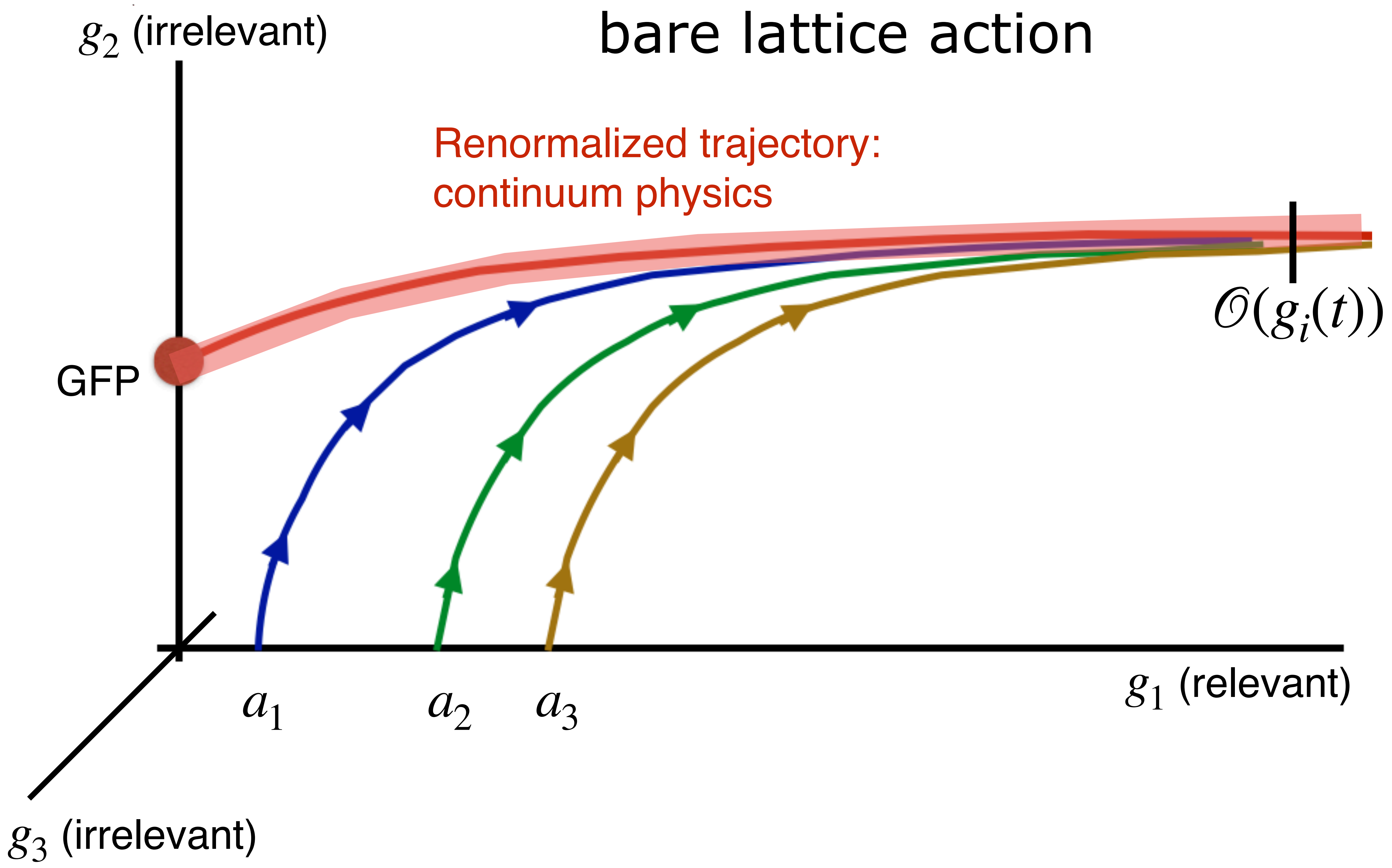

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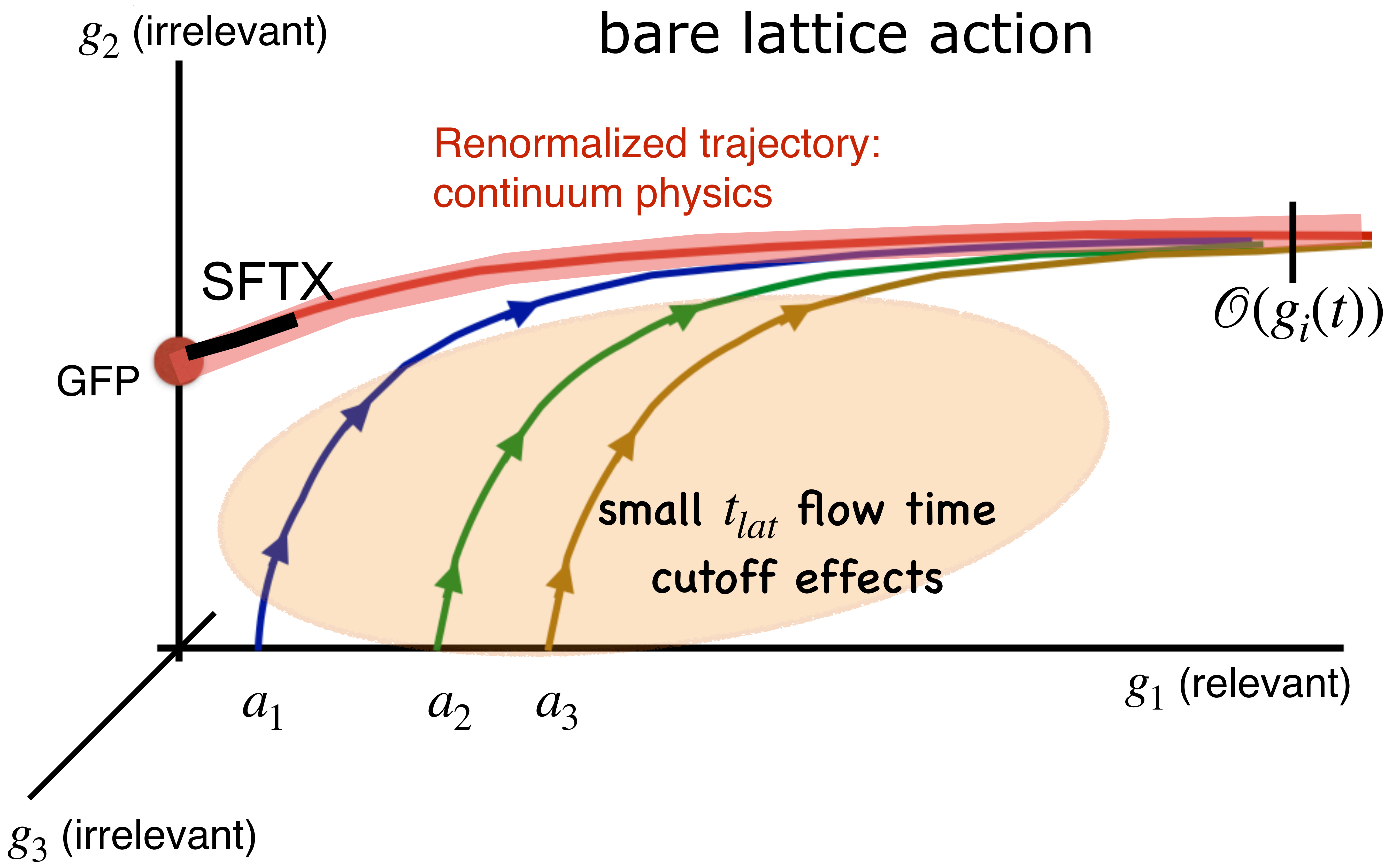
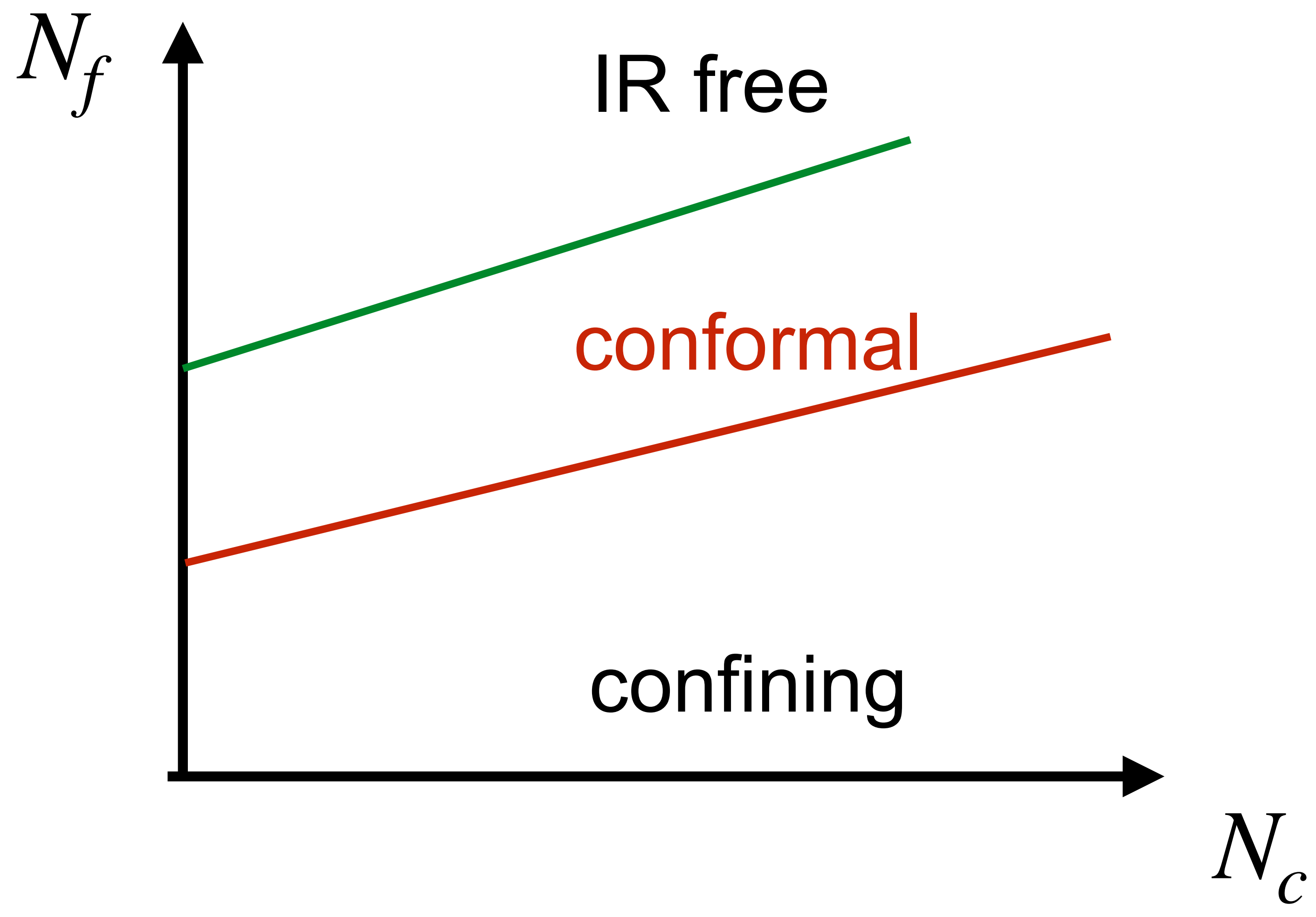


Vary flow time to map RT and $\mathcal{O}(t)$

- small t_{lat} flow time: cutoff effects
- short flow time expansion is continuum, on RT

Lattice and SFTX are both on the RT
They should match!

In the conformal window



$SU(N_c)$ gauge + N_f fermions within the conformal window

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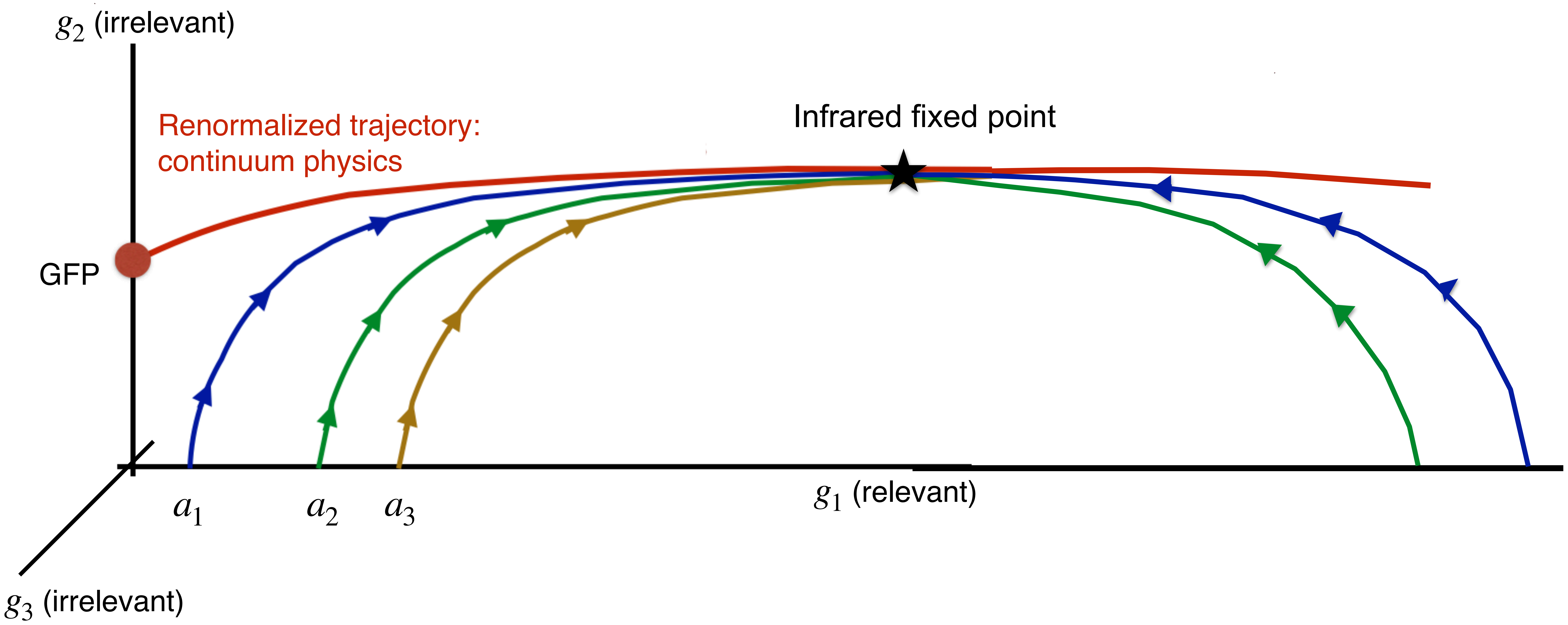

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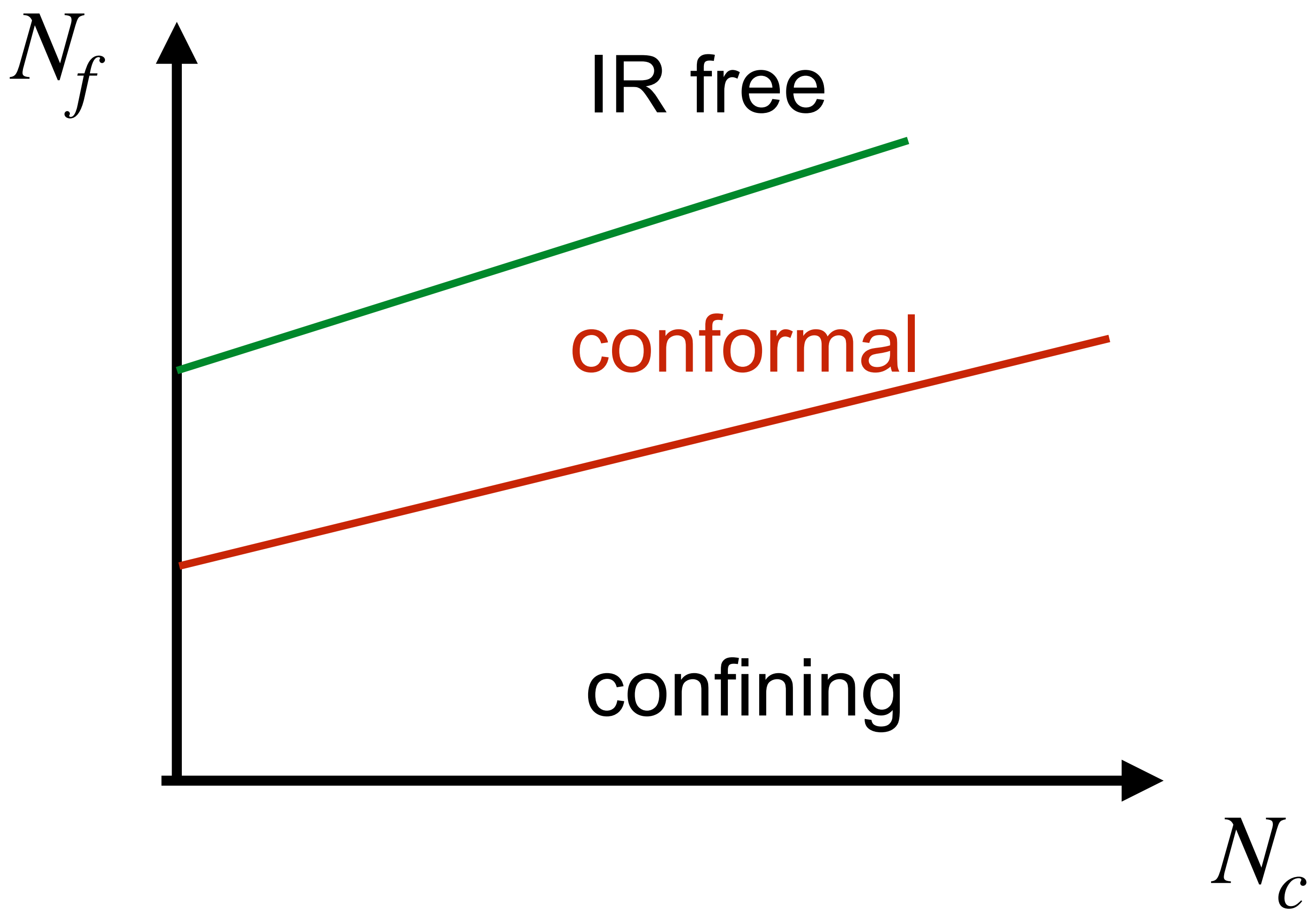

