Universität Zürich UZH

PSI Center for Neutron and Muon Sciences

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The Path to Multiloop Amplitudes: From Feynman Diagrams to a Number

LTPhD Seminar

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December 09, 2024

Lagrangian and Perturbation Theory

- Lagrangian defines our theory, e.g., the Standard Model

$$\mathcal{L}_{\mathrm{SM}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\mathrm{i}\bar{\psi} \not{D} \psi + \mathrm{h.c.}) + (\bar{\psi}_i y_{ij} \psi_j \phi + \mathrm{h.c.}) + |D_\mu \phi|^2 - V(\phi)$$

- From it we can calculate observables, e.g., cross sections

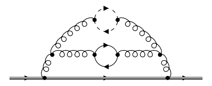
$$\sigma = \int \mathrm{d}\Pi_{\mathrm{PS}} |\mathcal{M}|^2$$

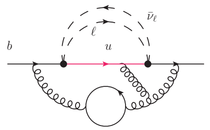
  where $\int \mathrm{d}\Pi_{\mathrm{PS}}$ is an integral over phase space and $\mathcal{M}$ the *amplitude* of the process
- Exact solution only known for toy theories
- Instead:
  - Perturbation theory: expand in small parameters, e.g., in the electromagnetic coupling constant g_e:

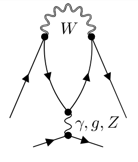
$$\mathcal{M} = \mathcal{M}_0 + g_e^2 \mathcal{M}_1 + g_e^4 \mathcal{M}_2 + \ldots,$$
$$\sigma = \sigma_0 + g_e^2 \sigma_1 + g_e^4 \sigma_2 + \ldots$$

  - Discretize space-time on a lattice and run numerical simulation: mainly applicable to low energies

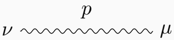
Feynman Rules and Feynman Diagrams



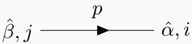




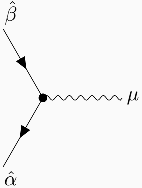
- Illustrative *and* practical method: Feynman diagrams



$$= \quad -\mathrm{i}\delta_{\mu\nu}\frac{1}{p^2},$$



$$= \quad \mathrm{i}\frac{(\not{p}+m)_{\hat{\alpha}\hat{\beta}}}{p^2-m^2}$$

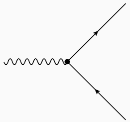


$$= \quad -\mathrm{i}g_e\left(\gamma_\mu\right)_{\hat{\alpha}\hat{\beta}}$$

- Straightforward counting of perturbative order in g_e through vertices

Tree-Level Amplitude

- Let's consider the process $\gamma(q) \to e^-(p_1)e^+(p_2)$
- Leading diagram:



- Kinematics: momentum conservation $q = p_1 + p_2$ and scale $\hat{s} = \frac{q^2}{m^2}$
- Insert Feynman rules into diagram:

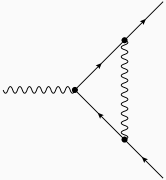
$$\mathcal{M} = \epsilon_\mu(q)\bar{u}(p_2)\Gamma^\mu u(p_1),$$
$$\Gamma^\mu = g_e\gamma^\mu$$

- Polarization of external photon $\epsilon_\mu(q)$ and spinors of external electron $u(p_1)$ and positron $\bar{u}(p_2)$ dealt with when squaring amplitude $|\mathcal{M}|^2$

$\Rightarrow$ Ignore them for amplitude and only look at Γ^μ

One-Loop Amplitude

- Again one diagram



- Inserting Feynman rules:

$$\Gamma^{\mu}(\hat{s}) = \int \frac{\mathrm{d}^D k}{(2\pi)^D}(-\mathrm{i}g_{\mathrm{e}}\gamma^{\mu_1})\frac{\mathrm{i}(\not{p}_2 + \not{k} + m)}{(p_2+k)^2 - m^2}(-\mathrm{i}g_{\mathrm{e}}\gamma^{\mu})\frac{\mathrm{i}(\not{p}_1 + \not{k} + m)}{(p_1+k)^2 - m^2}(-\mathrm{i}g_{\mathrm{e}}\gamma^{\mu_2})\frac{-\mathrm{i}g_{\mu_1\mu_2}}{k^2}$$

  where $D = 4 - 2\epsilon$ to regulate divergencies
- This looks intimidating ...
  - Loop integral $\int \frac{\mathrm{d}^D k}{(2\pi)^D}$
  - Tensor integral with index μ
  - Non-trivial matrix in Dirac space

Tensor Decomposition (I)

