Electronic noise in the time domain

LTP seminar

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- Introduction / Motivation
- RC-filter with noise as a 'Stochastic Process'
  - equation of motion
  - Noise
  - Fokker-Planck equation
- application : pixel with threshold
  - time evolution, a comment on S-curves
  - rate of noise hits
  - time resolution

electronic noise

- all outputs / nodes of electronic exhibit fluctuations = noise
- fudamental and unavoidable, e.g.
  - Johnson-Nyquist noise in a resistor
  - shot-noise of currents
- when signals are "small", noise must be taken into consideration

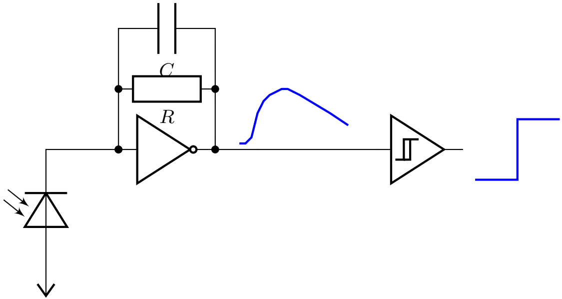
textbook receipe for noise analysis

- introduce (fictituous) noise sources in addition to existing components
- analyze the circuit in the frequency domain
  - determines the amplitude, $v(f)$, of your output assuming that everything happens at a single fixed frequency, f
  - sum up all frequencies and noise sources incoherently

$$v^2 = \int v(f)^2 df$$

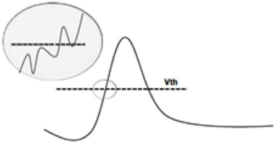
- repeated measurements of the output voltage:
  $\longrightarrow$ gaussian with width $\sqrt{v^2}$
- all we care about in practice, but actually in some cases not enough

a typical pixel detector



- charge- (or current-) sensitive amplifier connected to sensor
  - fluctuates around “zero” ((continuous) reset)
  - particle through sensor creates signal excursion
- a comparator produces a logic signal when the amplitude crosses the threshold
- it then stays insensitive for some time !

- the time of the threshold crossing is important for us
  - LHC-like : 25 ns bunch-crossing clock: only hits registered during a particular (triggered) bunch crossing are read out
  - future detectors will make precise measurements of the arrival time
- electronic noise affects the amplitude of the charge we measure . . .
- . . . but it also affects the threshold-crossing time



Sadrozinsky et al, Rep. Prog. Phys. 81(2018) 026101

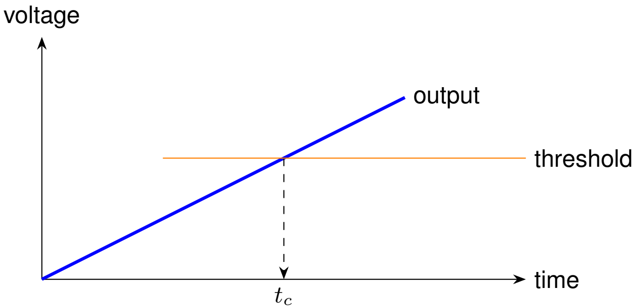
- it is apparently obvious that from this figure one can conclude

$$\sigma_t = \frac{\sigma_v}{\frac{dV}{dt}}$$

- $\frac{dV}{dt}$ output slope of the amplifier
- σ_v voltage noise of the amplifier output
- σ_t time resolution

why is $\sigma_t = \sigma_v/(\frac{dV}{dt})$? (and why wasn't it obvious to me?)