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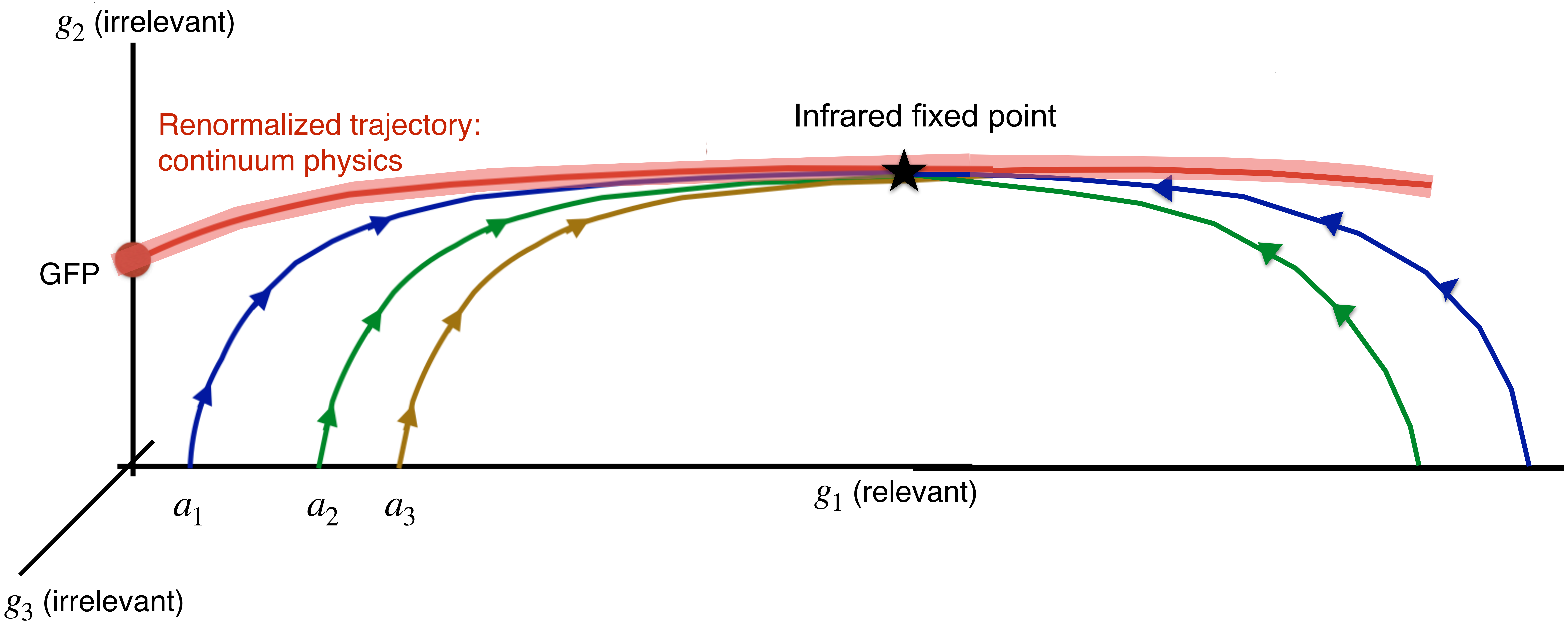

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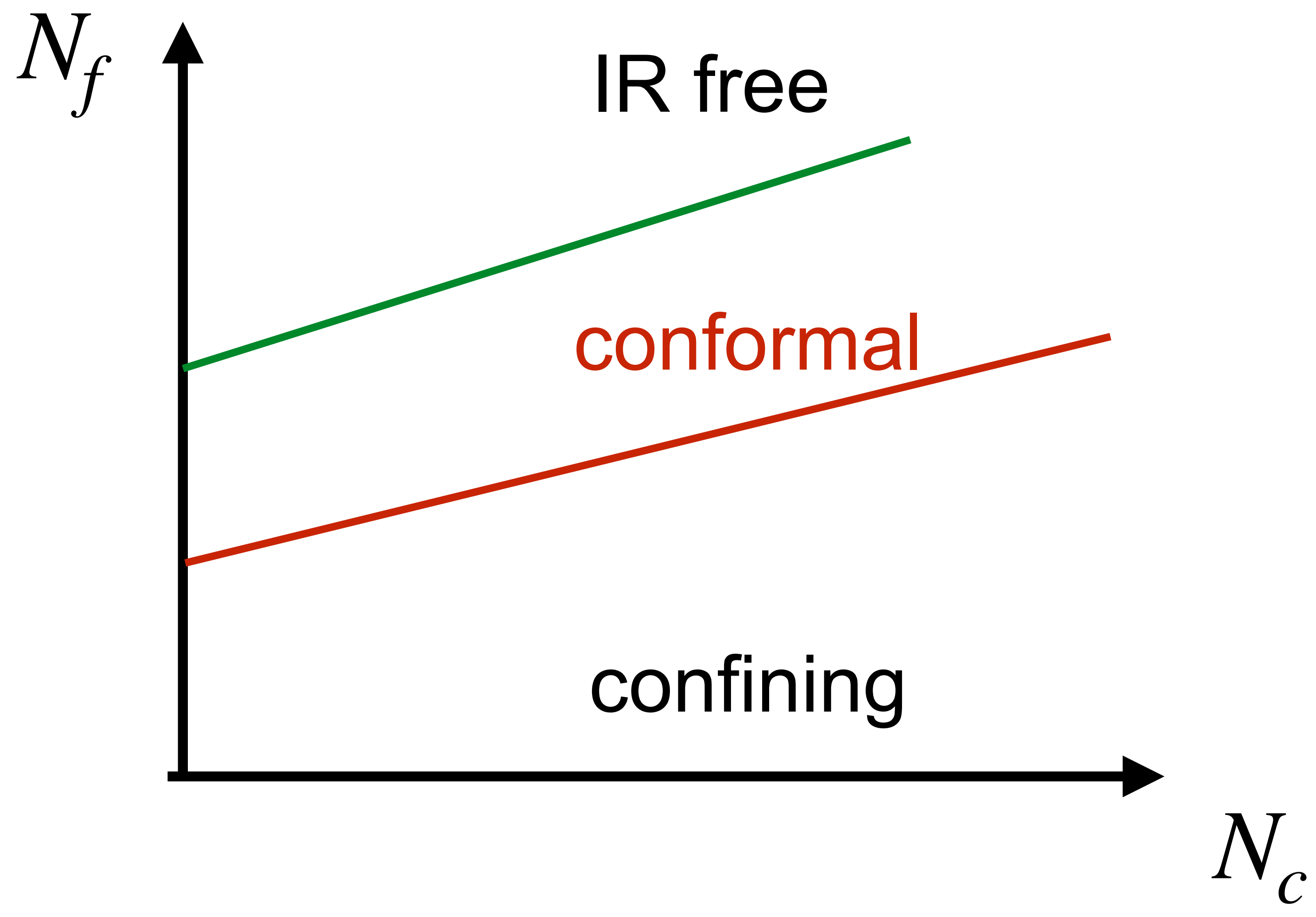

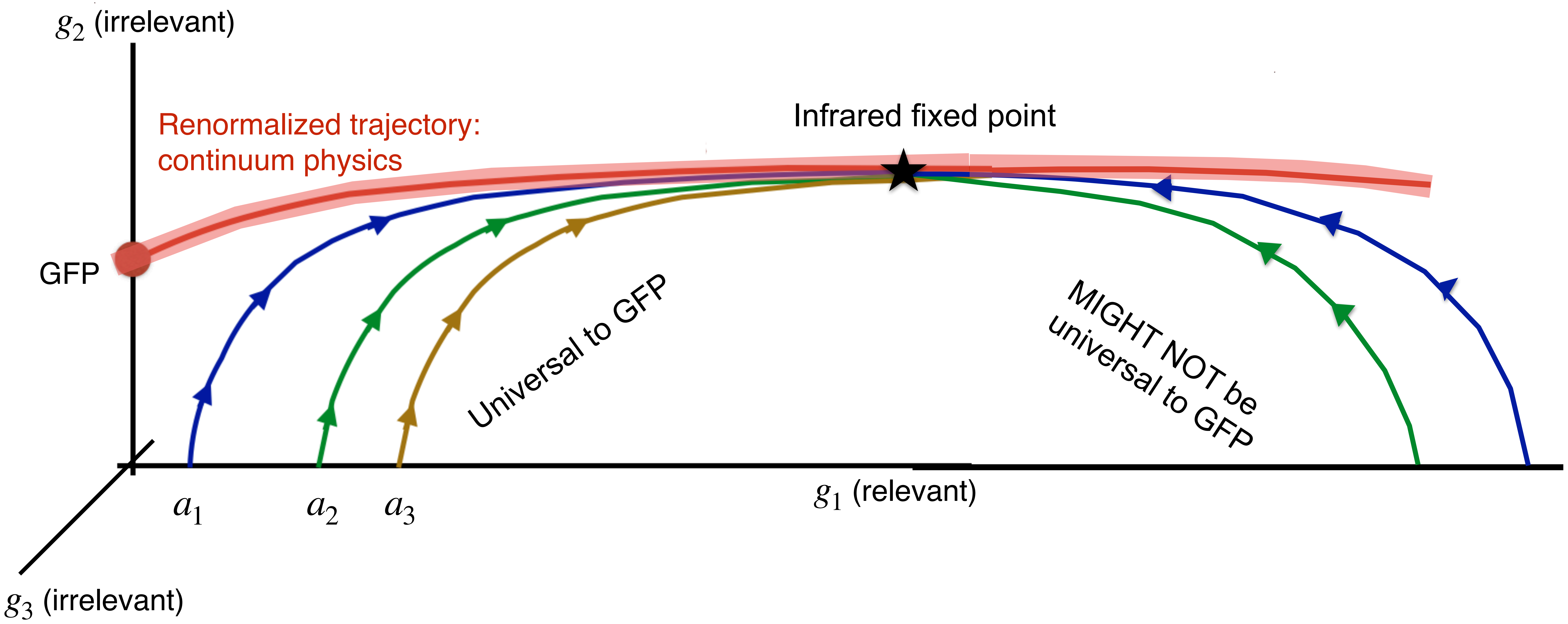

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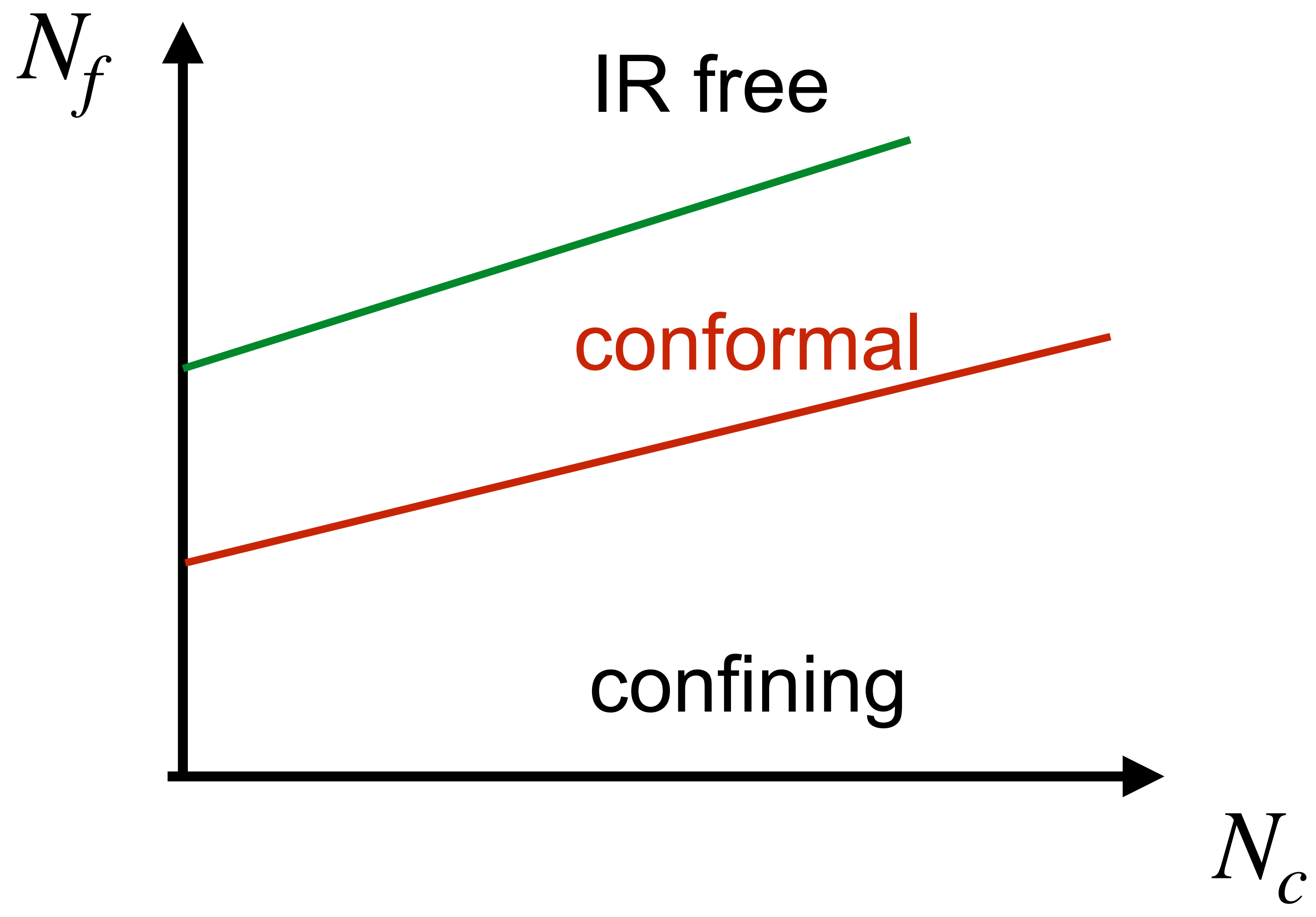

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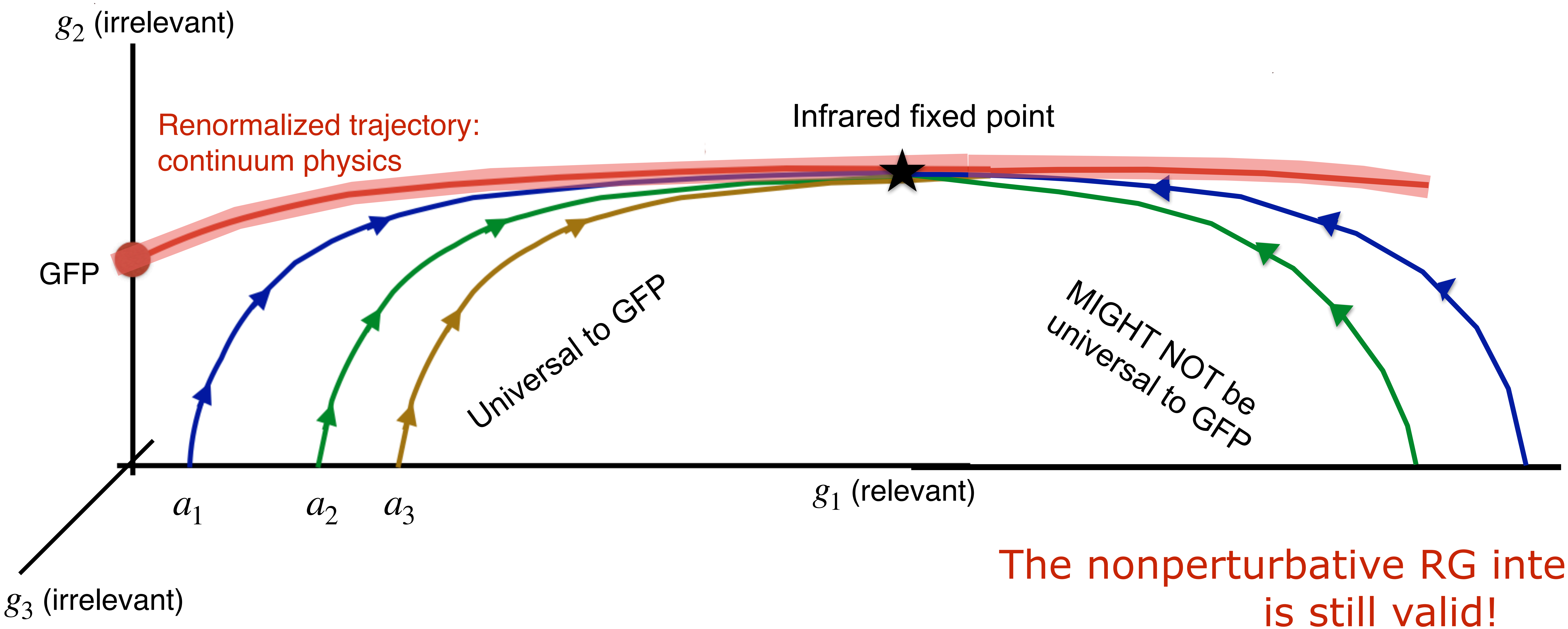


The nonperturbative RG interpretation
is still valid!

Operators:

Gauge flow:

Luscher *JHEP* 08 (2010) 071

$$\mathcal{O} \equiv g^2_{GF} = \mathcal{N} t^2 < E(t) >$$

➡ $g^2_{GF}(t)$ is dimensionless, $\gamma_{g^2} = 0$

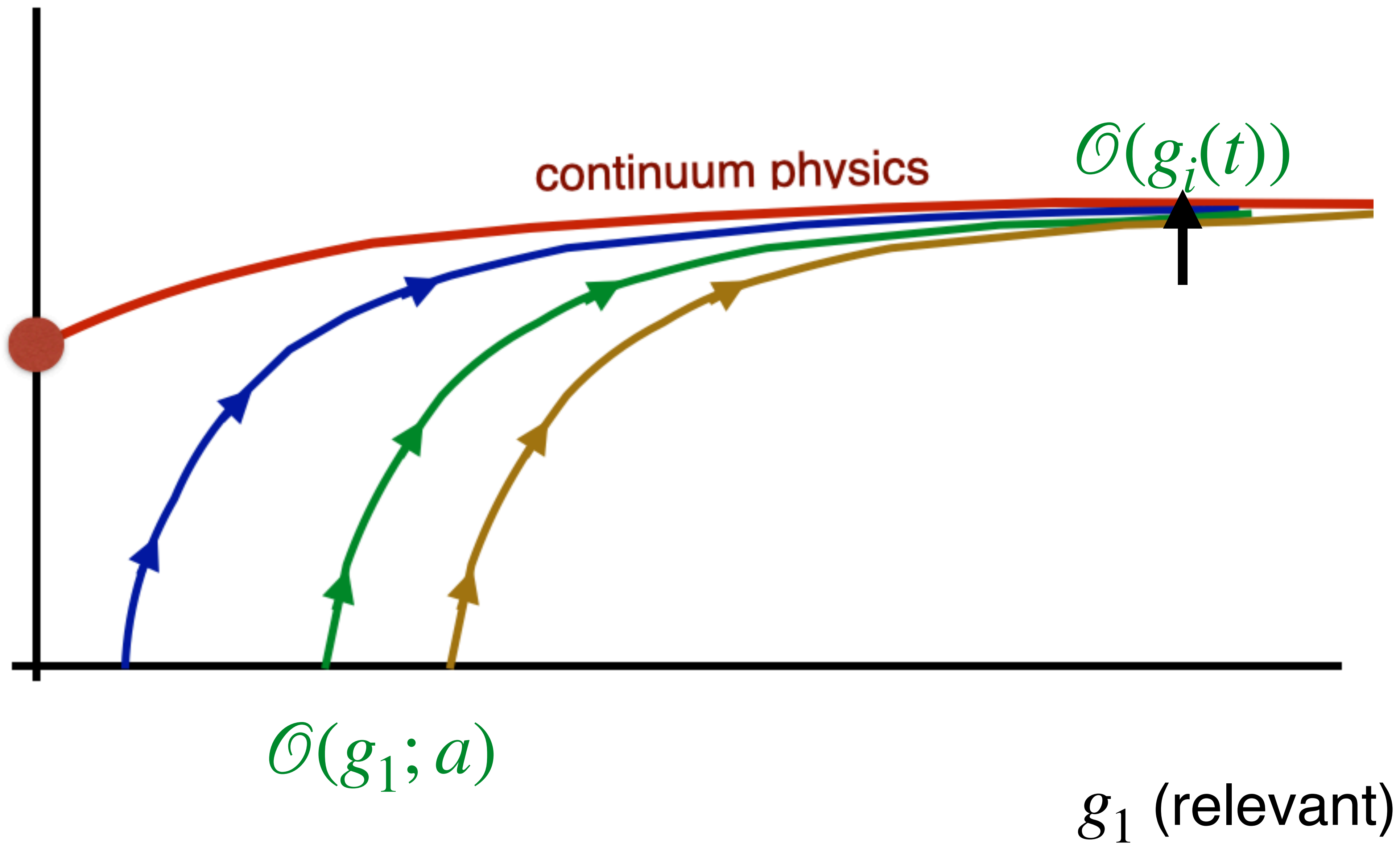
- along the RT it measures the flow
- renormalized coupling

$$g^2_{\overline{MS}} = g^2_{GF} + c g^4_{GF} + \ldots$$

➡ RG β function: $\beta(g^2) = -t \dfrac{d\, g^2_{GF}}{d\, t}$

- Applications:
  - QCD-like: $\Lambda_{GF}, \Lambda_{\overline{MS}} \rightarrow \alpha_{strong}$
  - conformal: $\beta(g^2_{IRFP}) = 0 \rightarrow$ IRFP