- Properties and symmetries of process allow us to write general expression

$$\Gamma^{\mu}(\hat{s}) = g_e F_1(\hat{s})\gamma^{\mu} - \frac{i g_e}{2m} F_2(\hat{s})\sigma^{\mu\nu} q_{\nu} \qquad \text{with} \qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]$$

  valid at *all loop orders*
- γ^{μ} and $\sigma^{\mu\nu} q_{\nu}$ form complete basis and all other structures linearly combinations of the two
- $F_1(\hat{s})$ and $F_2(\hat{s})$ called *form factors* which are scalar functions of kinematic scale $\hat{s}$
- Side remark: $F_2(\hat{s} = 0) \widehat{=} \frac{g-2}{2}$
- We have seen at leading order $F_1(\hat{s}) = 1$ and $F_2(\hat{s}) = 0$
- Can be obtained by applying *projectors* to amplitude:

$$\mathcal{P}_i[\Gamma_{\mu}] = \mathrm{Tr}[P_i^{\mu}(\not{p}_2 + m)\Gamma_{\mu}(\not{p}_1 - m)]$$

  where P_i^{μ} chosen such that

$$\mathcal{P}_i[\Gamma_{\mu}] = F_i(\hat{s})$$

Tensor Decomposition (II)

- Applying the projectors we find

$$F_1^{(1)}(\hat{s}) = c_1^1 I(0,1,1) + c_1^2 I(-1,1,1) + \text{nine more terms},$$
$$F_2^{(1)}(\hat{s}) = c_2^1 I(0,1,1) + c_2^2 I(-1,1,1) + \text{nine more terms}$$

where, e.g.,

$$c_1^1(\hat{s}) = 2\left(\left(D^2 - 10D + 16\right)\hat{s} + 4(D-4)\right)$$

are rational functions and

$$I(a_1, a_2, a_3) = \int \mathrm{d}^D k \, \frac{1}{(k^2)^{a_1}((k+p_1)^2 - m^2)^{a_2}((k+p_2)^2 - m^2)^{a_3}}$$

are scalar Feynman integrals

$\Rightarrow$ Tensor structures dealt with

Integration-by-Parts Relations

- Chetyrkin and Tkachov noticed

$$\int \mathrm{d}^D k \, \frac{\partial}{\partial k^\mu} \frac{\ell^\mu}{P_1^{a_1} P_2^{a_2} P_3^{a_3}} = 0$$

  with ℓ an arbitrary momentum [Tkachov 1981; Chetyrkin, Tkachov 1981]

- Proof: Gauß's law
- But the derivative can also be applied explicitly, for our example:

$$0 = (D - 2a_1 - a_2 - a_3) I(a_1, a_2, a_3) - a_1 I(a_1 - 1, a_2 + 1, a_3) - a_2 I(a_1 - 1, a_2, a_3 + 1)$$

⇒ Linear relations between different integrals!

- Exploiting relations allows us to express all integrals through a finite (and small) number of master integrals

Reduced Expressions

- Exploiting the relations, only two integrals remain:

$$F_1^{(1)}(\hat{s}) = \tilde{c}_1^1(\hat{s}) I(0,1,0) + \tilde{c}_1^2(\hat{s}) I(0,1,1),$$
$$F_2^{(1)}(\hat{s}) = \tilde{c}_2^1(\hat{s}) I(0,1,0) + \tilde{c}_2^2(\hat{s}) I(0,1,1)$$

  where, e.g.,

$$\tilde{c}_1^1(\hat{s}) = -\frac{4(D-2)\left(D^2(\hat{s}+4) + D\left(\hat{s}^2 - 5\hat{s} - 28\right) - 2\hat{s}^2 + 4\hat{s} + 40\right)}{(D-4)(D-3)(\hat{s}-4)}$$

  are now more complicated

- In the end interested only in result in four dimensions, not $D = 4 - 2\epsilon$

⇒ Expand in ϵ:

$$\tilde{c}_1^1(\hat{s}) = \frac{1}{\epsilon}\frac{8(\hat{s}^2-4)}{\hat{s}-4} - \frac{8(3\hat{s}+8)}{\hat{s}-4} + \epsilon\frac{8(\hat{s}^2-\hat{s}-4)}{\hat{s}-4} + \mathcal{O}(\epsilon^2)$$

- Only have to solve the two integrals $I(0,1,0)$ and $I(0,1,1)$ now

Differential Equations [Kotikov 1991 (×3); Remiddi 1997]

- Take derivative of master integrals w.r.t. $\hat{s}$:

$$\frac{\mathrm{d}}{\mathrm{d}\hat{s}} I(0,1,0) = \frac{1}{(\hat{s}-4)} I(-1,2,0) + \frac{2-\hat{s}}{(\hat{s}-4)\hat{s}} I(0,1,0) - \frac{2}{(\hat{s}-4)\hat{s}} I(0,2,-1),$$
$$\frac{\mathrm{d}}{\mathrm{d}\hat{s}} I(0,1,1) = \frac{1}{(\hat{s}-4)} I(-1,2,1) + \frac{2-\hat{s}}{(\hat{s}-4)\hat{s}} I(0,1,1) - \frac{2}{(\hat{s}-4)\hat{s}} I(0,2,0)$$