- no noise: $V(t) = \frac{dV}{dt} t \quad \Rightarrow \quad t_c$



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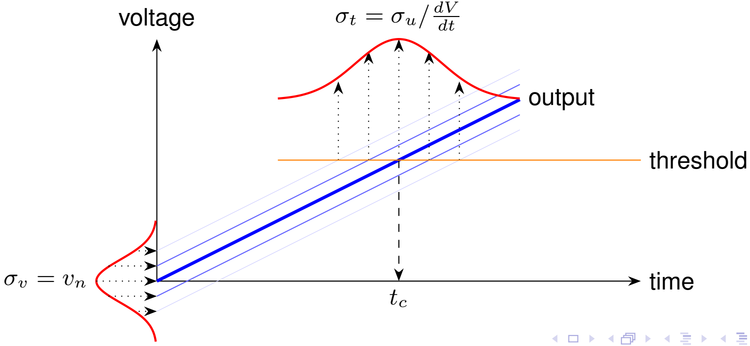
- no noise: $V(t) = \frac{dV}{dt}t \quad \Rightarrow \quad t_c$
- $V \rightarrow V + \delta v \quad \Rightarrow \quad t_c \rightarrow t_c - \delta V/(\frac{dV}{dt})$
  noisy voltage σ_v superimposed $\rightarrow \sigma_t = \sigma_v/(\frac{dV}{dt})$



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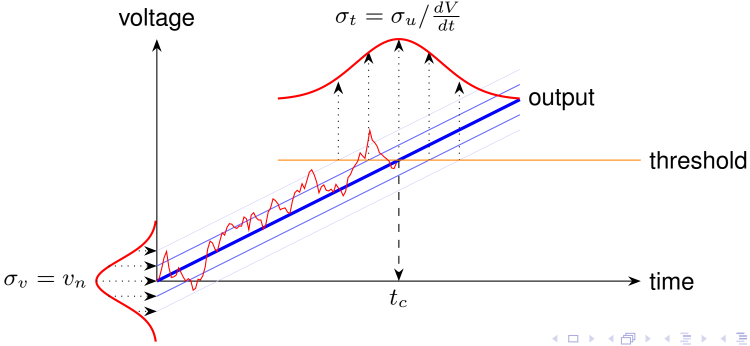
- really true? Assumes 'static' noise from $t = 0$ to $\sim t_c$
  A trajectory that reaches the threshold at t_1 may have crossed already before. In this case the later crossing will be ignored



How to take this into account?

- traditional noise analysis tells us the amplitude of noise fluctutations
- we need to know when the output crosses a threshold for the first time
- well know in the context of stocastic processes 'first-passage-time'

physisicists approach to stochastic processes

- think about and ensemble of systems, each with a random trajectory $v(t)$
- described by distribution

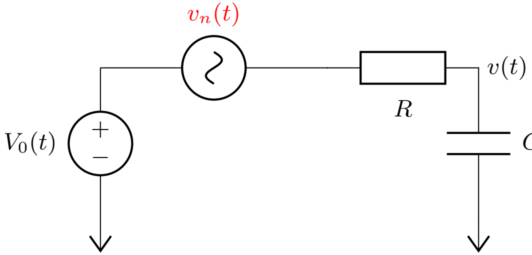
$$P(v,t)dv$$

  number of trajectories between v and $v + dv$ at time t
- time evolution of $P(v,t)$ according to equation of motion + noise
- the requirement of not having crossed the threshold before some time t becomes a bondary condition

$$P(v,t') = 0 \quad \text{for} \qquad t' < t, \quad v > \text{threshold}$$

- $P(v,t)$ keeps track of all trajectories that haven't crossed before t

Example : RC-filter with noise



equation of motion

$$\dot{v}(t) = -\frac{1}{RC}v(t) + \frac{1}{RC}V_0(t) + \frac{1}{RC}v_n(t)$$

$v_n(t)$ is the noise

- stochastic, every value is a random number
  not your regular function
- ensemble = all possible realiziations of $v_n(t)$ and hence $v(t)$
- distribution $P(v,t)$ describes ensemble

noise “function”

- noise function

$$v_n(t) = a \cdot \xi(t)$$

- with following properties ($\langle \ldots \rangle$ = ensemble averages)

$$\langle \xi(t) \rangle = 0$$
$$\langle \xi(t_1)\xi(t_2) \rangle = 2\delta(t_1 - t_2)$$