(continuous β fn vs step scaling)



AH, O. Witzel, *Phys.Rev.D* 101 (2020) 3

Fodor et al,EPJWeb Conf. 175, 08027 (2018)

Operators:

Fermion flow: Luscher *JHEP* 04 (2013) 123

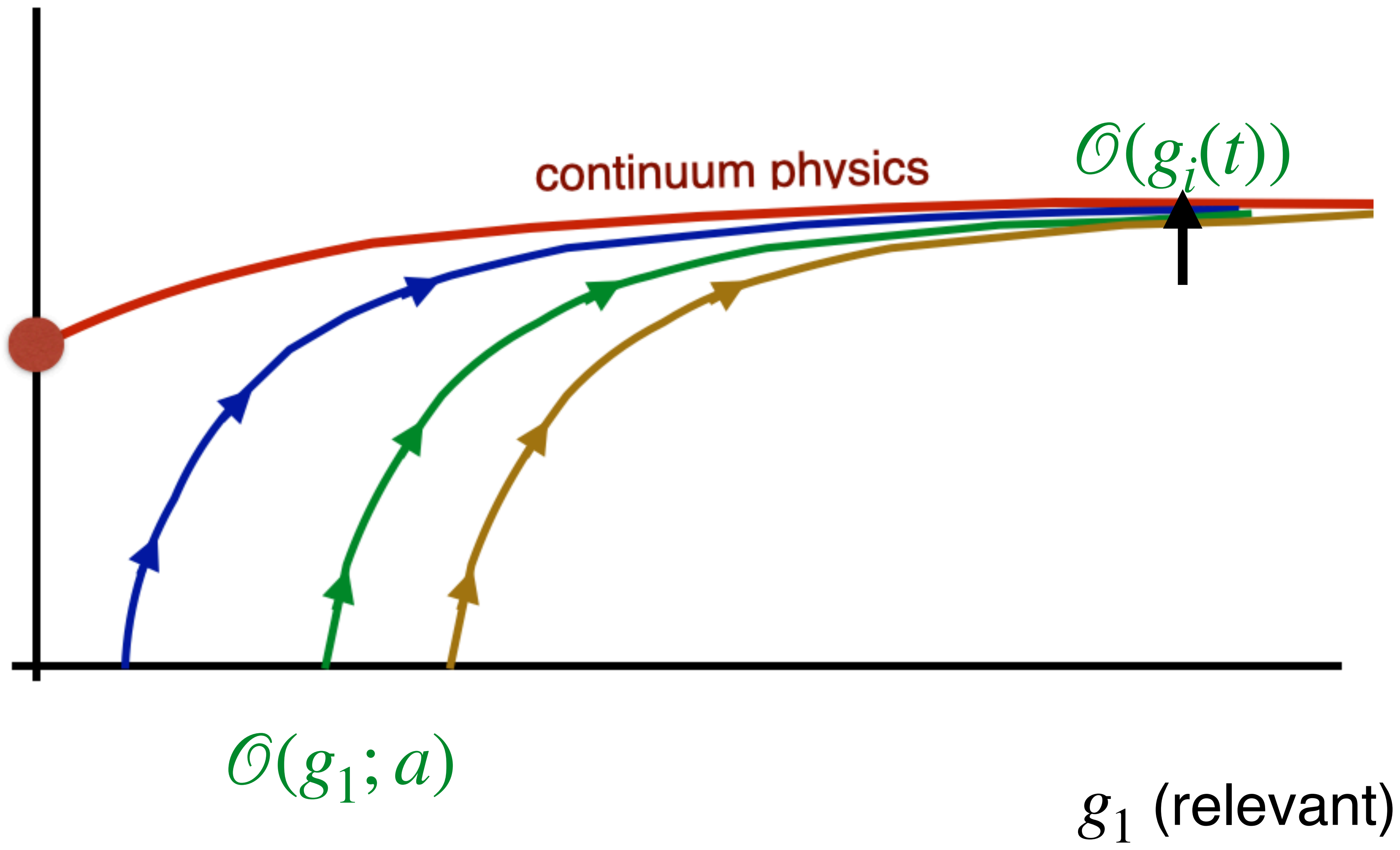
- bare operator: $\mathcal{O}(a, t=0) = \bar{\psi}(x)\Gamma\psi(x)$

➡ flowed operator is renormalized

  - running anomalous dimension $2t\dfrac{d\,\mathcal{O}(t)}{d\,t} \sim \gamma_{\mathcal{O}}(t)$
  - match to SFTX
  - RG scheme: matching factor

    $Z_{\mathcal{O}}^{-1}(a,t)\,\mathcal{O}(g_i(t),t) \;=\; \mathcal{O}(g_i(t),t)^0$ tree level

- Applications:
  - QCD-like: running γ_O, RG scheme
  - conformal: predict $\gamma_{\mathcal{O}}^*$ at IRFP



Hasenfratz.,Neil, Shamir, Svetitsky, Witzel, *Phys.Rev.D* 108 (2023) 7
Hasenfratz, Monahan, Schindler, Rizik, Witzel, in prep.

Fermion Operators:

A. Carosso, AH, E. Neil,
PRL 121,201601 (2018)

Often $\langle \mathcal{O}_\Gamma(t) \rangle = 0$; consider a GF two-point function

$$G_{\mathcal{O}}(x_4, t) = \int d^3x d^3x' \langle \mathcal{O}(\mathbf{x}, x_4; t) \mathcal{O}(\mathbf{x}', 0; t = 0) \rangle$$

- Only one operator is flowed, GF-RG equivalence (coarse graining) is OK
- Also need $x_4 \gg \sqrt{8t}$
- The scaling dimension of $G_{\mathcal{O}}(x_4, t)$ is $\Delta_{\mathcal{O}} = d_{\mathcal{O}} + \gamma_{\mathcal{O}} + \eta$

canonical dimension

anomalous dimension

wave function renormalization $Z_\chi = e^{t\,\eta/2}$

The vector and axial charge operators have no anomalous dimension

→ in the ratio $\mathcal{R}_{\mathcal{O}}(x_4; t) = \dfrac{G_{\mathcal{O}}(x_4; t)}{G_V(x_4; t)}$ both Z_χ and $d_{\mathcal{O}}$ cancel

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Anomalous dimension and RG

$\gamma_{\mathcal{O}}$ is the logarithmic derivative

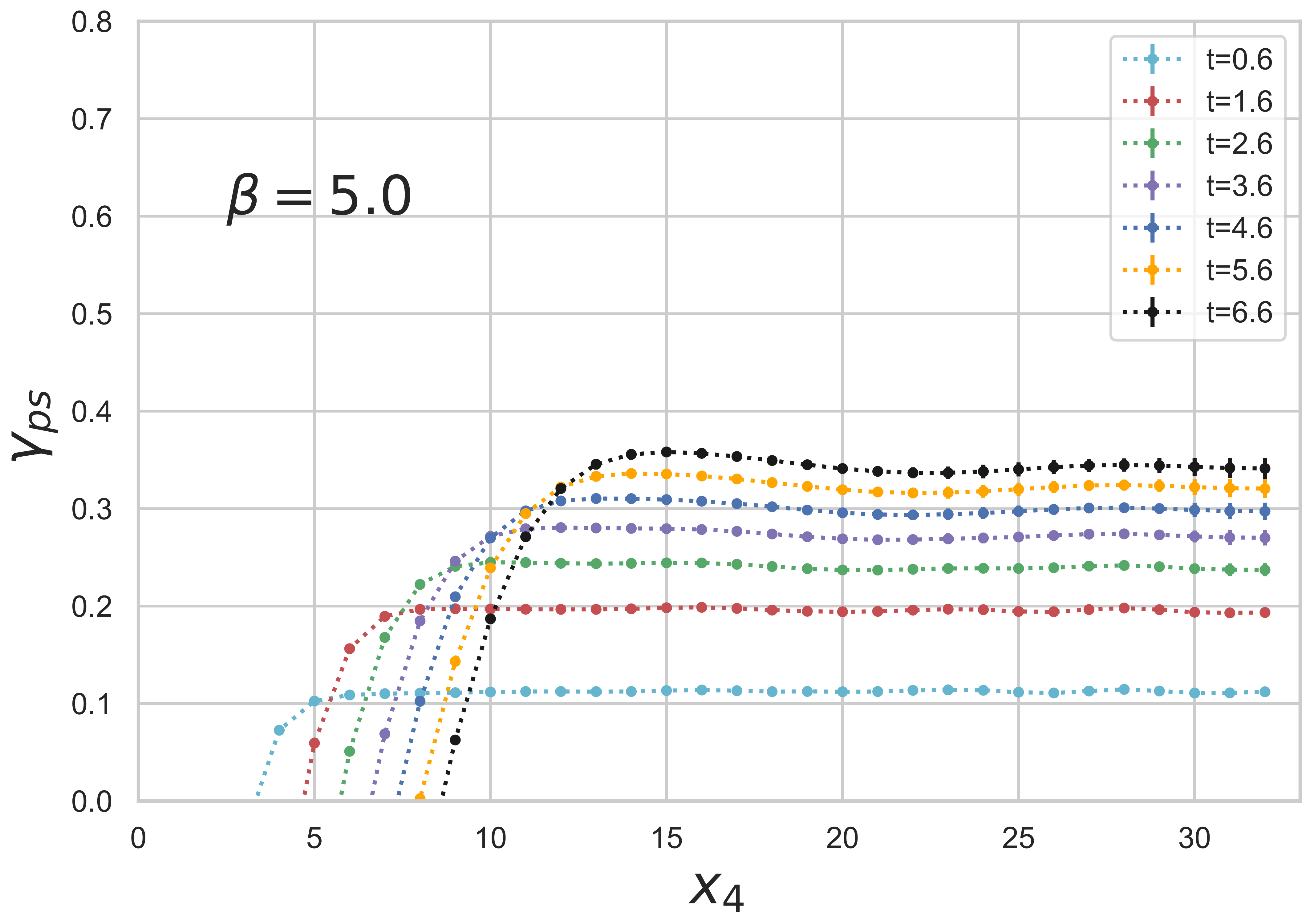
$$\gamma_{\mathcal{O}}(a;t) = 2t\frac{d\log\mathscr{R}_{\mathcal{O}}(\mathrm{a};\mathrm{x}_4,\mathrm{t})}{dt} \qquad ; \qquad \mathscr{R}_{\mathcal{O}}(x_4;t) = \frac{G_{\mathcal{O}}(x_4;t)}{G_V(x_4;t)}$$

Typical correlator when $x_4 \gg \sqrt{8t}$

$$G_{\mathcal{O}}(t) = A_1(t)e^{-m_1x_4} + A_2(t)e^{-m_2x_4} + \dots$$

$$2t\frac{\mathrm{dlogG}_{\mathcal{O}}(\mathrm{t})}{dt} = \frac{\mathrm{dlogA}_1(\mathrm{t})}{dt} + \mathcal{O}\left(e^{-(m_2-m_1)x_4}\right)$$

- $\gamma_{\mathcal{O}}(t)$ is independent of x_4 if $x_4 \gg \sqrt{8t}$
- $\gamma_{\mathcal{O}}(t)$ corresponds to the lightest state;
  all others die out



Some recent applications:

- SU(3) gauge, no fermions
- SU(3) gauge + $N_f = 2$ fundamental fermions; domain wall lattice fermions
- SU(3) gauge + $N_f = 12$ fundamental fermions; staggered lattice fermions
- SU(3) gauge + $N_f = 10$ fundamental fermions; Wilson lattice fermions
- SU(4) gauge + $N_f = 4 + 4$ fundamental+sextet fermions; Wilson lattice fermions
- SU(3) gauge + $N_f = 8$ fundamental fermions; staggered lattice fermions
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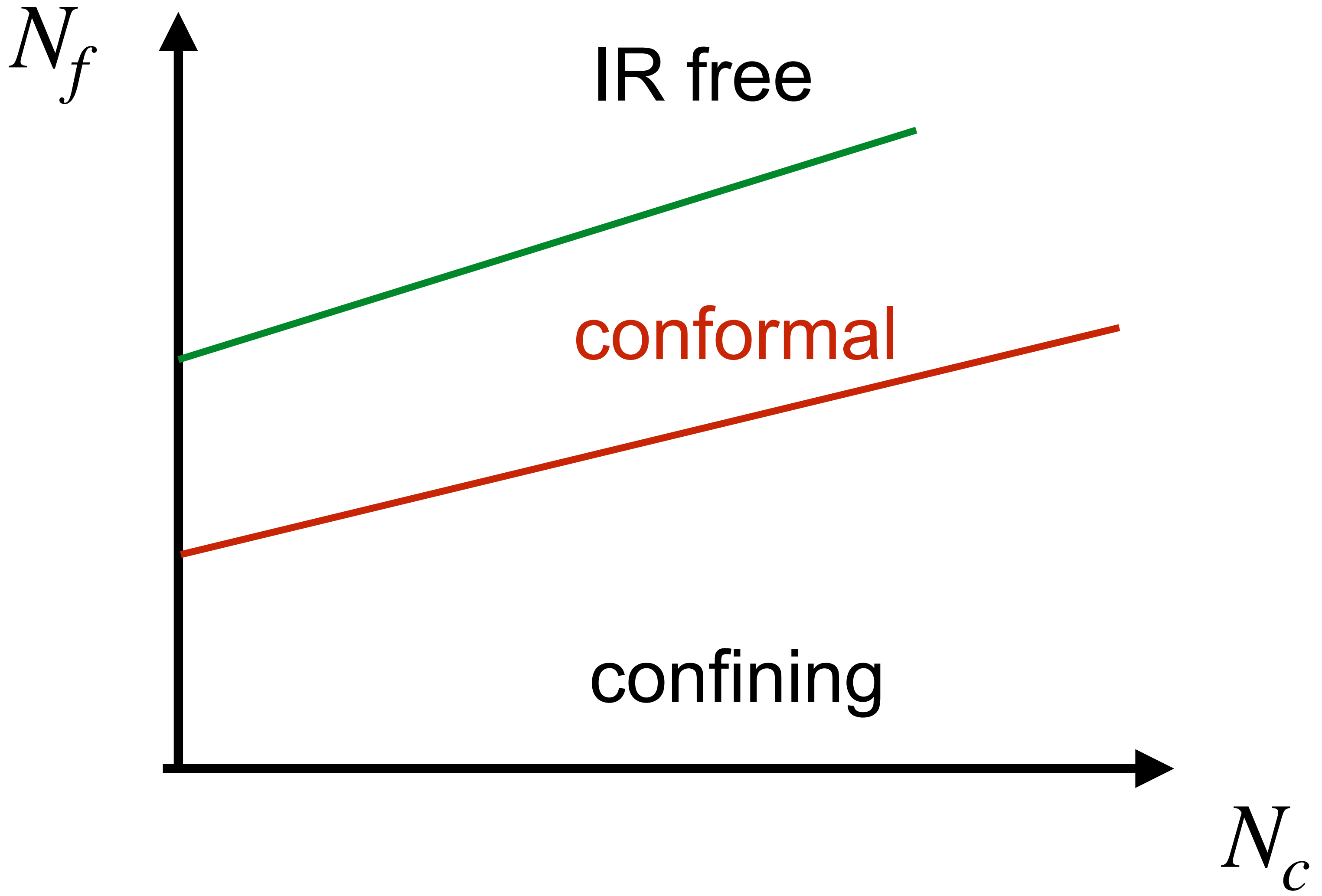

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QCD-like: scale setting, Λ_{QCD}, etc

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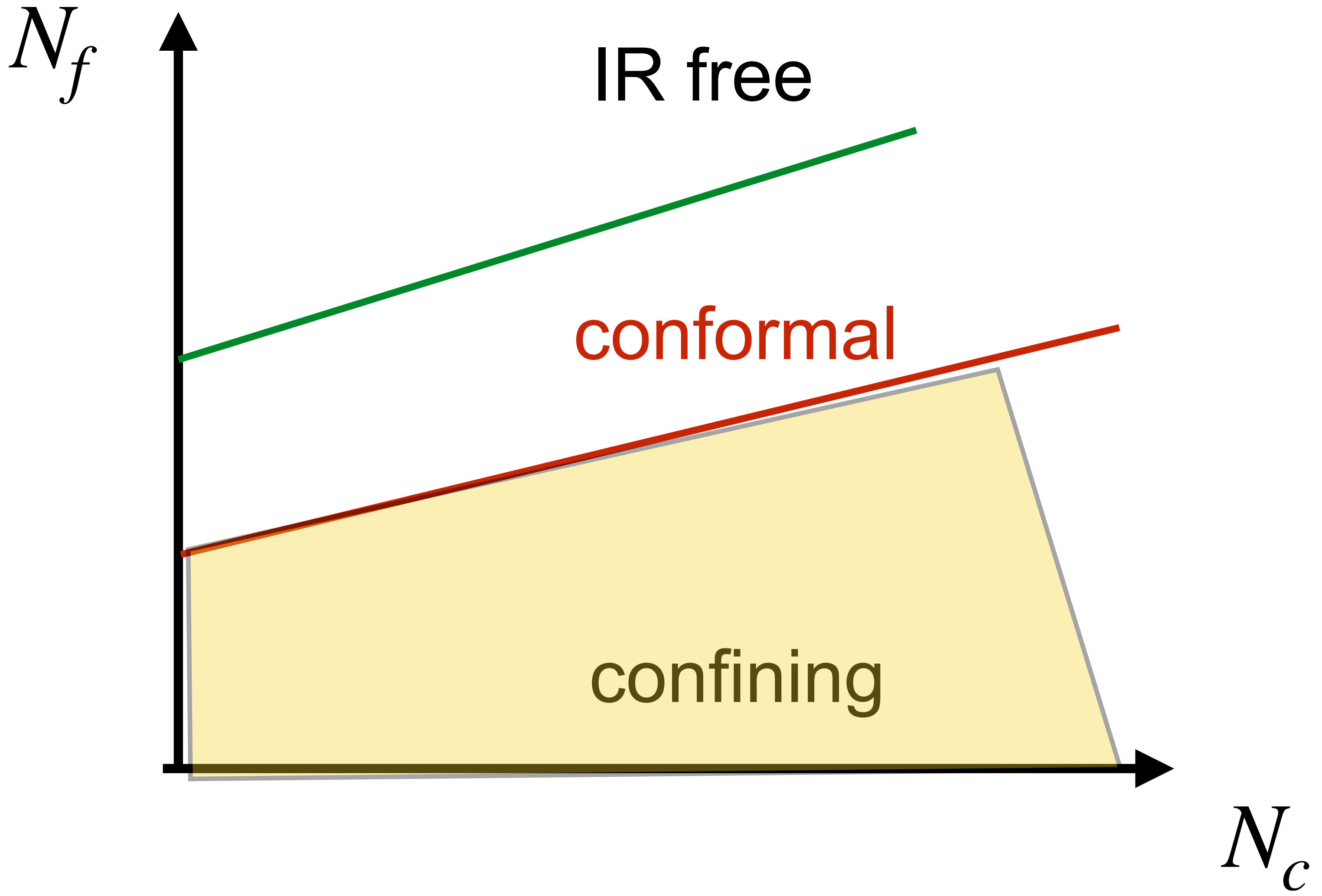