- Can reduce r.h.s. to master integrals again:

$$\frac{\mathrm{d}}{\mathrm{d}\hat{s}} I(0,1,0) = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}\hat{s}} I(0,1,1) = \frac{2-\epsilon\hat{s}}{(\hat{s}-4)\hat{s}} I(0,1,1) + \frac{-2+2\epsilon}{(\hat{s}-4)\hat{s}} I(0,1,0)$$

⇒ System of ordinary differential equations

Solution of the Differential Equations

$$\frac{\mathrm{d}}{\mathrm{d}\hat{s}} I(0,1,0) = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}\hat{s}} I(0,1,1) = \frac{2-\epsilon\hat{s}}{(\hat{s}-4)\hat{s}} I(0,1,1) + \frac{-2+2\epsilon}{(\hat{s}-4)\hat{s}} I(0,1,0)$$

- Solution can be obtained order-by-order as a Laurent expansion in ϵ after fixing boundaries:

$$I(0,1,0) = -\frac{1}{\epsilon} - 1 + \epsilon\left(-1 - \frac{\pi^2}{12}\right) + \mathcal{O}(\epsilon^2),$$

$$I(0,1,1) = \frac{1}{\epsilon} + 2 + \frac{1+x}{1-x}H(0,x) + \epsilon\Bigg(4 - \frac{\pi^2(1+3x)}{12(1-x)} + 2\frac{1+x}{1-x}H(0,x)$$
$$- 2\frac{1+x}{1-x}H(-1,x)H(0,x) + \frac{1+x}{2(1-x)}(H(0,x))^2 + 2\frac{1+x}{1-x}H(0,-1,x)\Bigg) + \mathcal{O}(\epsilon^2)$$

where

$$x = \frac{2+\sqrt{\hat{s}-4}\sqrt{\hat{s}}-\hat{s}}{2}$$

and $H(\ldots,x)$ are *multiple polylogarithms* [Remiddi, Vermaseren 1999]

Final Results

$$F_1(\hat{s}) = 1 + \frac{g_e^2}{(4\pi)^2}\Bigg[\frac{1}{\epsilon}\left(\frac{2\left(x^2+1\right)H(0,x)}{(x-1)(x+1)} - 2\right) + \frac{\left(x^2+1\right)H(0,x)^2}{(x-1)(x+1)}$$
$$+ \frac{\left(3x^2+2x+3\right)H(0,x)}{(x-1)(x+1)} - \frac{4\left(x^2+1\right)H(-1,x)H(0,x)}{(x-1)(x+1)} + \frac{4\left(x^2+1\right)H(0,-1,x)}{(x-1)(x+1)}$$
$$+ \frac{\pi^2\left(-x^2-1\right)}{3(x-1)(x+1)} - 4 + \mathcal{O}(\epsilon)\Bigg] + \mathcal{O}(g_e^4),$$

$$F_2(\hat{s}) = 0 + \frac{g_e^2}{(4\pi)^2}\left[\frac{4x}{(x-1)(x+1)}H(0,x) + \mathcal{O}(\epsilon)\right] + \mathcal{O}(g_e^4)$$

- Poles in ϵ cancel against poles from renormalization and infrared subtraction
- Can evaluate numerically:

$$F_1(\hat{s}=2.3) \approx 1 + \frac{g_e^2}{(4\pi)^2}\left[\frac{1}{\epsilon}2.522 - 3.349 + \mathcal{O}(\epsilon)\right] + \mathcal{O}(g_e^4),$$
$$F_2(\hat{s}=2.3) \approx 0 + \frac{g_e^2}{(4\pi)^2}\left[3.482 + \mathcal{O}(\epsilon)\right] + \mathcal{O}(g_e^4)$$

$\Rightarrow$ Ready to be plugged into Monte Carlo codes like `McMule`

Summary of How to Calculate Multiloop Amplitudes

1. Generate all diagrams
   - Almost trivial with automated codes
2. Rewrite amplitude in terms of scalar Feynman integrals
   - Usually straightforward and cheap, but not always
3. Reduce scalar integrals to master integrals
   - No principle problem, but can be computationally expensive or even prohibitive
4. Solve master integrals
   - Many different methods
   - Numerical methods in principle always applicable, but can be computationally expensive or even prohibitive
   - Analytic solutions of state-of-the-art problems usually not straightforward and might result in new function classes that are on the forefront of math research

$\Rightarrow$ Automated and efficient tools necessary for higher orders