- noise values at different times are uncorrelated (white noise)
- the factor 2 is just one possible normalization
- note that $\langle v_n^2(t) \rangle = 2a^2\delta(0)$ as such is not a meaningful number
  unfiltered noise voltages are not observables

Connection with frequency domain noise

- this is nothing new, same as in the frequency domain
- unfiltered white noise is infinite when integrated over all frequencies
- in this case

$$v_n(f_1, f_2) \equiv \int_{f_1}^{f_2} a \cdot \hat{\xi}(f) df$$
$$\longrightarrow \quad \langle v_n^2(f_1, f_2) \rangle = 4a^2 \cdot (f_2 - f_1)$$

- a in $v_n(t) = a\xi(t)$ is 2x the traditional 'noise per $\sqrt{Hz}$'

example : thermal resistor noise (Nyquist-Johnson)

$$v_n = \sqrt{4k_B T R \Delta f} \quad \Rightarrow \quad a = \sqrt{k_B T R}$$

Evolution of $P(v,t)$

- the formalism below was developed for diffusion. To conform with existing literature, call voltages x from now on.
- equation of motion('Langevin equation') usually written as

$$\dot{x} = -A(x,t) + B(x,t) \cdot \xi(t)$$

- RC-filter (without external voltage $V_0(t)$ for now)

$$\dot{x}(t) = -\frac{1}{RC} x(t) + \frac{1}{RC} x_n(t)$$

$\Rightarrow$

  - $A(x,t) = -\frac{1}{RC}$
  - $B(x,t) = B = a/RC$

Evolution of $P(x,t)$

- expand $P(x,t)$ and $x(t)$ in t to get $P(x,t+\Delta)$ from $P(x,t)$ (Kramers-Moyal expansion)
- for $P(x,t)$ with constant B

$$\frac{\partial}{\partial t}P(x,t) = \underbrace{-\frac{\partial}{\partial x}\left(A(x,t)P(x,t)\right)}_{\text{drift}} + \underbrace{B^2\frac{\partial^2}{\partial x^2}P(x,t)}_{\text{diffusion}}$$

'Fokker-Planck equation'

- in this case

$$\frac{\partial}{\partial t}P(x,t) = \frac{\partial}{\partial x}\left(\frac{x}{RC}P(x,t)\right) + \frac{a^2}{(RC)^2}\frac{\partial^2}{\partial x^2}P(x,t)$$

Stationary solution

- leave the RC-filter alone for a long time with no boundary condition
- $\longrightarrow$ becomes stationary: $\frac{\partial}{\partial t} P(x,t) = 0, \quad P(x,t) = P(x)$

$$0 = \frac{\partial}{\partial x}(xP(x)) + \frac{a^2}{RC}\frac{\partial^2}{\partial x^2}P(x)$$

- the only equation I could solve analytically

$$P(x) = e^{-\frac{x^2}{2\sigma^2}} \quad \text{with} \quad \sigma^2 = \frac{a^2}{RC}$$

- finite (unlike v_n^2) with a gaussian distribution
- if the resistor is the only noise source $a = \sqrt{k_B T R}$

$$\sigma^2 = \frac{a^2}{RC} = \frac{k_B T R}{RC} = \frac{k_B T}{C}$$

  'kTC'-noise
- in general, will replace a^2/RC by σ^2 now

Numerical studies

- Fokker-Planck equation with σ and $\tau = RC$:

$$\tau \frac{\partial}{\partial t} P(x,t) = \frac{\partial}{\partial x}(xP(x,t)) + \sigma^2 \frac{\partial^2}{\partial x^2} P(x,t)$$