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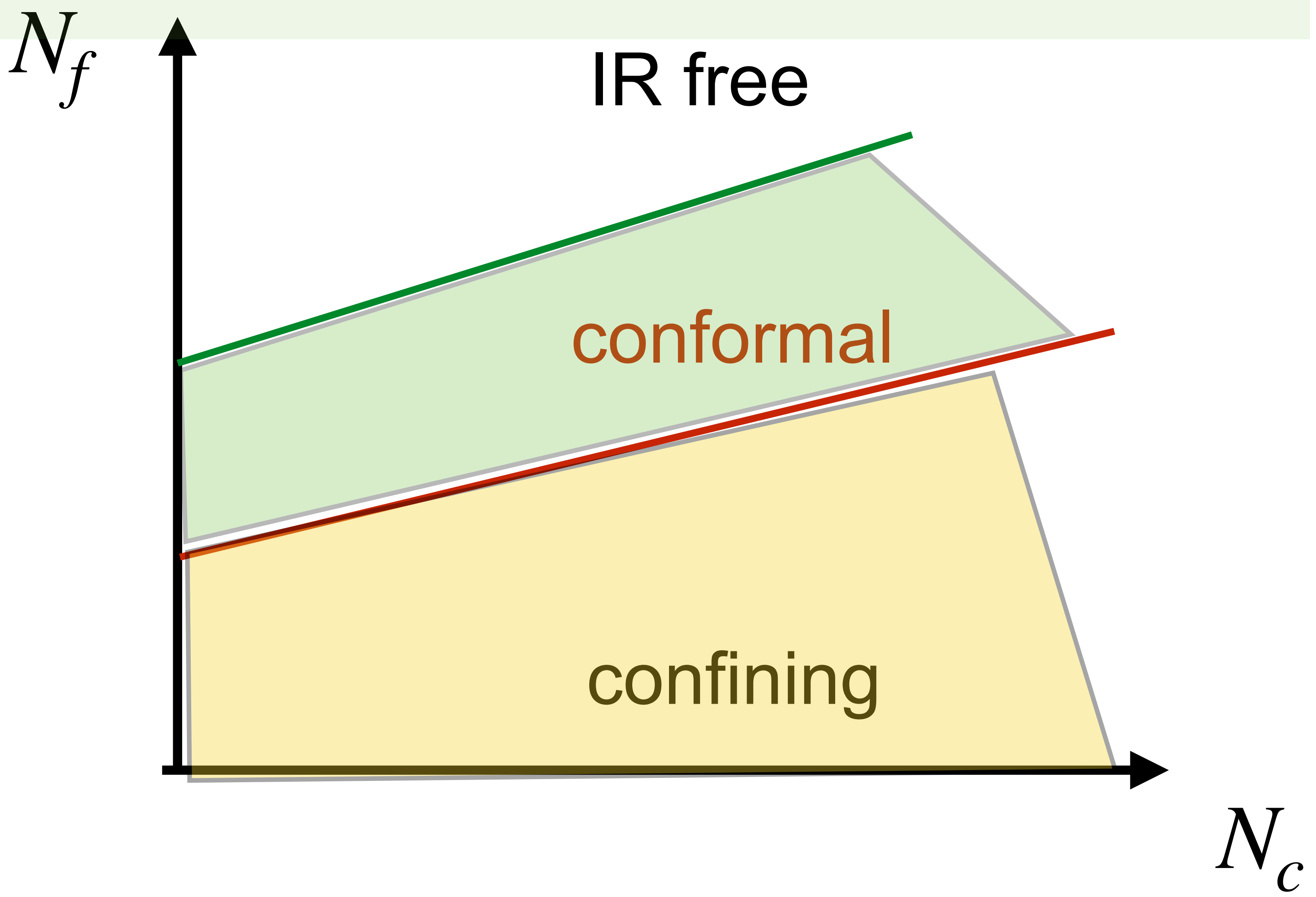


Details, details, ……
(lattice)

Limits

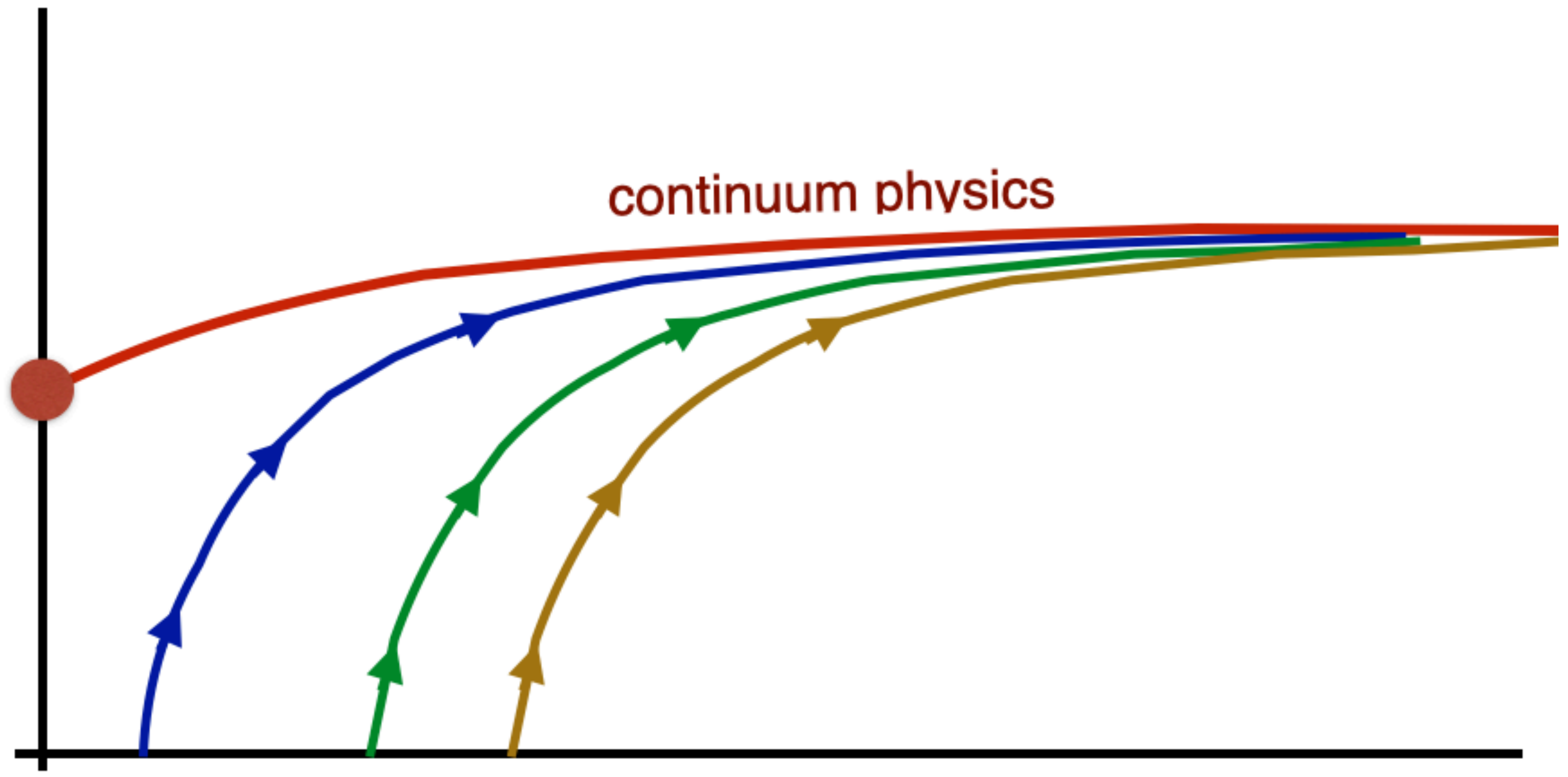
The RG picture I discussed is valid in infinite volume:

- take the $1/L \to 0$ limit $\left(g^2(L=\infty) = g^2(L) + \frac{c}{L^4} + \ldots\right)$

Cancel small-flow time cutoff effects:

- take the $a^2/t \to 0$ continuum limit
  (forces the bare coupling g_0^2 UVFP)

Can be delicate but straightforward



Improved actions

Improved actions reduce cutoff effects:

- perturbative : Symanzik program - useful in the weak coupling
- non-perturbative: (empirical)
  - -cutoff effects can trigger unphysical bulk phase transitions :

➡ **Pauli-Villars improvement:**

AH, Shamir, Svetitsky, PRD104, 074509 (2021)

Add *heavy* PV bosons

- -same interaction as fermions but with bosonic statistics
- - $S_{eff} < 0$, $\beta = 2N_c/g_0^2$ increases : UV fluctuations decrease
  - in the IR the heavy flavors decouple, do not change physics
  - equivalently: range of effective gauge action is $\sim exp(-2am_{PV})$

This only modifies the gauge action

Add many PV bosons reduce the lattice fluctuations

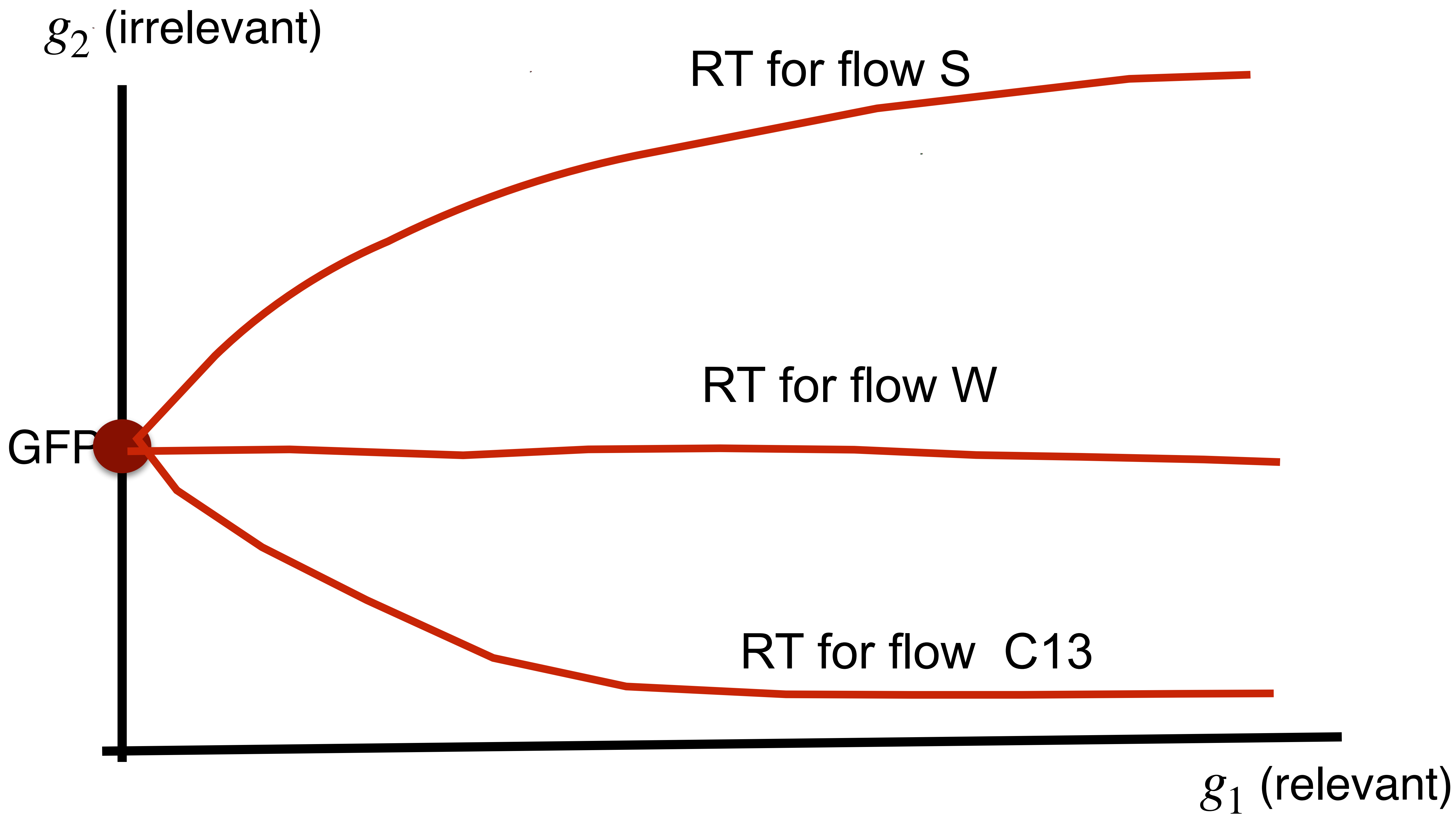
PV improvement was essential in the β function results with $N_f = 12$, 10, 8

Improved flows

When the RT is too far from the action, lattice artifacts of the flow are severe

➡ improve the flow

- perturbative : Zeuthen flow
- empirical non-perturbative:
  Symanzik-like flow with different coefficients c_p, c_r
  constraint: $c_p + 8c_r = 1$
  - $c_p = 1$: Wilson flow
  - $c_p = 5/3$: Symanzik flow
  - $c_p = 1/3$: C13 flow
  - etc but $c_p > 0$

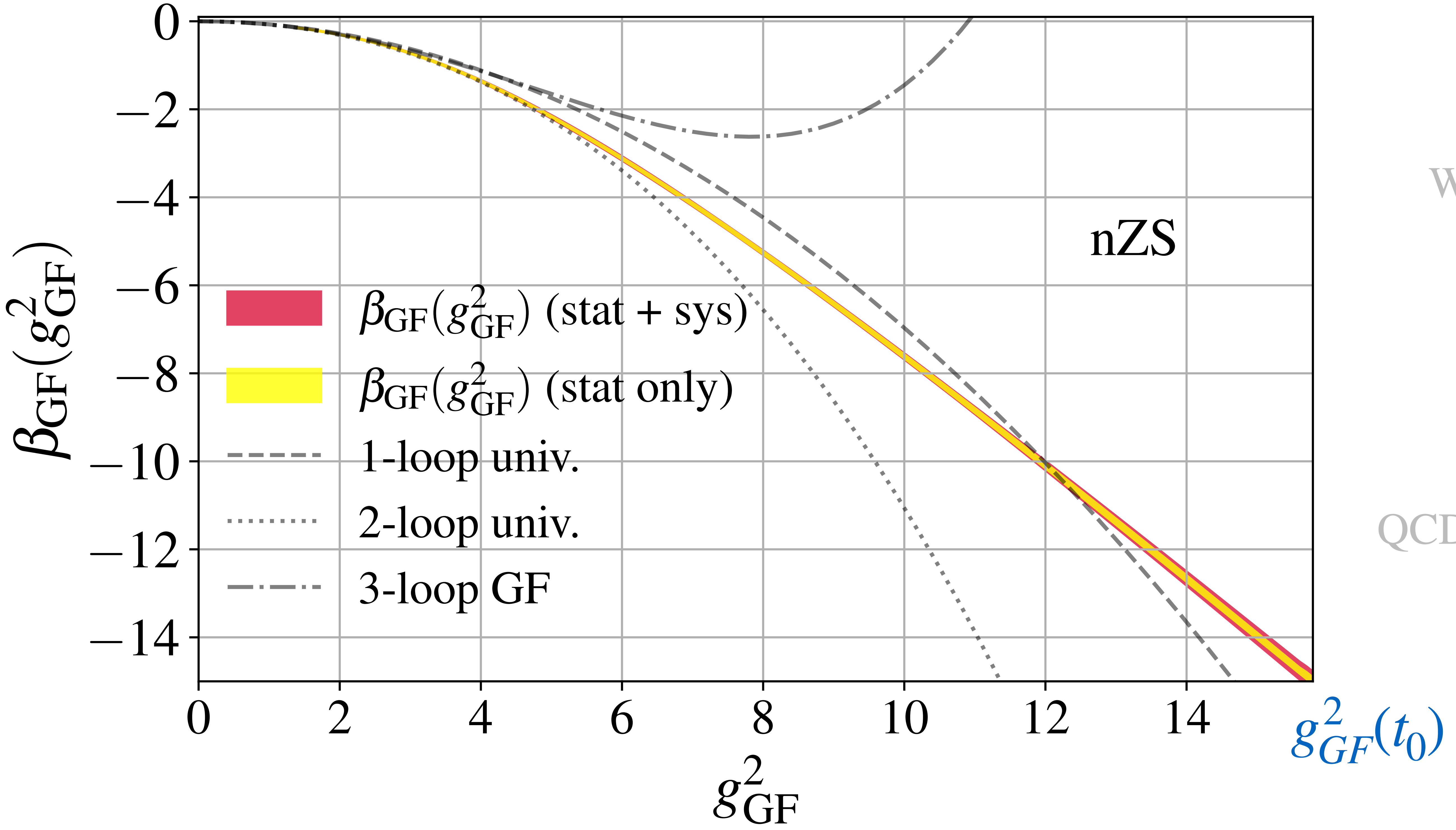


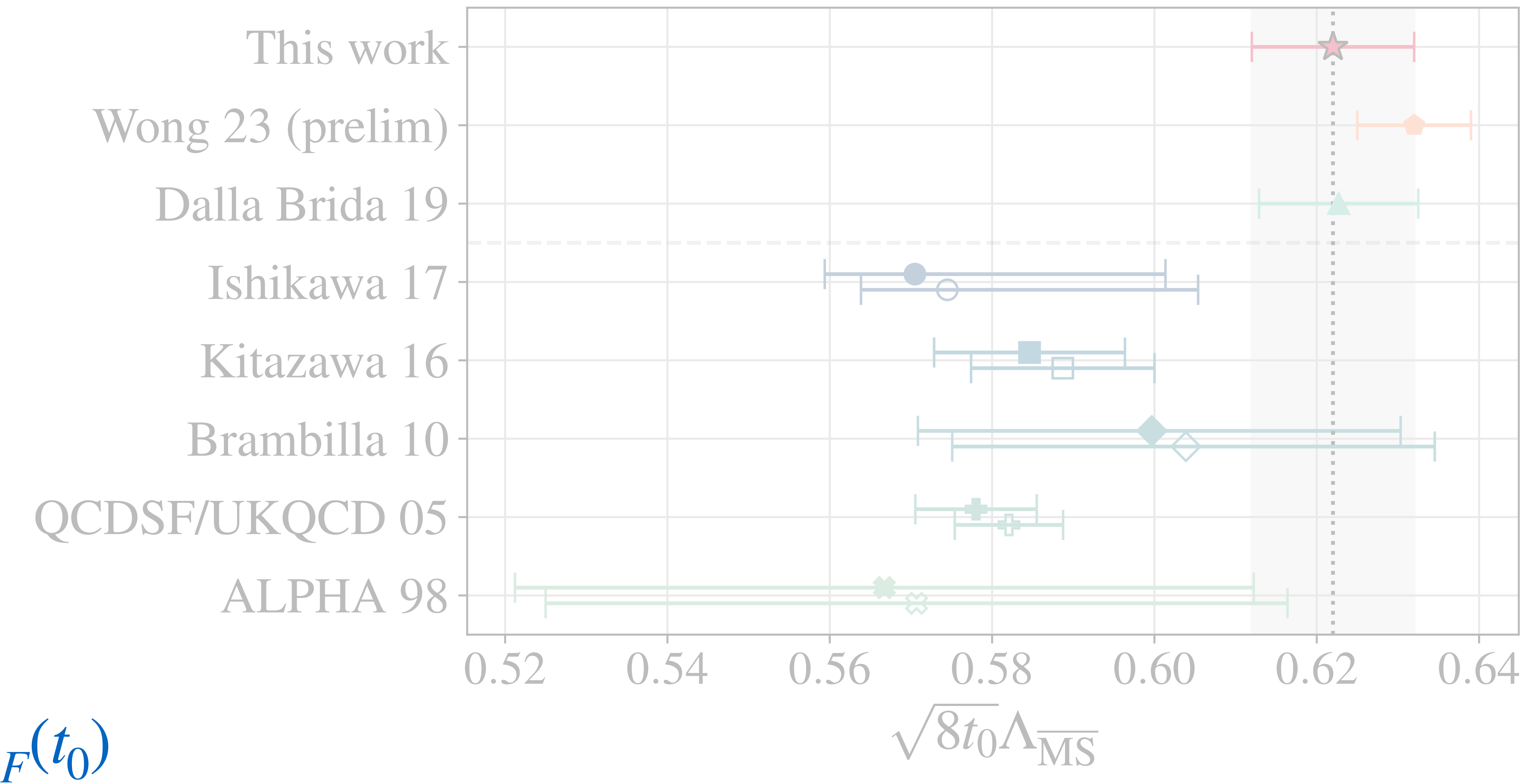
A few recent results

SU(3) Yang-Mills

(AH,C.Peterson,J.VanSickle,O.Witzel, arXiv: 2303.00704)

Wong et al - next talk





$\Lambda_{\overline{MS}}$ consistent with other GF results
in tension with other (older) results

RG β function in the gradient flow scheme

- maps into perturbative curves at weak coupling
- very linear in the strong coupling (why?)

