The Vortex Effect in High-Intensity Cyclotrons and Isochronous FFA's

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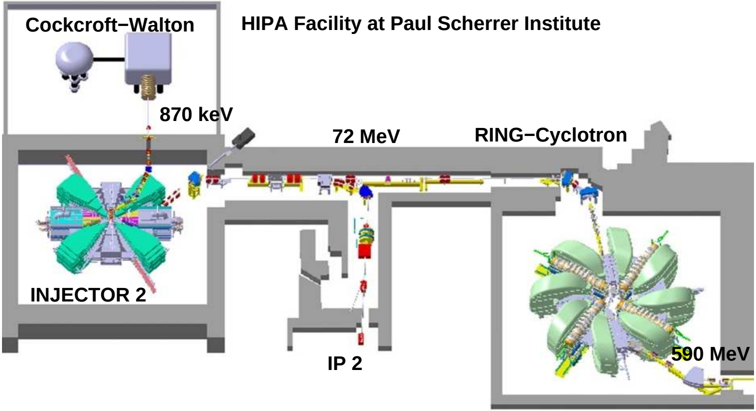
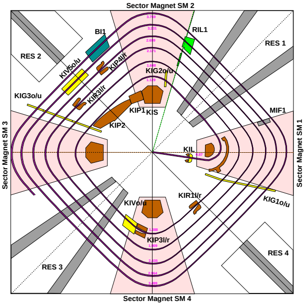
Outline

1. PSI / Injector 2
2. Visualizing the Vortex Effect
3. Matching
4. Simple linear (symplectic) model
5. Conditions for space charge induced "longitudinal focusing".
6. RF considerations.
7. Model versus OPAL [5, 6] simulations.
8. Summary.

PSI High Intensity Proton Acc. (HIPA)



- Axial injection with $E = 870\,\mathrm{keV}$.
- Considerable number of collimators (horz. + vert.)
- High accelerating voltages ($72\,\mathrm{MeV}$ after 80 turns).
- Max. Current so far $I \leq 2.7\,\mathrm{mA}$.
- 3rd harmonic resonators (formerly known as flat top) used for acceleration.
- Max. Current so far $I \leq 2.7\,\mathrm{mA}$.
- Various proofs of Vortex effect: bunch shape measurements [2].



Visualization I

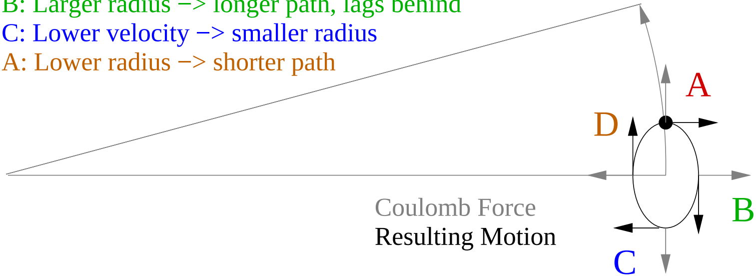
Cause and (orthogonal) Effect of Space Charge in Vortex Motion

A: Higher velocity –> larger radius

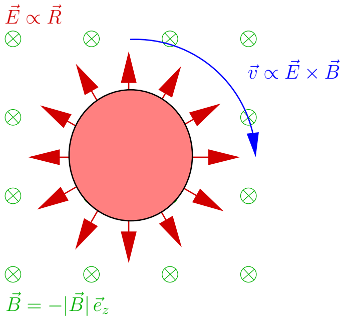
B: Larger radius –> longer path, lags behind

C: Lower velocity –> smaller radius

A: Lower radius –> shorter path



Visualization II ($E \times B$-Drift)