- or with normalized coordinates x/σ and t/τ

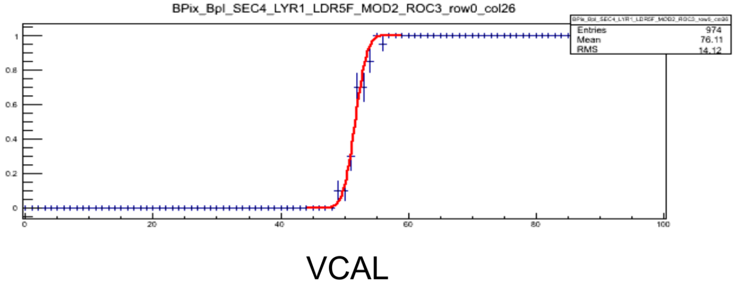
$$\frac{\partial}{\partial t} P(x,t) = \frac{\partial}{\partial x}(xP(x,t)) + \frac{\partial^2}{\partial x^2} P(x,t)$$

very convenient, covers all situations

- numerical solutions
  - starting from some $P(x, t=0)$, e.g. a gaussian
  - $P(x, t+\Delta t) = P(x,t) + \Delta t \times \left[\frac{\partial}{\partial x}(xP(x,t)) + \frac{\partial^2}{\partial x^2} P(x,t)\right]$
  - implement threshold at x_c by forcing $P(x \geq x_c, t) = 0$

example: threshold scans ("S-curve")

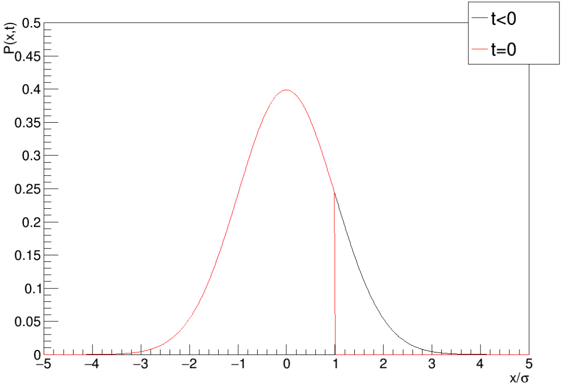
- one of our favorite calibration measurements
- inject a test signals of known, variable pulse-height
- count the fraction of pulses for which the comparator fires



- fit with an gaussian upper tail (aka error function) measures threshold x_c and noise (width)
- equivalent (illustration only): lower the threshold from ∞ to x_C

Time evolution with fixed threshold at $1\ \sigma$

- no threshold for $t < 0$: start with a gaussian centered at 0
- threshold turns on instantaneously at $t = 0$
- voltages above threshold fire the comparator, curves show 'survivors'



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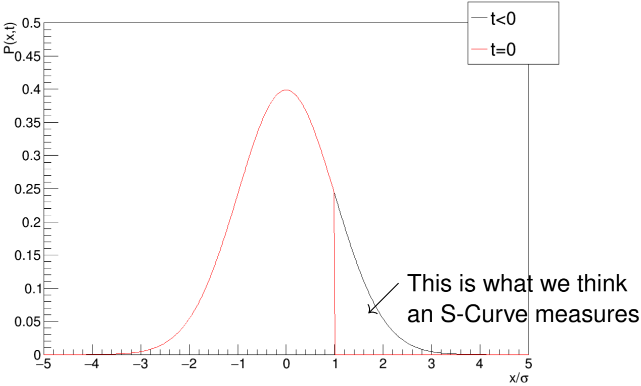
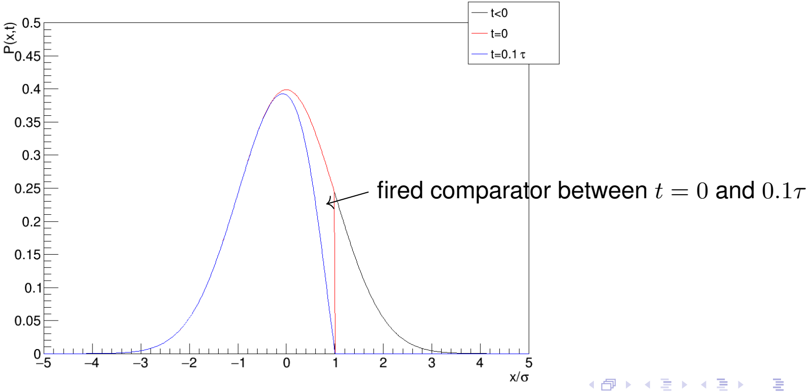