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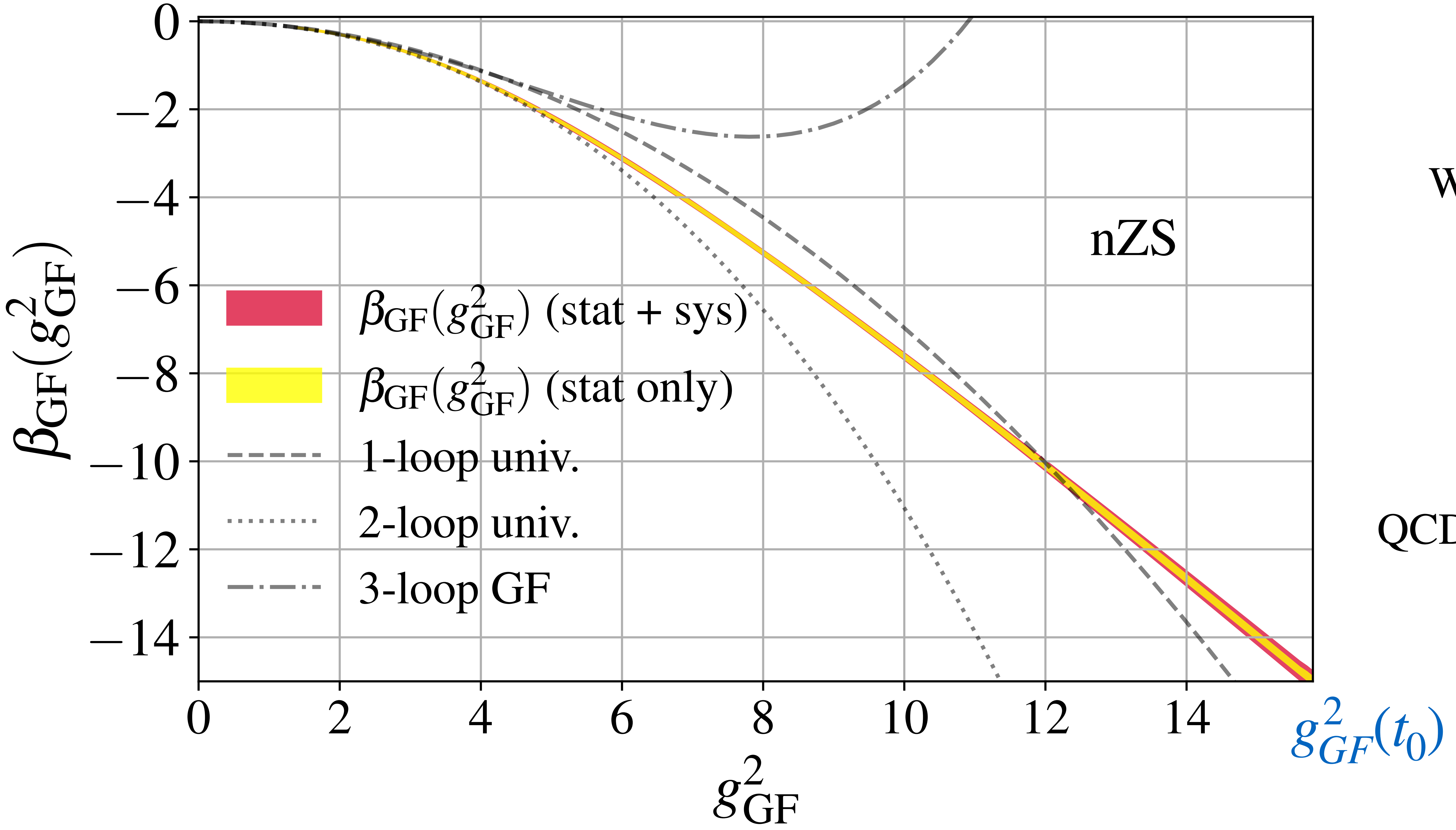

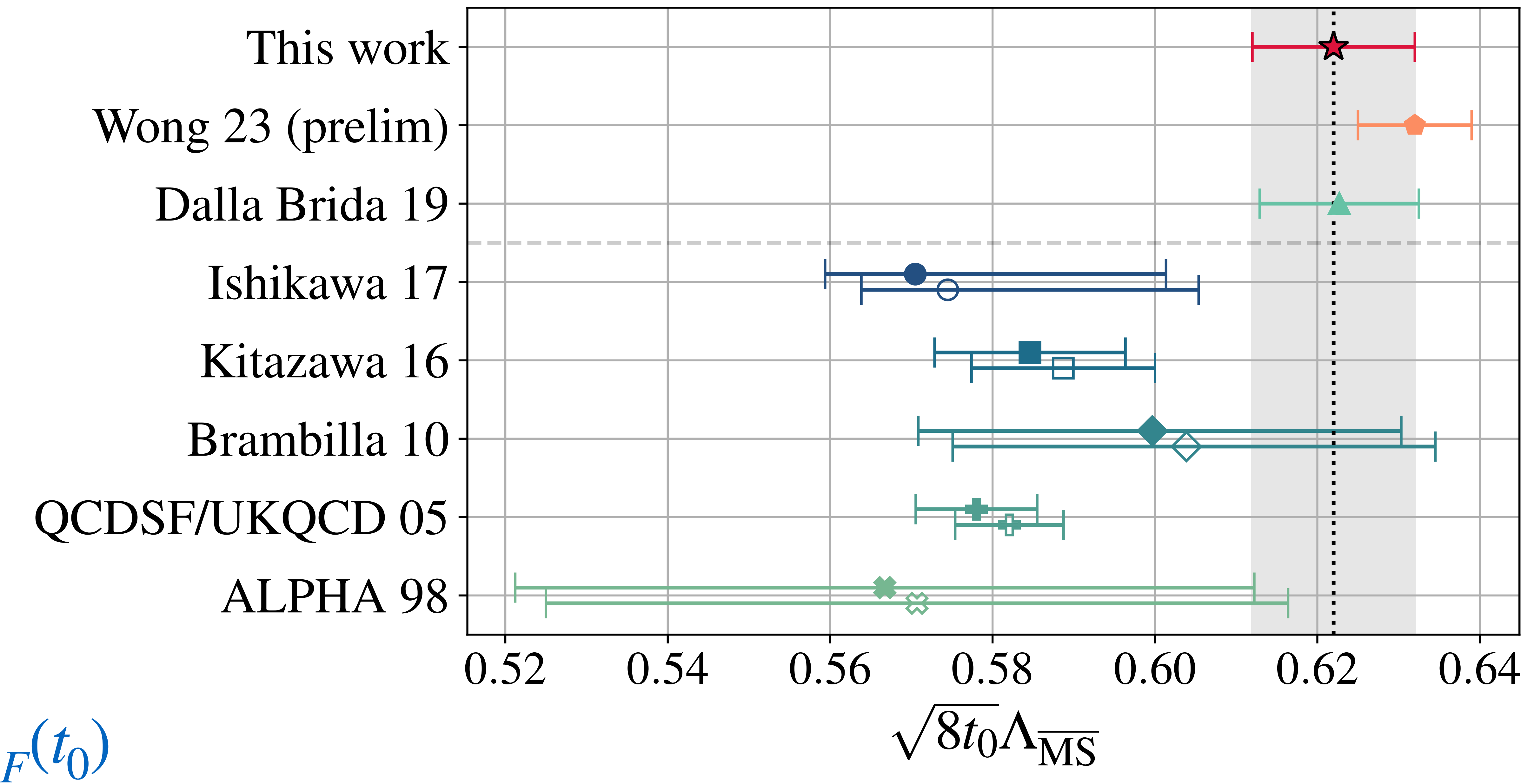

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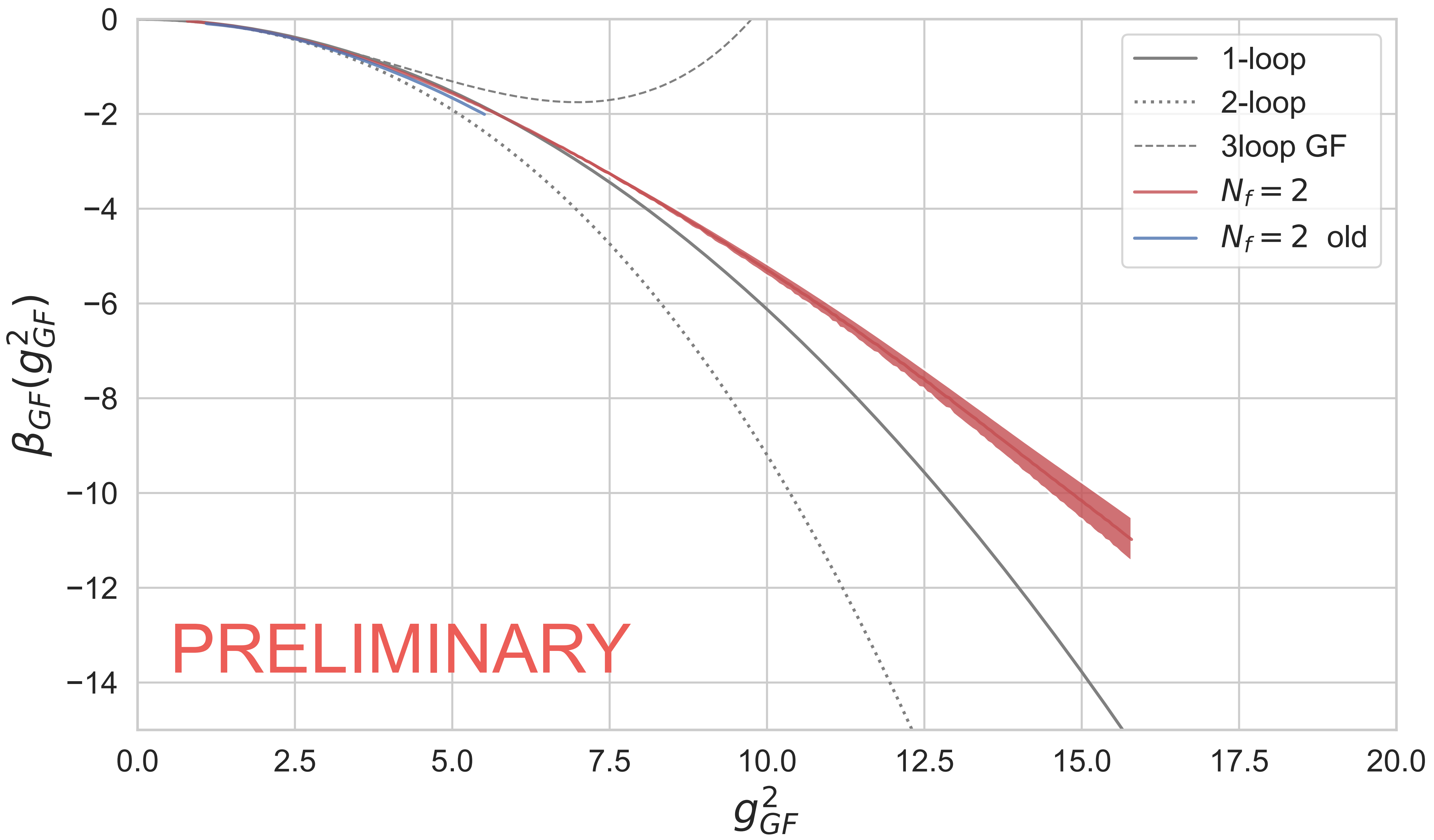
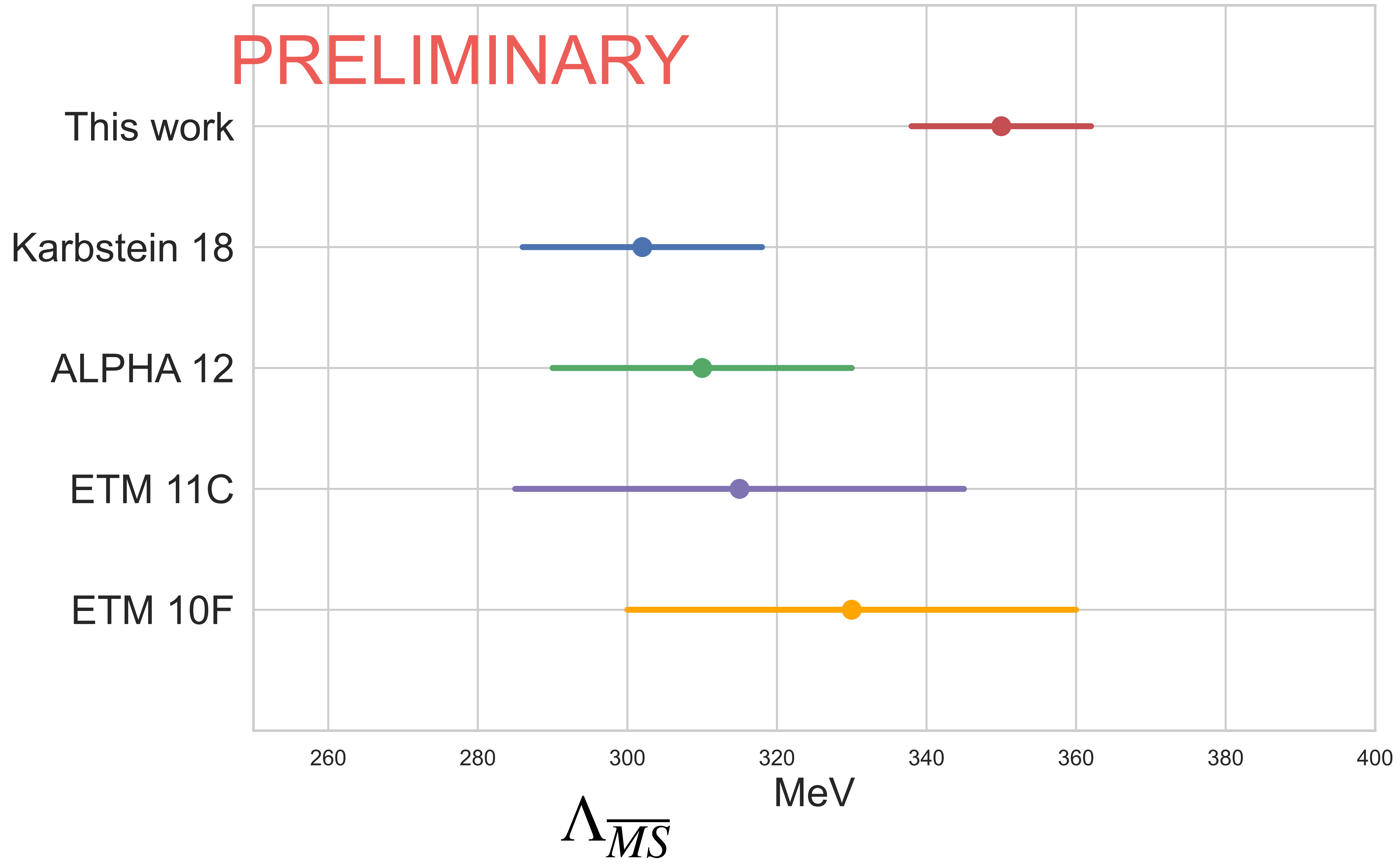
SU(3)+ $N_f = 2$

A.H.,C. Monahan, M. Rizik,
A. Shindler and O. Witzel
Lattice’21 (arXiv:2201:09740)
+ in preparation

Pilot study with domain wall fermions





RG β function in the gradient flow scheme

- maps into perturbative curves at weak coupling
- very linear in the strong coupling (why?)