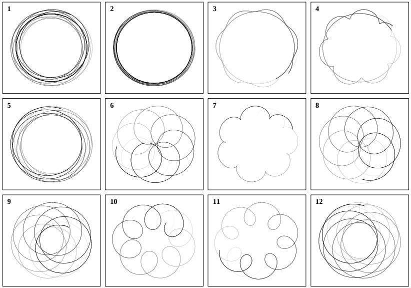
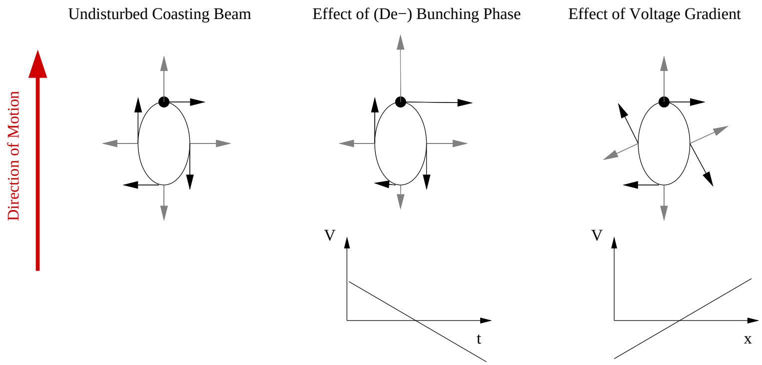


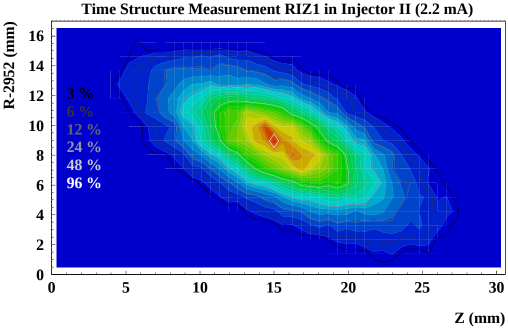
Figure: Left: Scetch of the principle. Right: More "realistic" orbits for various starting conditions.

Possible (linear) RF-Effekts



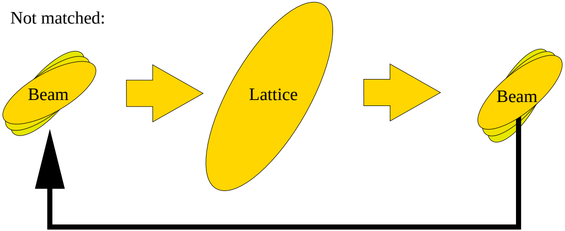
Vortex Effect in Injector II:

- Two bunchers in front of cyclotron provide high space charge density.
- PSI Injector II with 2.4 mA without flattop and low losses.
- Time Structure Measurement shows round bunches.

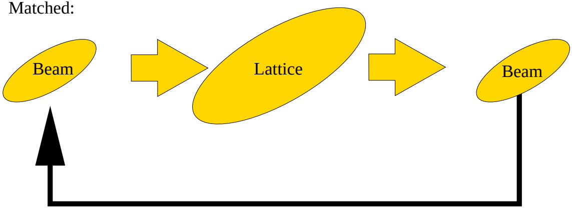


Coasting beam: Matching without Space Charge

Not matched:



Matched:



Coasting beam: Matching without Space Charge

- Hill-Type equation of motion: $\psi' = \mathbf{F}(s)\,\psi$.
- One-turn transfer matrix **M** defined by lattice (Bend::Drift:Bend::Drift...).
  M is symplectic: $\mathbf{M}\,\mathbf{J}\,\mathbf{M}^T = \mathbf{J}$.
- $\mathbf{M} = \exp(\mathbf{F}\,s)$ is exponential of Hamiltonian matrix **F** [3].
- CFC (const foc. channel): $\psi' = \mathbf{F}\,\psi$ with $\mathbf{F} = \mathrm{const}$ substitutes "real" position-dependent $\mathbf{F}(s)$.
- Beam-matrix $\boldsymbol{\Sigma} = \langle\psi\psi^T\rangle$ changed by lattice: $\boldsymbol{\Sigma}_{n+1} = \mathbf{M}\,\boldsymbol{\Sigma}_n\,\mathbf{M}^T$.
- Use symplectic unit matrix **J** and define $\mathbf{S}_n \equiv \boldsymbol{\Sigma}_n\,\mathbf{J}$:
  $\mathbf{S}_{n+1} = \mathbf{M}\,\mathbf{S}_n\,\mathbf{M}^{-1}$: Symplectic transport is a similiarity transformation.
- Matched beam $\mathbf{S}_{n+1} = \mathbf{S}_n = \tilde{\mathbf{S}} \Rightarrow \tilde{\mathbf{S}}$ and **M** commute.
- $\Rightarrow \tilde{\mathbf{S}}$ and **M** share a system of **Eigenvectors**.
- $\Rightarrow \tilde{\mathbf{S}} = \tilde{\mathbf{S}}(\mathbf{M}, \varepsilon_i)$ (Emittances ε_i are Eigenvalues of **S**).