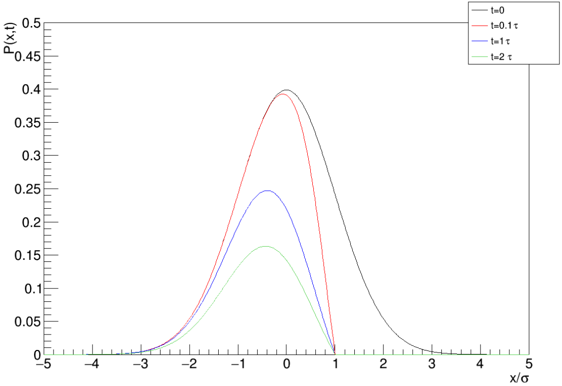
Time evolution with fixed threshold at $1\ \sigma$

- for $t > 0$ noise/diffusion :
  - some systems close to threshold go above
  - the region just below threshold is depleted
- additional noise hits make thresholds look lower than they really are !
- size depends on effective time spent near threshold (later)



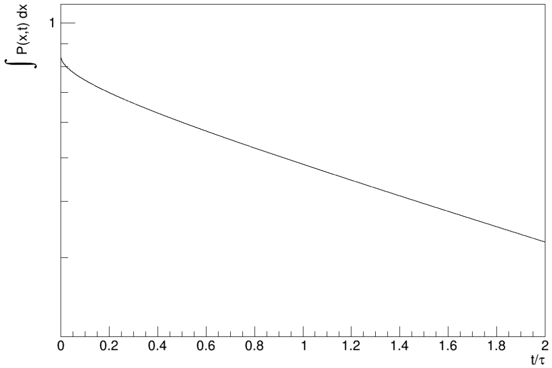
Time evolution with fixed threshold at $1\ \sigma$

- for long waiting times: total population continues to decrease (in the real pixel detector they would come back after some dead-time)

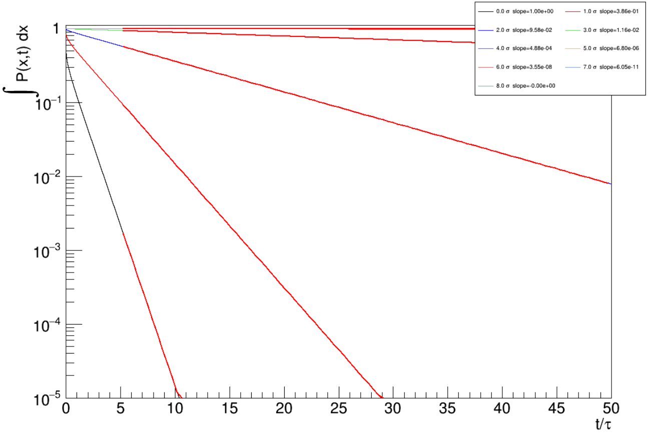


Survival probability vs time with fixed threshold at $1\ \sigma$

- instant drop from 1 to $1 - \Phi(\text{threshold})$ (=S-curve expectation)
- quick diffusion from threshold region (=time dependent correction)
- exponential decay later $\rightarrow$ log slope = noise hit rate

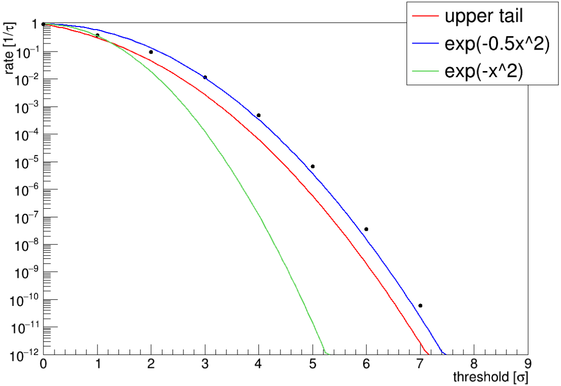


Survival probability with different thresholds $(0 \ldots 8\sigma$



- asymptotically exponential decay behaviour
- time constant = probability for a pixel to fire due to noise during τ