$\Lambda_{\overline{MS}}$ very preliminary

SU(3)+ $N_f = 2$

A.H.,C. Monahan, M. Rizik,
A. Shindler and O. Witzel
Lattice’21 (arXiv:2201:09740)
+ in preparation

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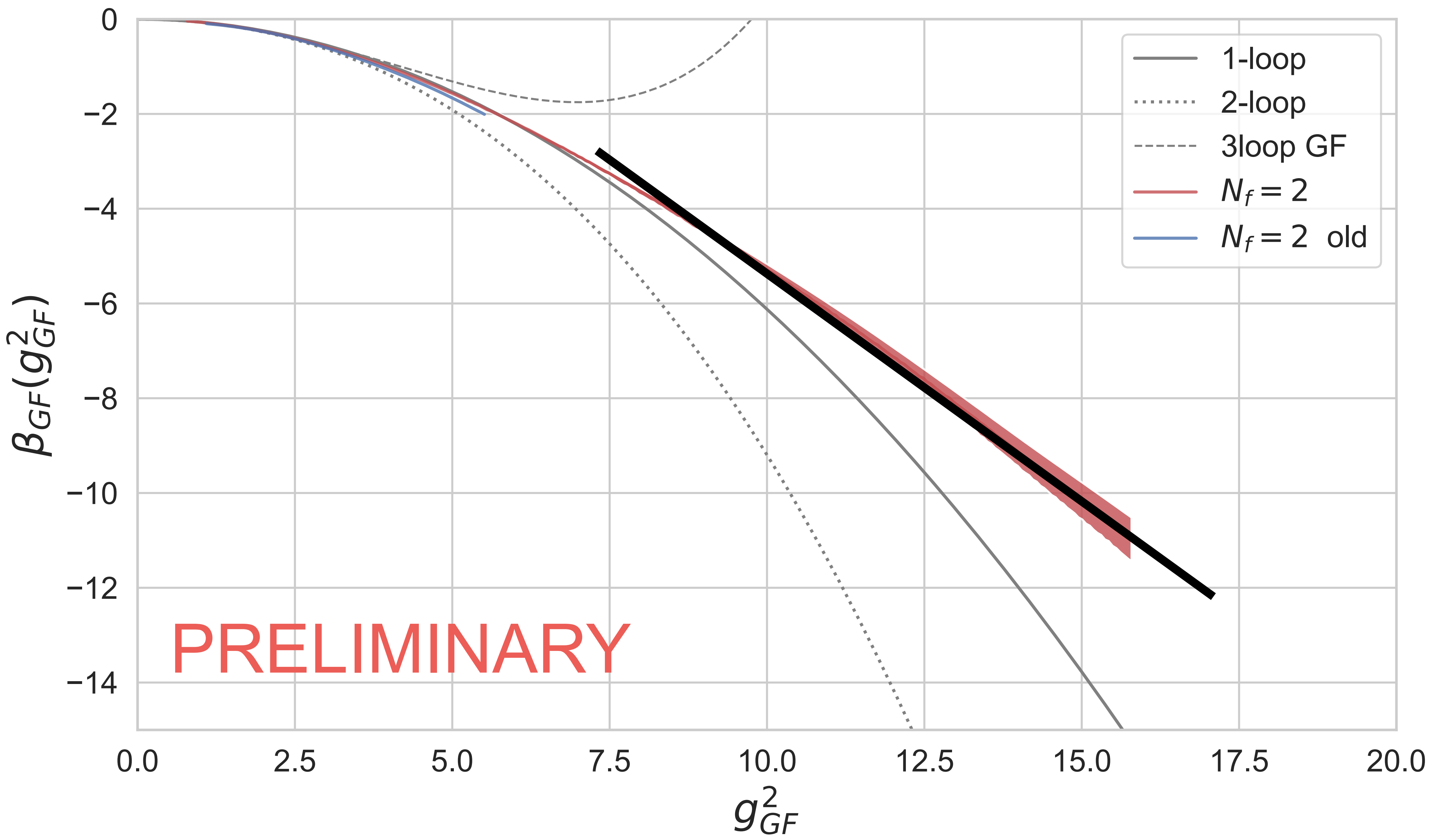

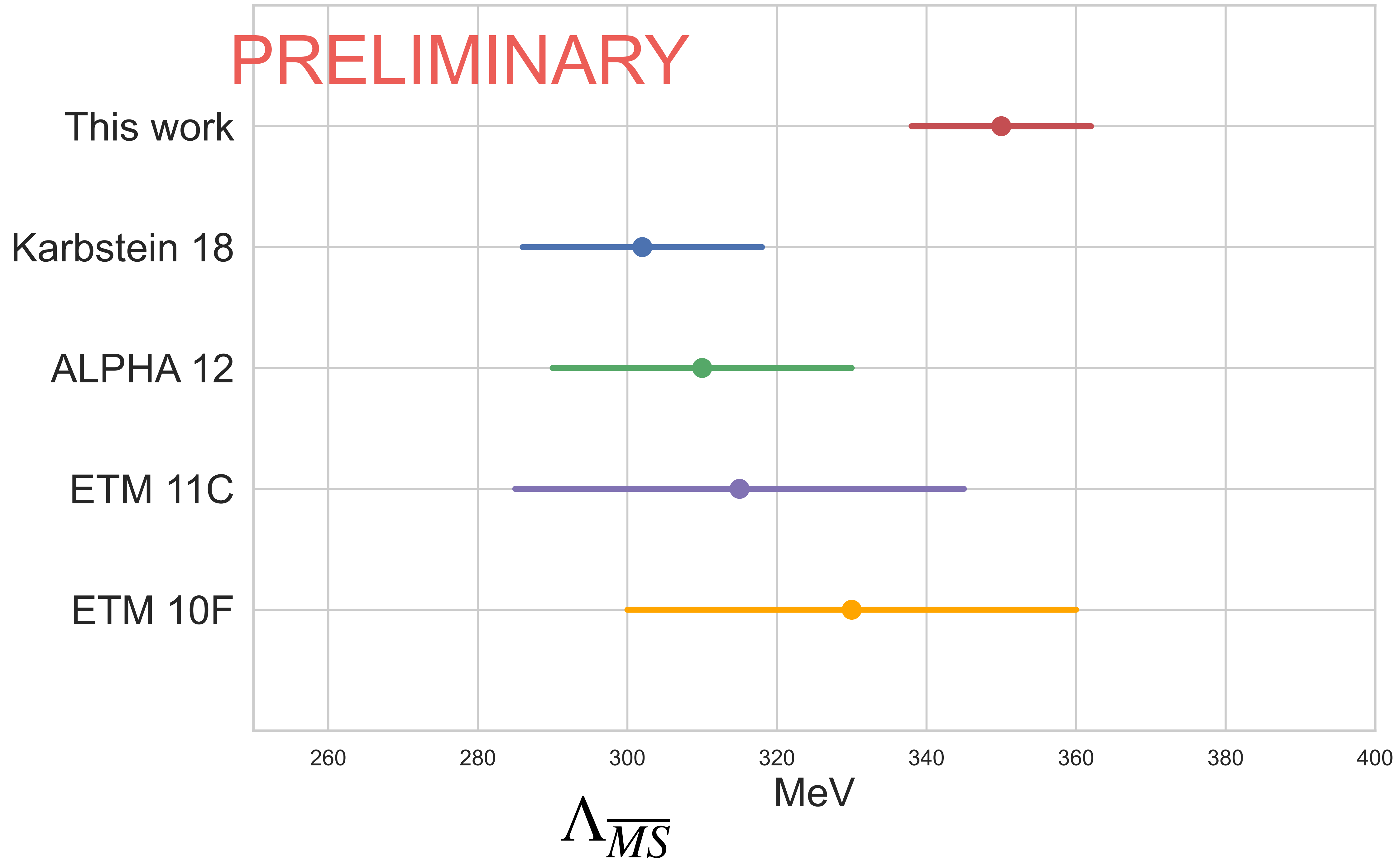

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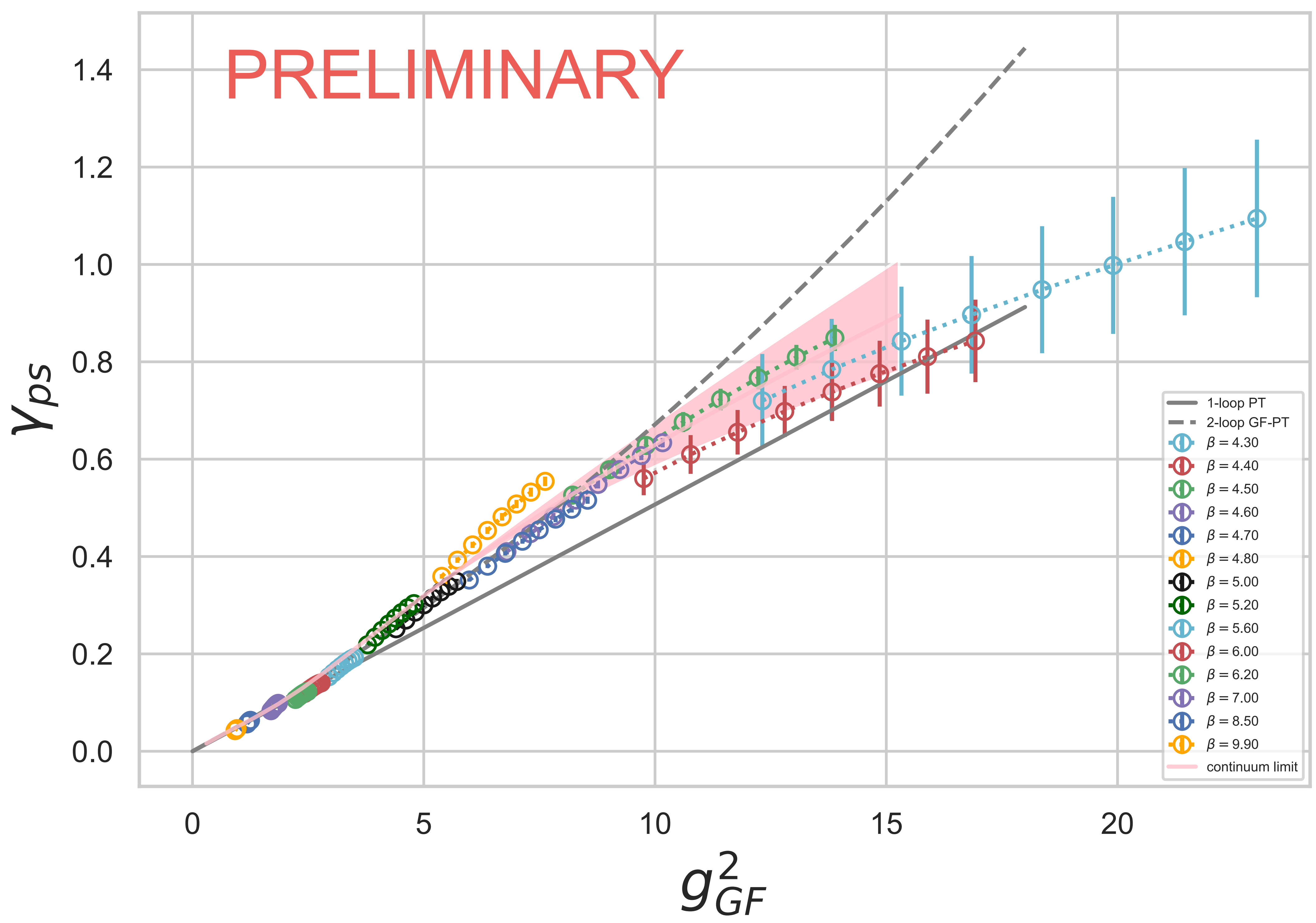


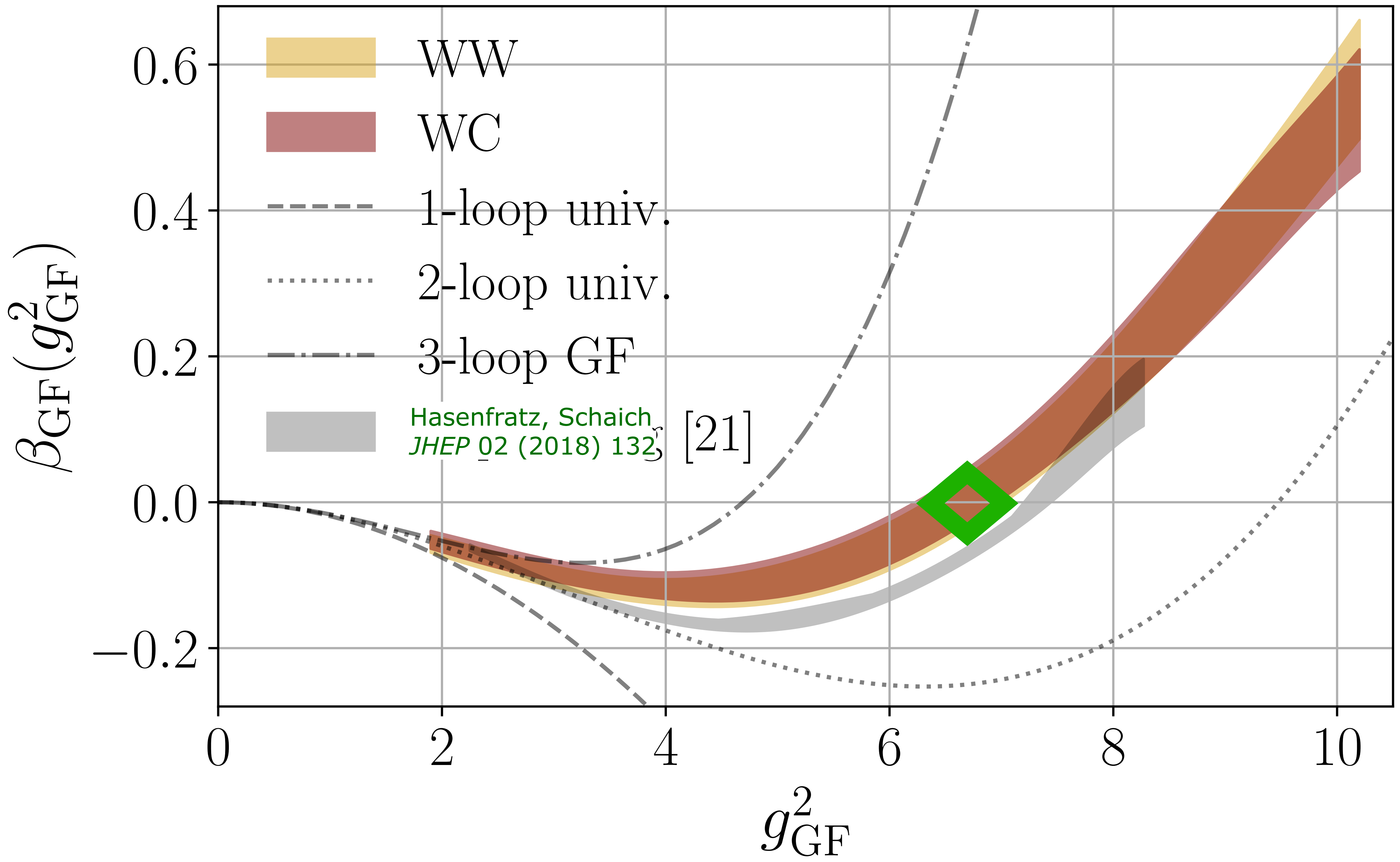
colored points show raw data:
at fixed bare coupling, changing flow time

Anomalous dimension of $\bar{\psi}\gamma_5\psi$

- small cutoff effects
- close to 1-loop

SU(3)+$N_f = 12$, staggered

Peterson, Hasenfratz, 2402.1803



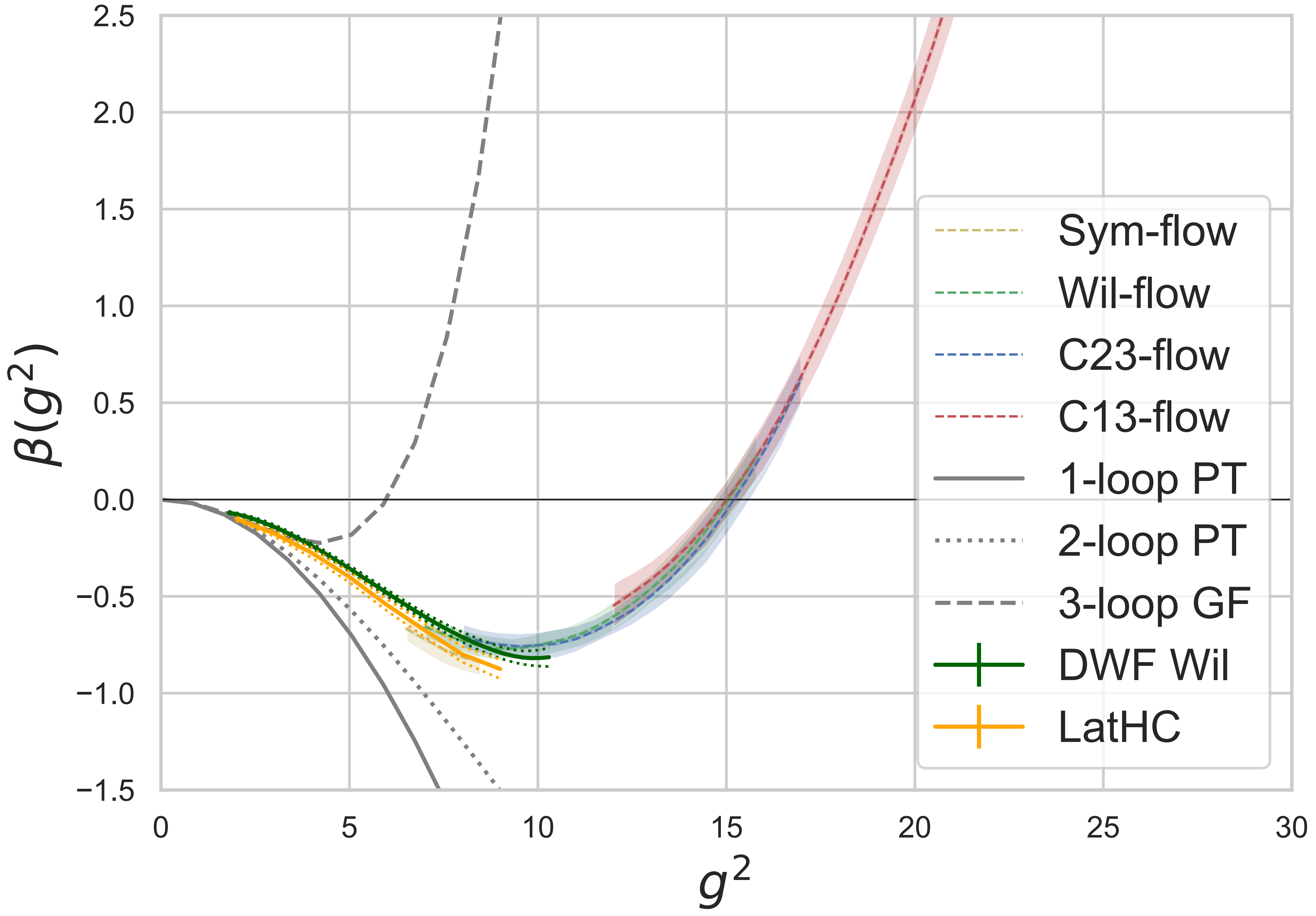
RG β function in the gradient flow scheme exhibits an IRFP

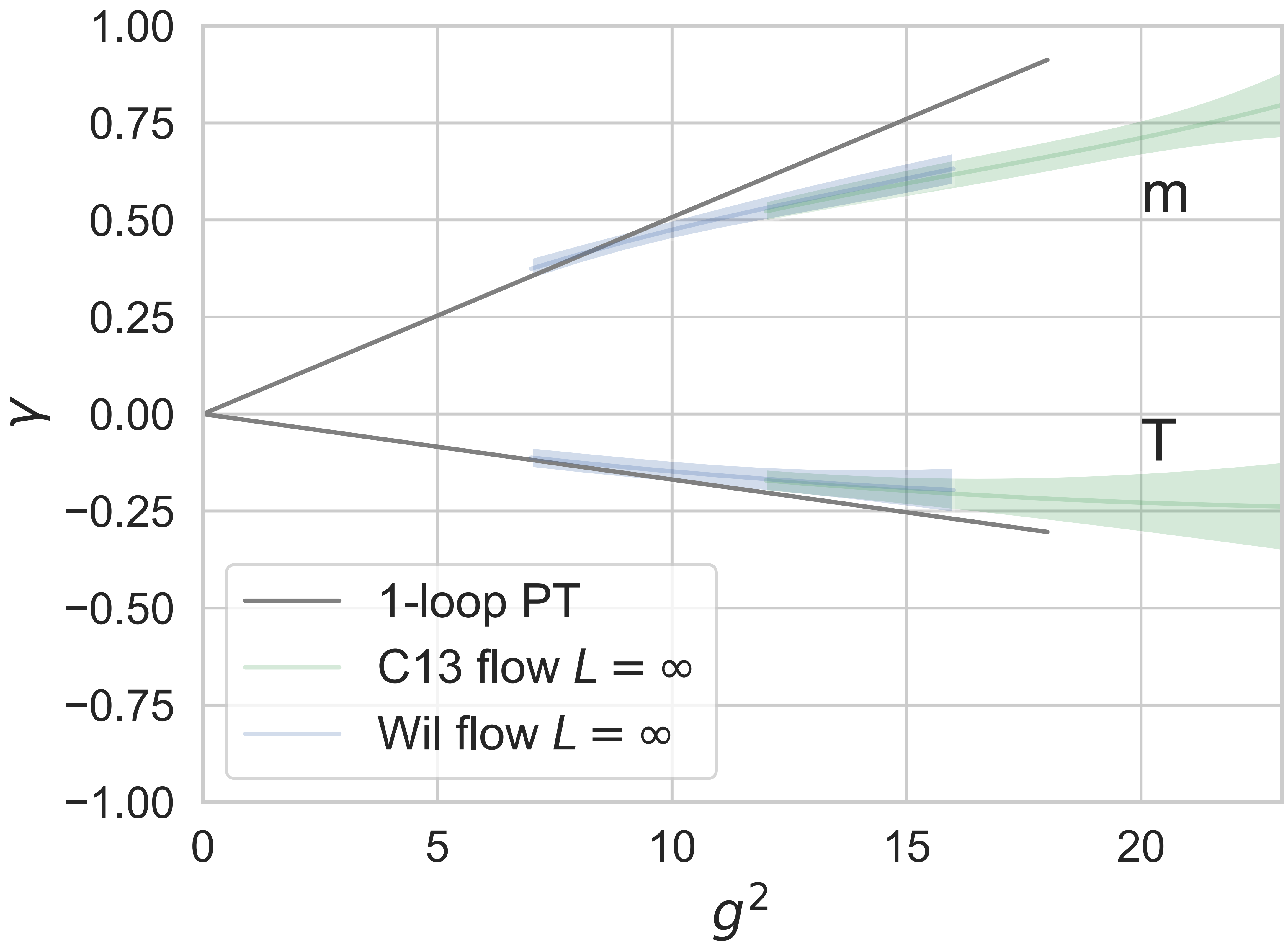
$g^2_{GF} \approx 6.8$

SU(3)+$N_f = 10$, Wilson

PV improved action allowed to reach $g^2_{GF} \gtrsim 20$

Hasenfratz.,Neil, Shamir, Svetitsky, Witzel,
Phys.Rev.D 108 (2023) 7





RG β function in the gradient flow scheme exhibits an IRFP at $g^2_{GF} \approx 15$, $\gamma_m \approx 0.6$