Matching with Space Charge

- **F** depends on lattice **and** beam size (self-interaction by space-charge).
- $\Rightarrow$ Transfer Matrix $\mathbf{M} = \mathbf{M}(\mathbf{F}, \mathbf{S})$.
- Matching becomes a "circular problem":
  We need to know $\tilde{\mathbf{S}}$ to determine $\mathbf{M}$.
  We need to know $\mathbf{M}$ to determine $\tilde{\mathbf{S}}$.
- Solution can (only?) be obtained iteratively:
  1. Knowing the emittances ε_i, guess $\mathbf{S}_0 = \Sigma_0\,\mathbf{J}$.
  2. Compute $\mathbf{M}_0(\mathbf{F}, \mathbf{S}_0)$.
  3. Compute $\mathbf{S}_1(\mathbf{M}_0, \varepsilon_i)$.
  4. Compute $\mathbf{M}_1(\mathbf{F}, \mathbf{S}_1)$
  5. Compute $\mathbf{S}_2(\mathbf{M}_1, \varepsilon_i)$ etc.
  6. If space charge is small enough, the sequence converges:
  7. Then $\mathbf{S}_{n+1} \approx \mathbf{S}_n$.

(see Ref. [10, 11, 12, 13]).

Computation I

Single particle dynamics (axial motion decoupled, treated separately):

- Pathlength along orbit s
- Radial coordinate $x = r(\theta) - r_0$ and $x' = \frac{dx}{ds}$.
- Longitudinal position $z = r_0\,(\theta - \theta_0)$.
- Momentum deviation $\delta = \frac{\Delta p}{p_0}$.
- Put in state vector $\psi = (x, x', z, \delta)^T$ in local co-moving curvilinear coordinates.
- Define $h = 1/r_0$ as curvature of orbit.
- Use CFC (const foc. channel) approximation: $\psi' = \mathbf{F}\,\psi$
- 1-Turn Transfer Matrix $\mathbf{M}$ defined by $\psi_{n+1} = \mathbf{M}\,\psi_n$

Computation II

The Hamiltonian matrix **F** is

$$\mathbf{F} = \begin{pmatrix} . & 1 & . & . \\ -k_x + K_x & . & K_{grad} & h \\ -h & . & . & \frac{1}{\gamma^2} \\ K_{grad} & . & K_z\,\gamma^2 + K_{rf} & . \end{pmatrix}$$

Focusing terms, space charge terms (defoc.), dispersive coupling $h = 1/r_0$, rf voltage grad/rf B-field, drift terms in black.

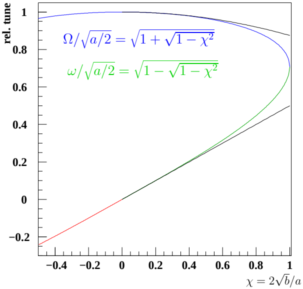
- $K_{rf} > 0$: "Debunching" phase.
- $K_{rf} < 0$: "Bunching" phase.

The motion is stable, if the Eigenvalues of **F** are purely imaginary. (and the *Eigenfrequencies* are real).

Compute the *Eigenfrequencies* Ω and ω of **F** [4]:

Technical Details:

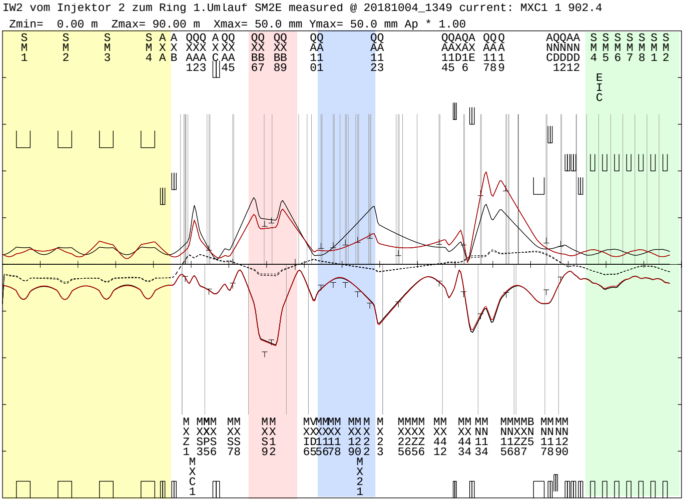
$$\begin{aligned}
a &\equiv -\mathrm{Tr}(\mathbf{F}^2)/2 = \Omega^2 + \omega^2 \\
b &\equiv \mathrm{Tr}(\mathbf{F}^2)^2/8 - \mathrm{Tr}(\mathbf{F}^4)/4 = \Omega^2\,\omega^2 \\
\chi^2 &= 4\,b/a^2 \\
\Omega &= \sqrt{\frac{a}{2} + \sqrt{\frac{a^2}{4} - b}} = \sqrt{a/2}\,\sqrt{1 + \sqrt{1 - \chi^2}} \\
\omega &= \sqrt{\frac{a}{2} - \sqrt{\frac{a^2}{4} - b}} = \sqrt{a/2}\,\sqrt{1 - \sqrt{1 - \chi^2}} \\
\omega &\in \mathbb{R} \Rightarrow b \geq 0 \\
b\,\gamma^2 &= K_x\,(K_{rf} + K_z\,\gamma^2) - K_{grad}^2 > 0
\end{aligned}$$



Ω is real, iff $b \leq a^2/4$.
ω is real, iff $0 \leq b \leq a^2/4$.

(Field error ε has been discussed before [10, 11], here neglected $\varepsilon \approx 0$).

Matched Beam Condition



Horizontal beam parameters: From matched beam. Vertical parameters σ_y,σ_{py} and $\mathrm{cov}(3,4)$ fitted to measurement.

Remarks on Decoupling

- Teng and Edwards suggested decoupling by *symplectic rotations*. (L.C. Teng: Concerning n-Dimensional Coupled Motions; NAL-Report FN-229 (1971).
  D.A. Edwards and L.C. Teng; (Cont. to PAC '73) IEEE Trans. Nucl. Sci. Vol 20, Issue 3, (1973), 885-888.)
- However, the matrix **F** describing the problem at hand, can not be decoupled by orthogonal transformations.
- A generalization that includes *symplectic boosts* has been formulated.
- A proper symplectic analysis based on Hamiltonian notions can be done in a straightforward way by the use of the (real) Clifford algebra $Cl(3,1)$. This is just a conveniant parameterization of the 4×4 matrix.
- For details have a look at:
  C. Baumgarten "A Geometrical Method of Decoupling" PRSTAB **15** (2012), 124001 [12].

Focusing Condition

Ignore RF Terms, then Focusing requires

$$\begin{aligned} b &= K_z\,(K_x + h^2\,\gamma^2 - k_x) > 0 \\ \Rightarrow & \quad K_x > k_x - h^2\,\gamma^2 \end{aligned}$$

The radial focusing force k_x is given by:

$$k_x = h^2\,(1+n) = h^2\,\left(1 + \frac{r}{B}\,\frac{dB}{dr}\right)$$

The isochronous field plus a typically small field error ε:

$$B(r) = B_0\,\gamma\,(1+\varepsilon) = B_0\,\frac{1+\varepsilon}{\sqrt{1-(r/a)^2}}\,,$$

This gives

$$k_x = h^2\,\gamma^2 + \frac{1}{r}\,\frac{d\varepsilon}{dr}\,.$$

Focusing condition:

$$K_x > \frac{1}{r}\,\frac{d\varepsilon}{dr}$$

Focusing Condition II

$\omega_0 = N_h \, \omega_{rf}$ is nominal orbital frequency, N_h is the harmonic number, ω real orbital frequency and ϕ is phase. Then:

$$\varepsilon \approx 1 - \frac{\omega_0}{\omega} = -\frac{1}{2\,\pi\,N_h}\,\frac{d\phi}{dE}\,\frac{dE}{dn}\,.$$

With $\frac{dE}{dn} = V\,\cos\phi$ and $\frac{dE}{dr} = m\,c^2\,\gamma^3\,r/a^2$ this gives:

$$\frac{1}{r}\,\frac{d\varepsilon}{dr} = \frac{d\varepsilon}{dE}\,\frac{dE}{dr} \approx -\frac{V\,m\,c^2\,\gamma^3}{2\,\pi\,N_h\,a^2}\left(\frac{d^2\phi}{dE^2}\,\cos\phi - \left(\frac{d\phi}{dE}\right)^2\,\sin\phi\right)\,.$$

Focusing condition ($\sin\phi \approx 0$, factors approx. const):

$$K_x > -\mathrm{const}\,\frac{d^2\phi}{dE^2}\,\cos\phi\,.$$

⇒ Longitudinal focusing depends on phase curve!