Noise hit rate vs threshold



- noise hit rate = $\frac{1}{\tau} e^{-\frac{x_c^2}{2\sigma^2}}$
- fun fact : with $E_c = \frac{1}{2}CV^2$ and $\sigma^2 = k_B T/C$ this becomes $e^{-E_c/(k_B T)}$ (Arrhenius)

Timing resolution

- dropped external voltage V_0 some slides ago , re-introduce it now as x_0
- inject pulse x_0 into the RC-Filter
- equation of motion

$$\dot{x} = -\frac{1}{\tau}(x - x_0) + \frac{\sigma^2}{\tau}\xi(t)$$

- Fokker-Planck becomes

$$\tau\frac{\partial}{\partial t}P(x,t) = \frac{\partial}{\partial x}((x-x_0)P(x,t)) + \sigma^2\frac{\partial^2}{\partial x^2}P(x,t)$$

- or, in normalized coordinates

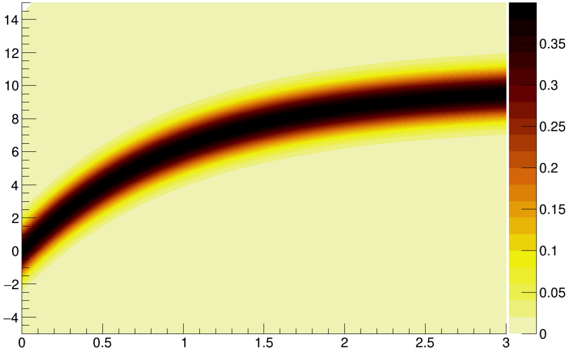
$$\frac{\partial}{\partial t}P(x,t) = \frac{\partial}{\partial x}((x-x_0)P(x,t)) + \frac{\partial^2}{\partial x^2}P(x,t)$$

RC-filter with signal injection, no threshold

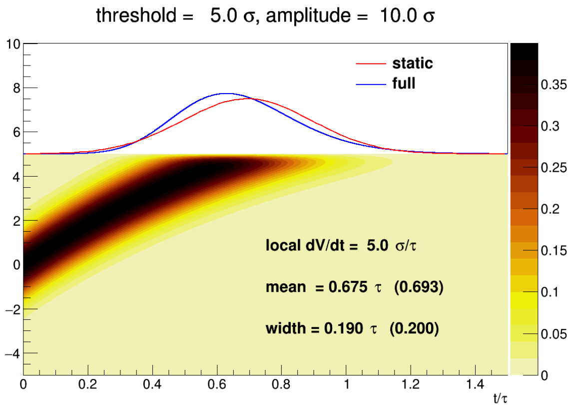
- starting from the stationary gaussian
- follows the expected

$$1 - e^{-t/\tau}$$

- use amplitude x_0 to select slope dV/dt

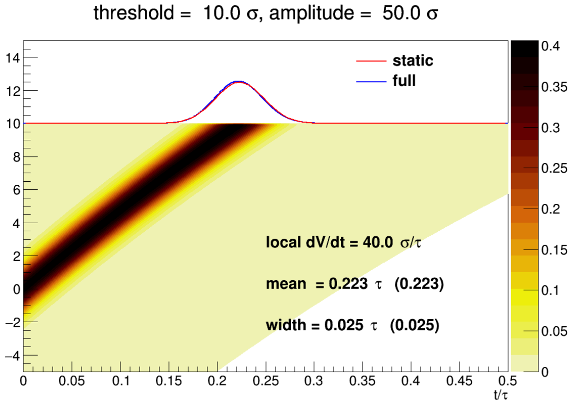


same signal with a threshold at 5 σ



- small effects on average crossing time and resolution

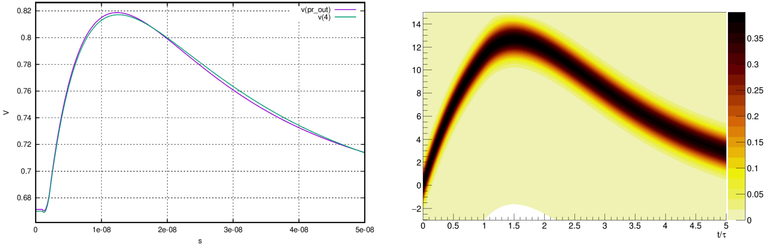
a more realistic situation



- almost no effect for realistic conditions
- e.g. $\sigma = 50\ e$, Signal $= 2500\ e$, threshold$= 500\ e$