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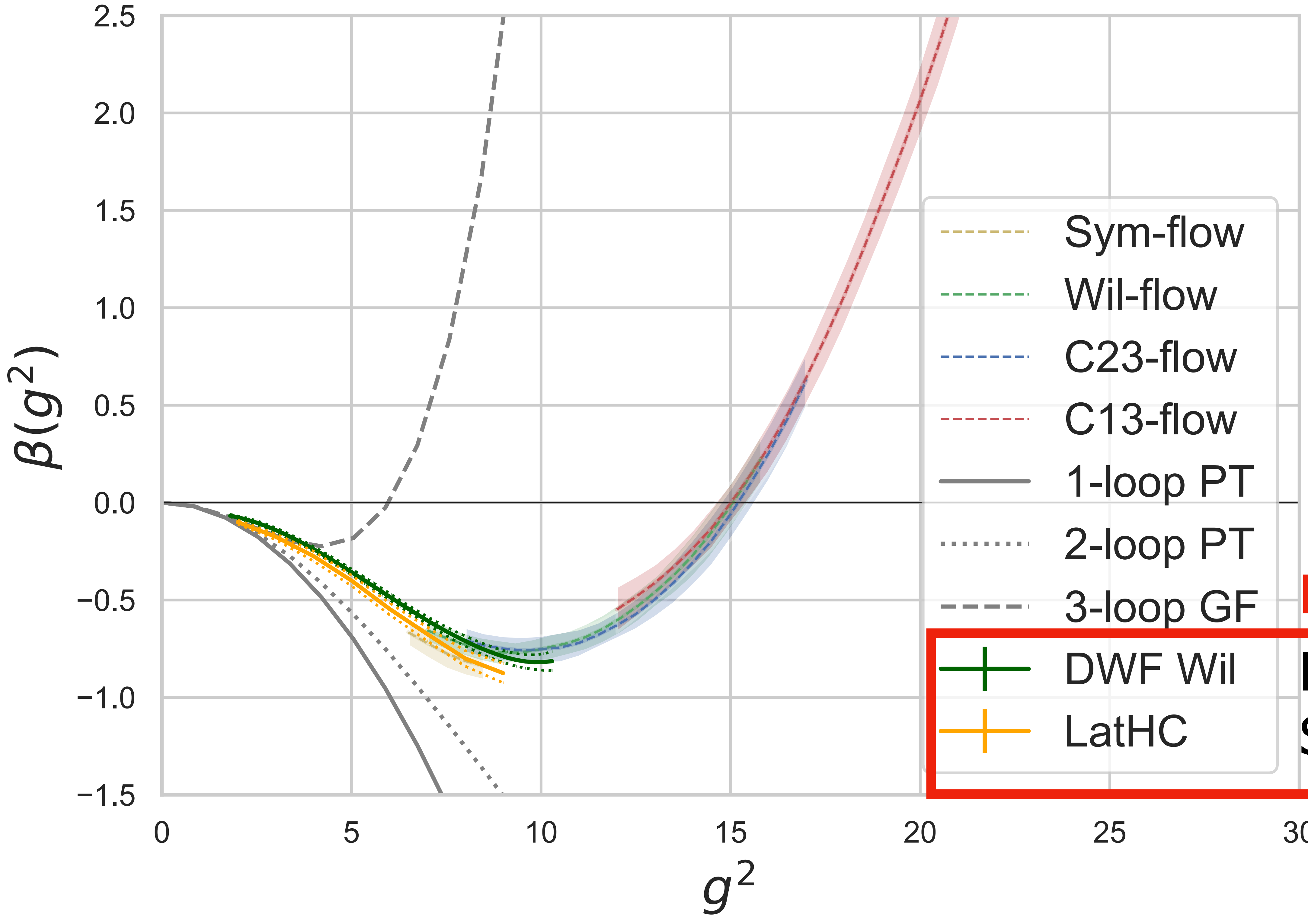
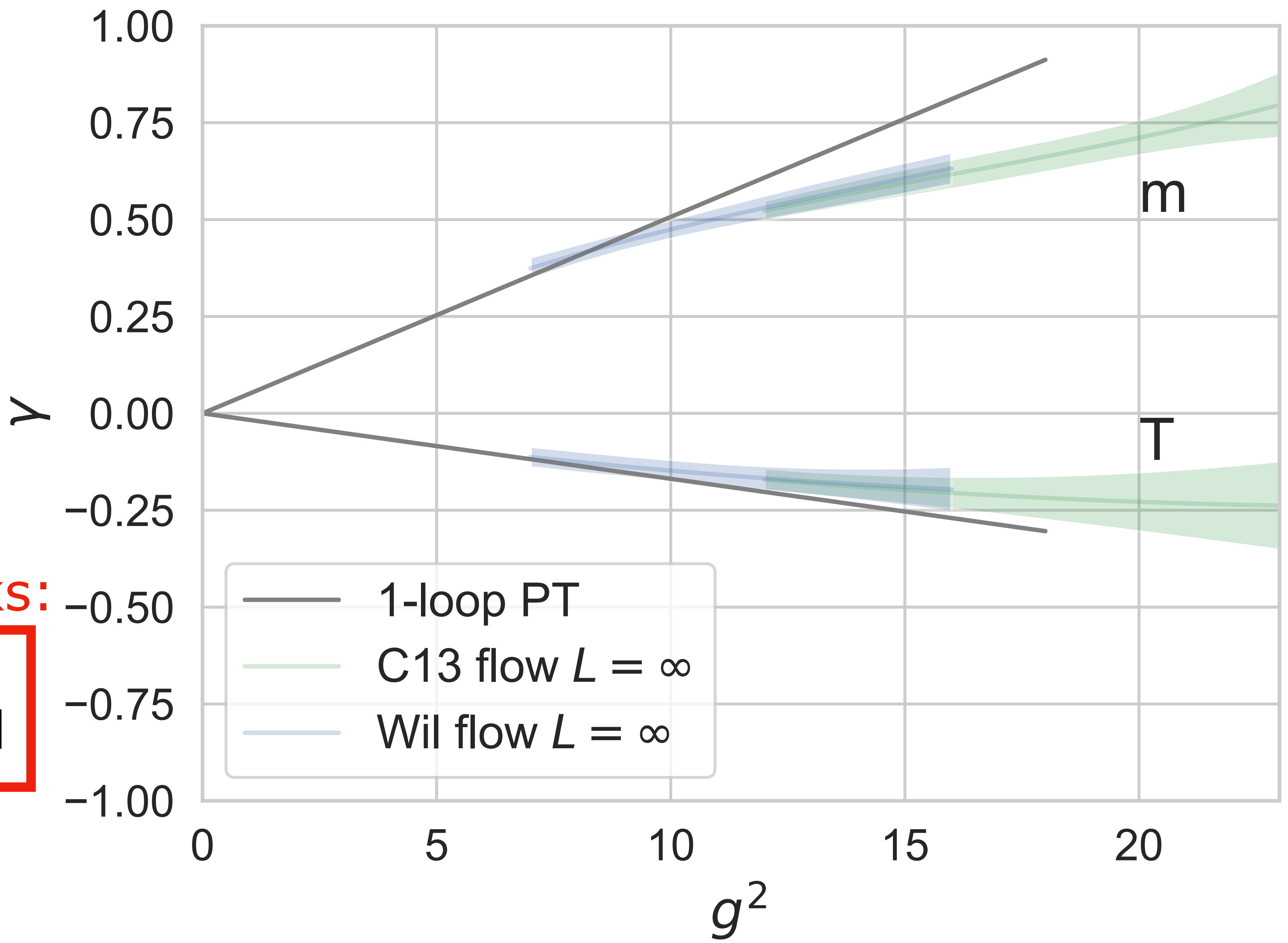


Prior works:
DWF
Staggered



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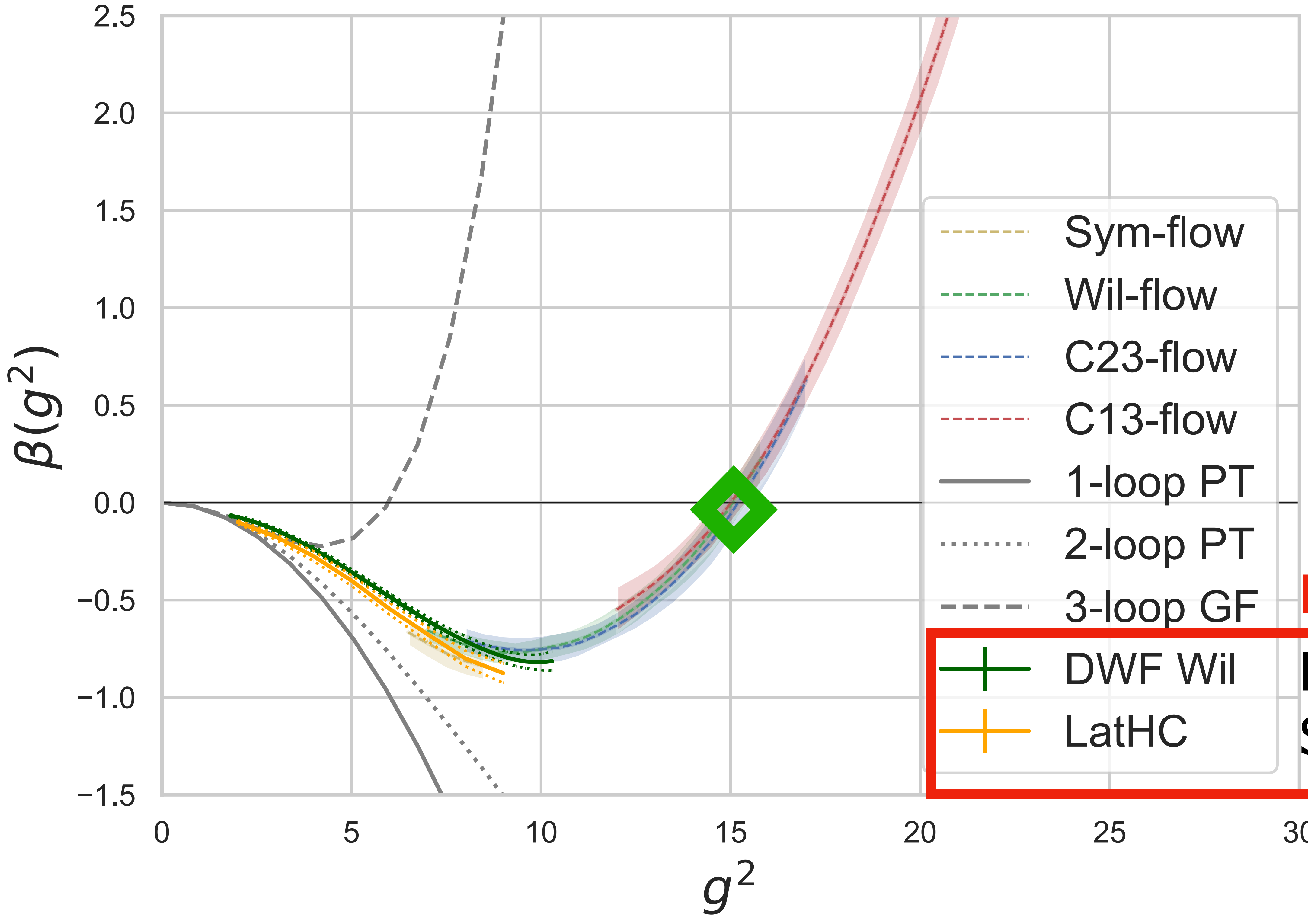

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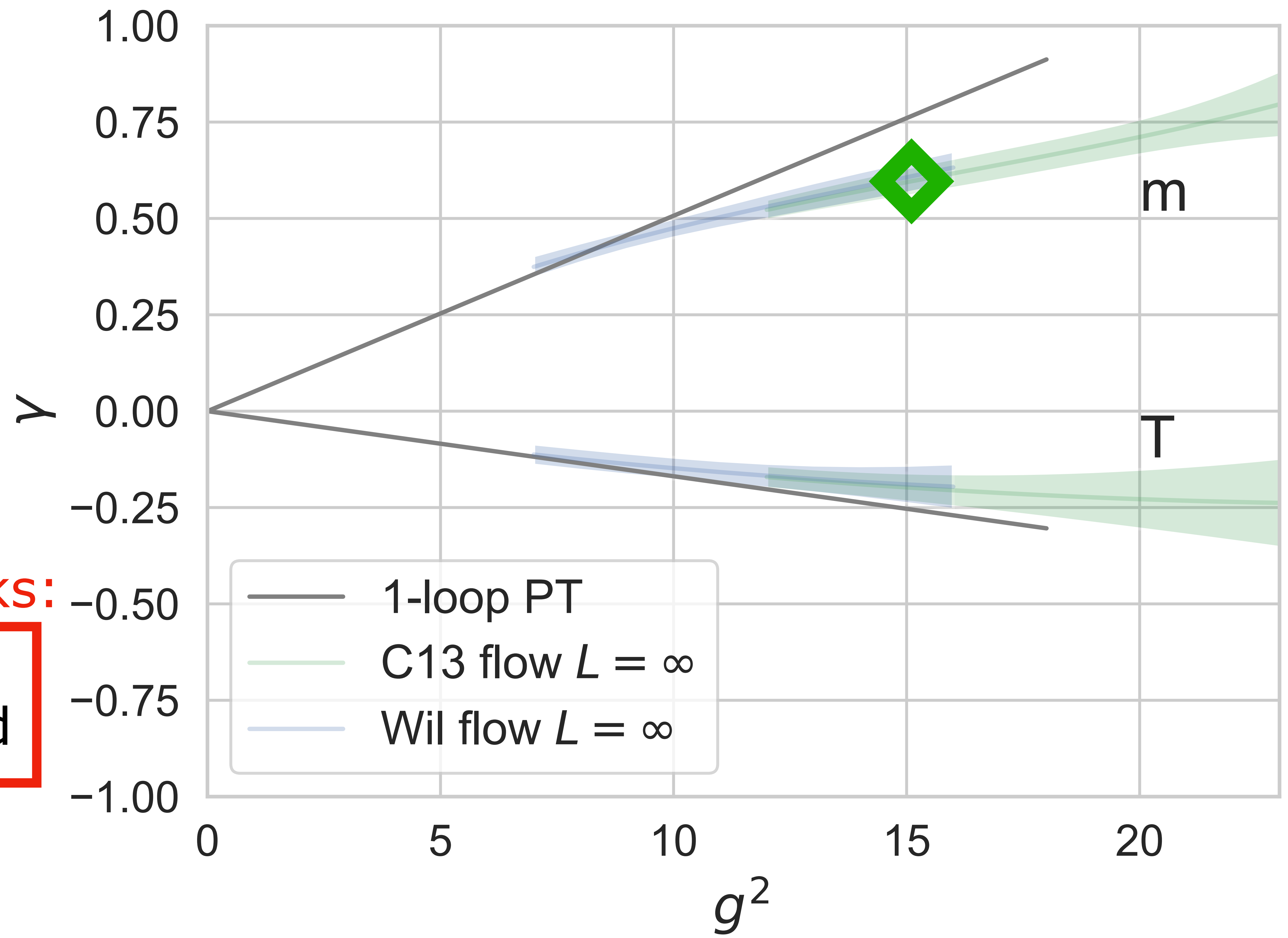

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SU(3)+$N_f = 8$, staggered

“But even Wikipedia says $N_f = 8$ is chirally broken!”

While it has been long favored as composite Higgs model with slowly walking coupling and a light scalar (Higgs), this is not the case

SU(3)+$N_f = 8$, staggered

LSD Collaboration 2306.06095,
Phys.Rev.D 108 (2023) 9

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SU(3)+$N_f = 8$, staggered

“But even Wikipedia says $N_f = 8$ is chirally broken!”

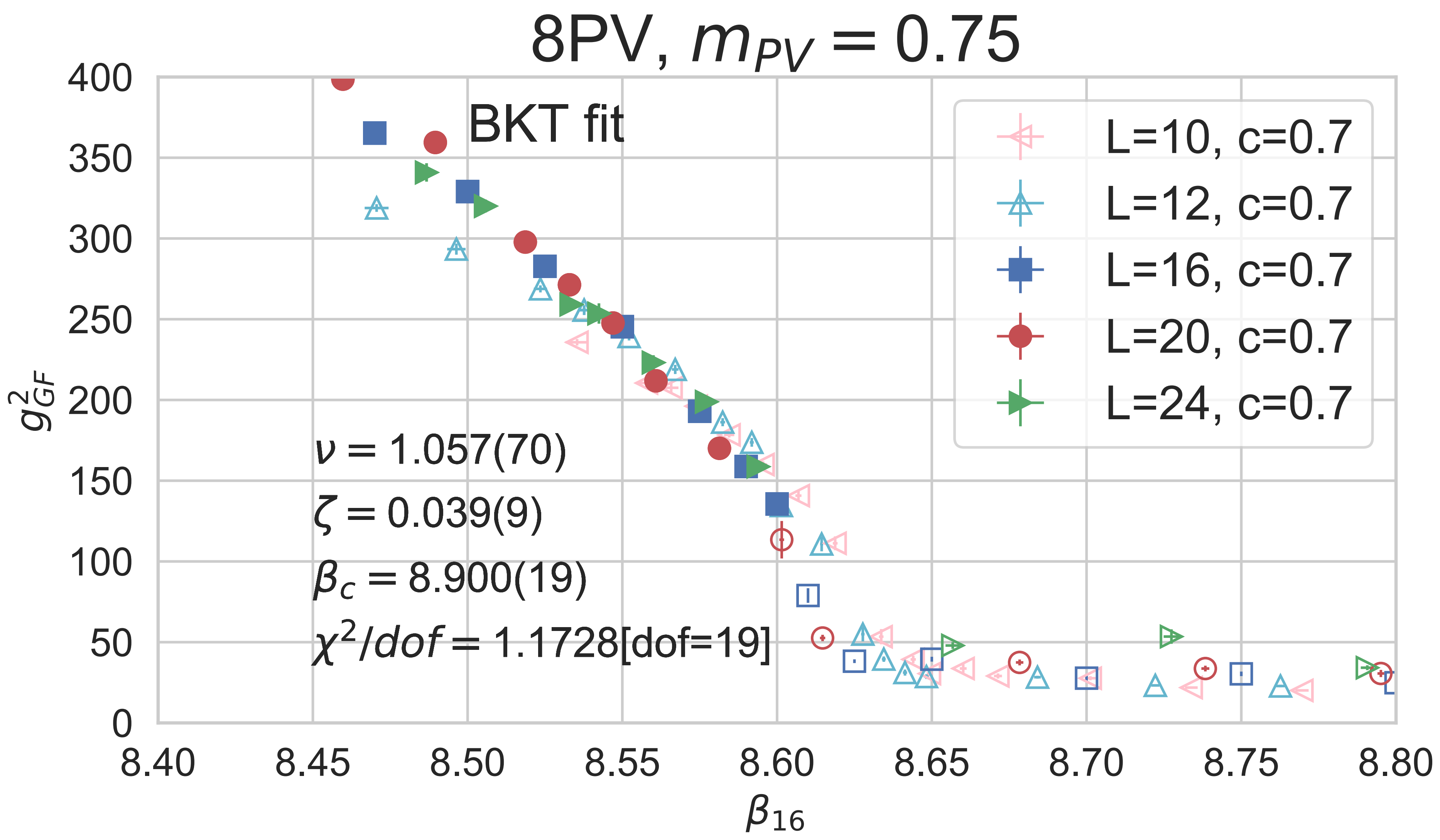
PV improved action reveals a different picture:

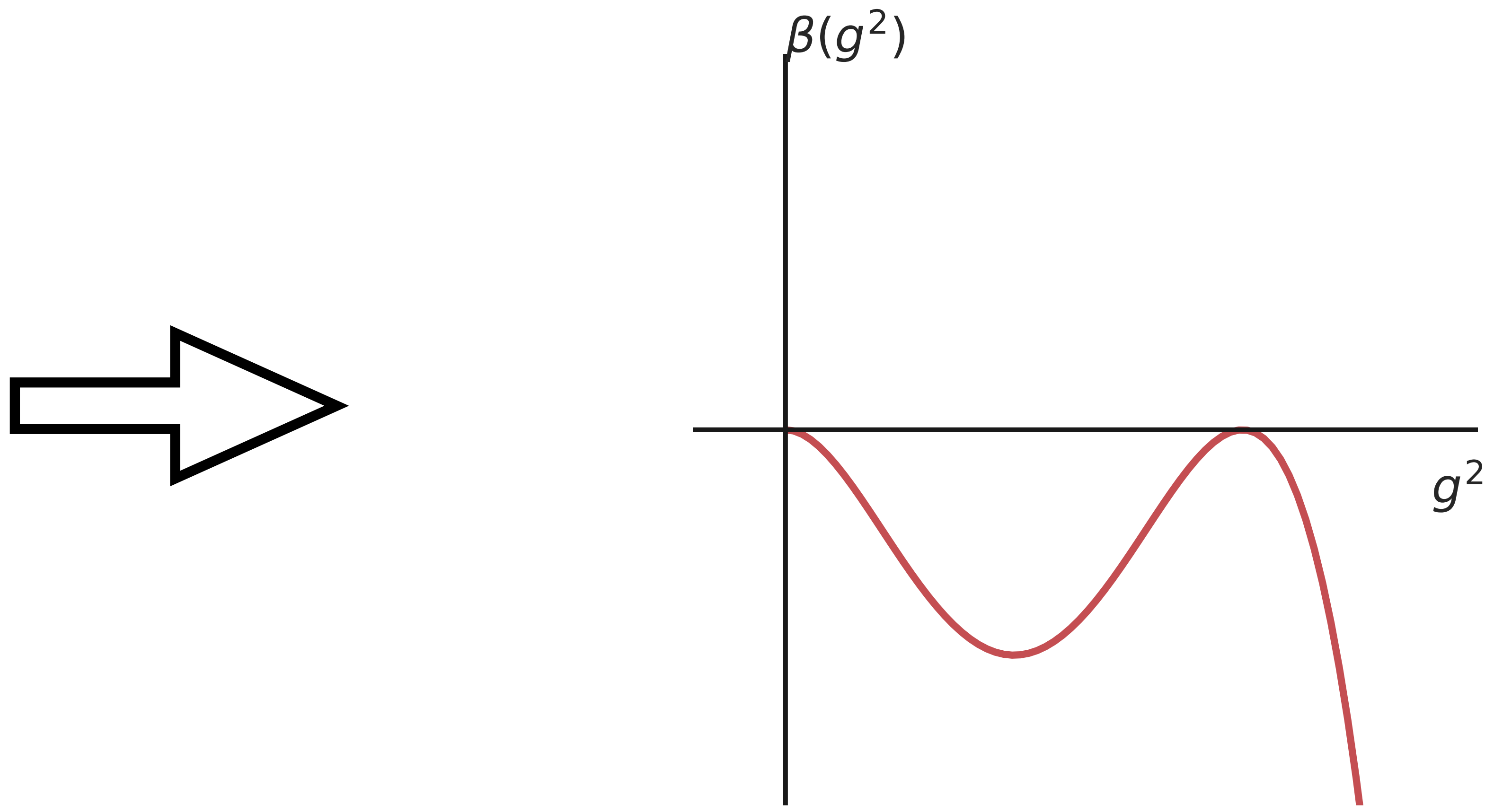
SU(3)+$N_f = 8$ could be the opening of the conformal window

Finite size scaling analysis using finite volume g^2_{GF} shows

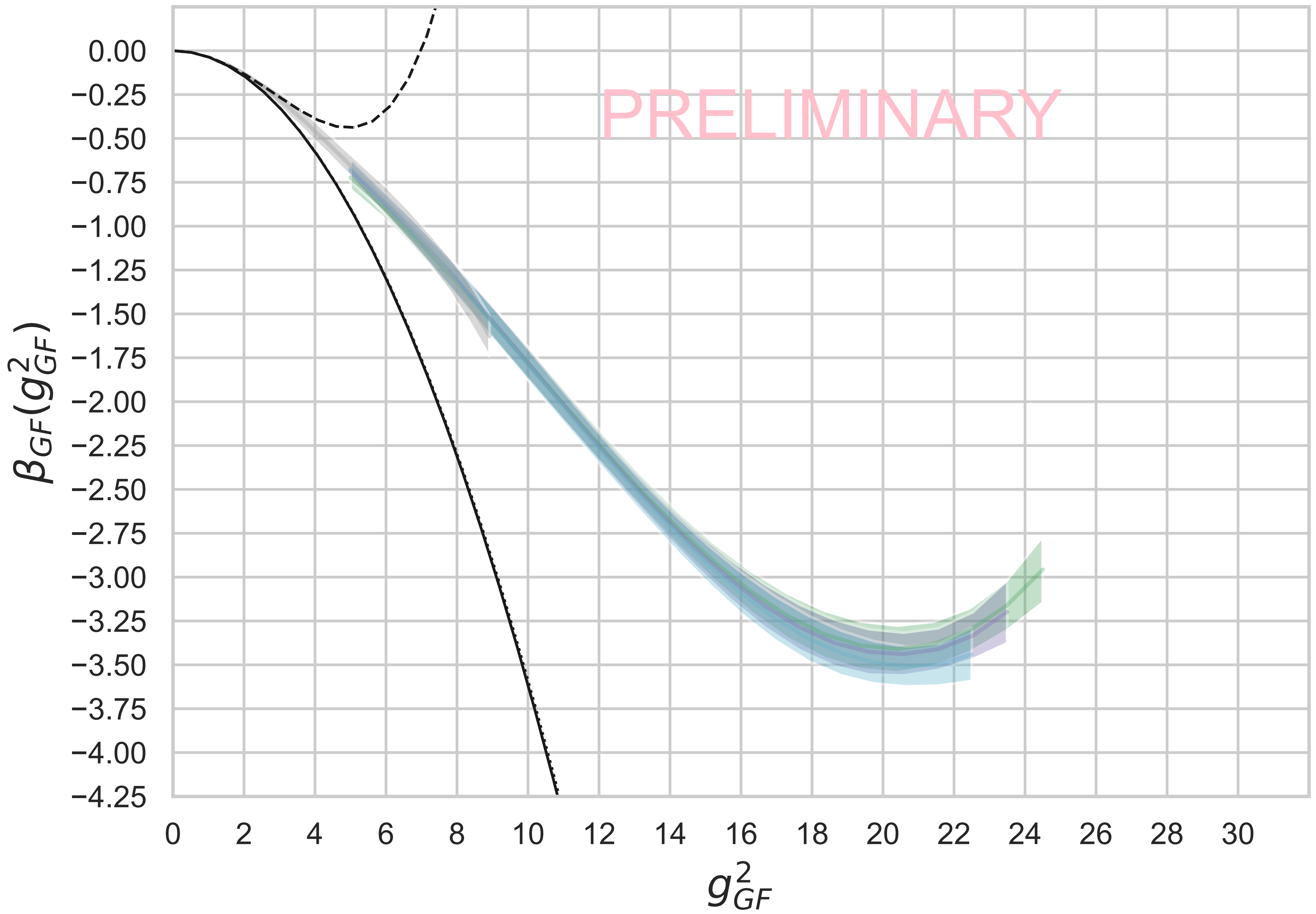
- continuous, likely BKT phase transition: $\xi \propto e^{-\zeta(\beta-\beta_c)^{-\nu}}$
- Symmetric Mass Generation phase

A.H. PRD 106 (2022) 014513
A.H, O. Witzel 2412.10322 (LSD)





SU(3)+$N_f = 8$, staggered



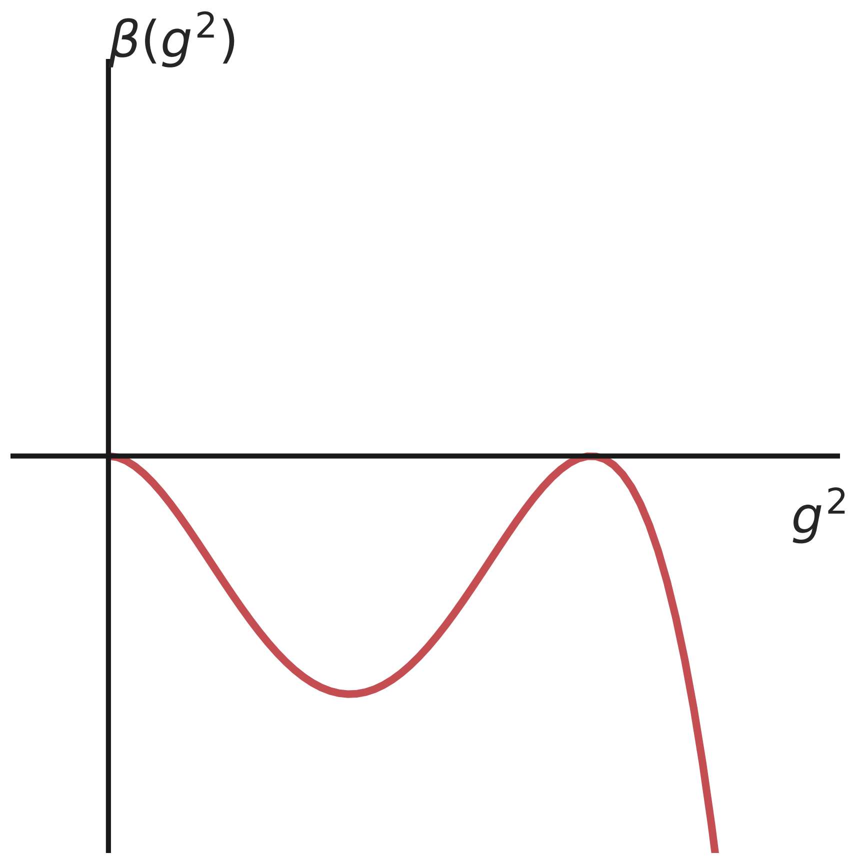
In practice, nonperturbative cutoff effects due to instantons make the predicted $|\beta(g^2)|$ too large

Hasenfratz, Peterson, in prep.

RG β function in the gradient flow scheme shows an uptick, but not a fixed point

We determine $\beta(g^2_{GF})$ while tuning $g_0^2 \to 0$

A BKT transition would require infinite flow time



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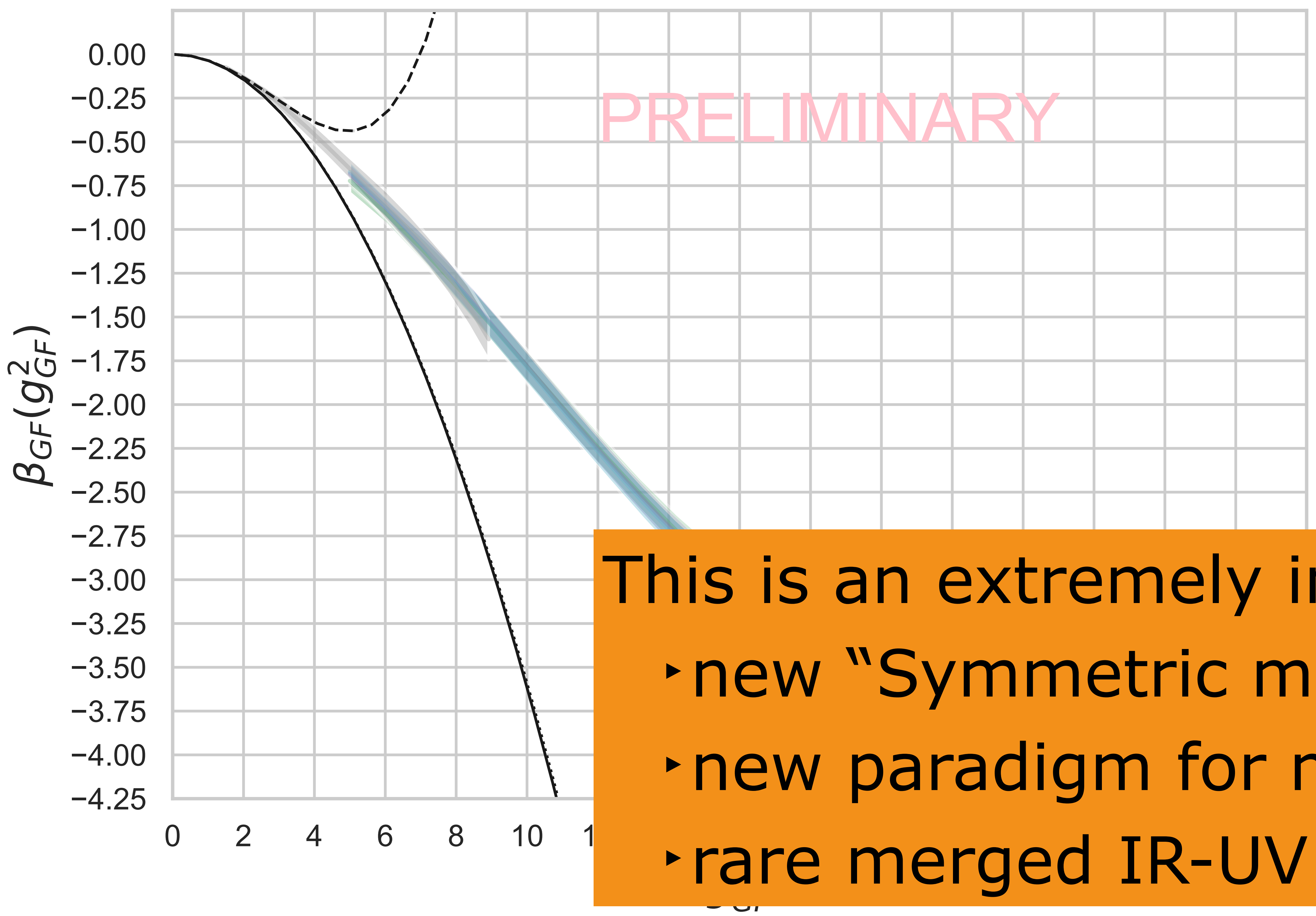

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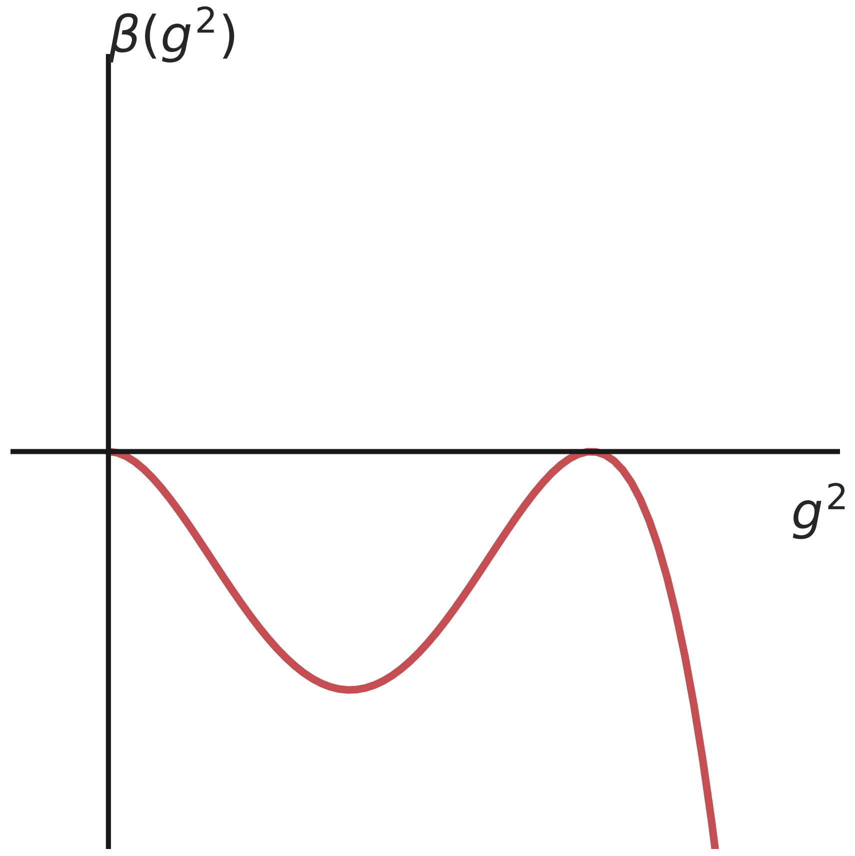
This is an extremely interesting system with a

- new “Symmetric mass generation” phase
- new paradigm for mass generation
- rare merged IR-UV fixed point

In practice, nonperturbative cutoff effects due to instantons make the predicted $|\beta(g^2)|$ too large



Summary and outlook

When gradient flow is interpreted as a continuous RG transformation, a plethora of non-perturbative methods open up:

- β and γ functions in the chirally broken regime: $\longrightarrow \Lambda_{QCD}$ and renormalization factors
- β and γ functions in conformal systems: $\longrightarrow$ IRFP and universal $\gamma*$
- g^2_{GF} in finite volume can be used in finite size scaling $\longrightarrow$ novel phase?

Future directions:

- Flowed observables can give new, fully non-perturbative renormalization scheme
- QCD α_{strong} could be determined with high accuracy
- Many applications in non-QCD-like systems