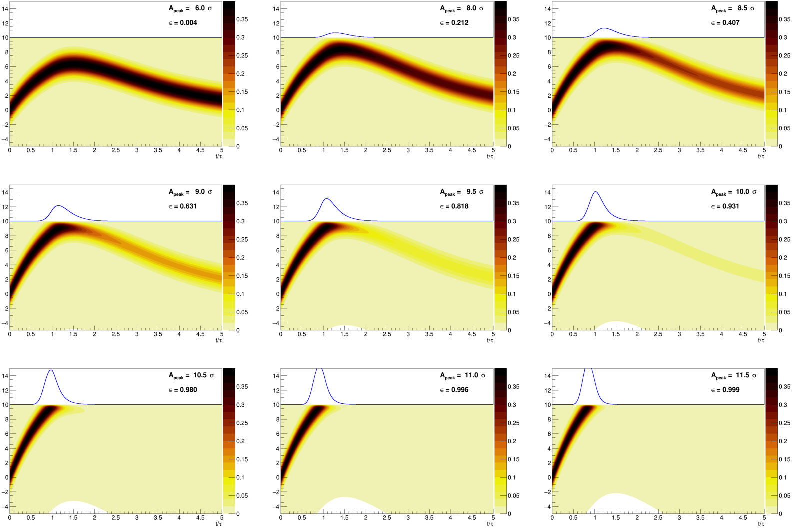
back to S-curves

- in contrast to the timing measurement, the peak amplitude can be close to the threshold
  $\Rightarrow$ spend significant time where noise evolution can have an effect
- the usual S-curve interpretation is for a measurement with no duration
- obviously an RC-filter does not have the peaking behaviour of a preamp emulate it by pulling the injected charge back out again after $1 \times \tau$

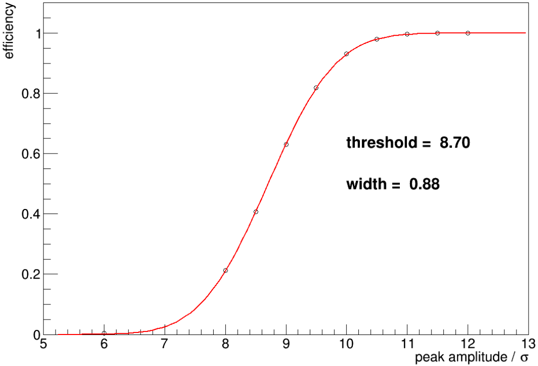


- more realistic analysis (probably) possible but more complicated

threshold = 10σ, pulse height = $6\ldots12\sigma$



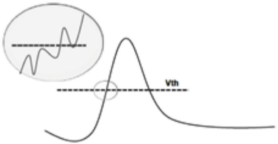
total number of hits vs amplitude



- apparent threshold lower than 10, as expected (real signals probably do more or less the same)
- width seems to be underestimated, too
- however, this is only a crude approximation

summary and conclusions

- methodology of stochastic processes can describe electronic noise
- straightforward to obtain results numerically for an RC filter
- reproduces $\exp(-0.5(\mathrm{threshold}/\mathrm{noise})^2$ law for noise hit rates
- negligible impact on time-resolution
  - frozen-noise approach appropriate if output runs through noise in $\Delta t \ll \tau$
  - this is of course the case in realistic situations
  - so the conclusion $\sigma_t = \sigma_v / \frac{dV}{dt}$ is correct,



  . . . just the picture is wrong

- S-curves are potentially sensitive to non-trivial effects
  quantitative study requires a more realistic model than just RC