How does this relate to transition gamma?

Slip factor η (used by synchrotron people;-):

$$\eta = \frac{p}{T}\,\frac{dT}{dp} = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\,,$$

where γ_t is the "transition gamma". In order to relate γ_t to the parameters ε and ϕ, we derive an expression for γ_t from the definition

$$\gamma_t^2 \equiv \frac{r}{p}\,\frac{dp}{dr}\,.$$

With $B(r) = B_0\,\frac{1+\varepsilon(r)}{\sqrt{1-\frac{r^2}{a^2}}}$ and $p = r\,q\,B(r)$ this gives:

$$\gamma_t^2 = \gamma^2 + r\,\frac{d\varepsilon}{dr}\,.$$

and so in first order:

$$\eta = -r\,\frac{d\varepsilon}{dr}\,.$$

How does this relate to transition gamma?

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}\,,$$

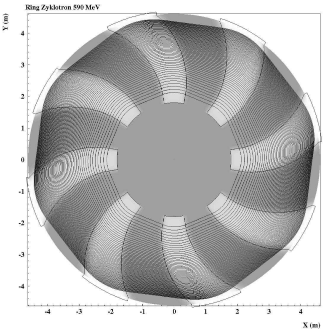
Above transition ($\gamma > \gamma_t$) one has $\eta > 0$, below transition, $\eta < 0$. Focusing condition is expressed with "slip factor" η:

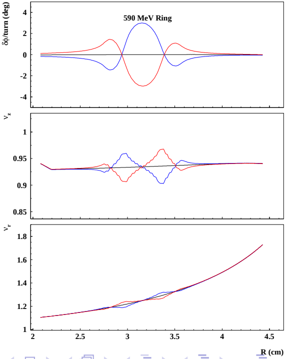
$$K_x > -\frac{\eta}{r^2}$$

Above transition ($\eta > 0$) we have focusing (stability?), below transition we have a threshold.

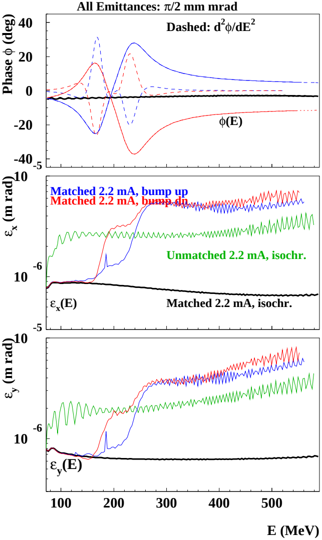
Example: Ring Cyclotron

1. Create "ideal" ring machine: Geometry similar to ring machine.
2. Compare 3 cases: perfect isochronism, positive, negative field bumps (see figure).
3. Compute matched σ-matrix for given emittances and beam current [10, 12].
4. Create random Gaussian distribution ($N = 10^5$) from σ-matrix [13].

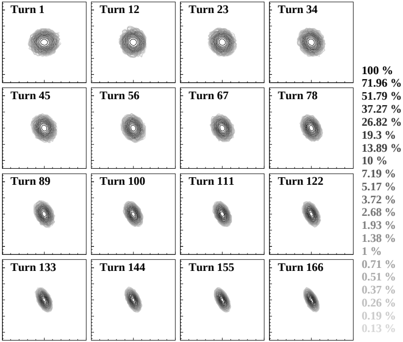




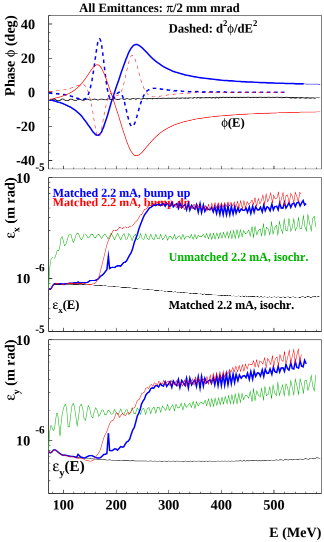
Matched Round Beam in Ideal Cyclotron I



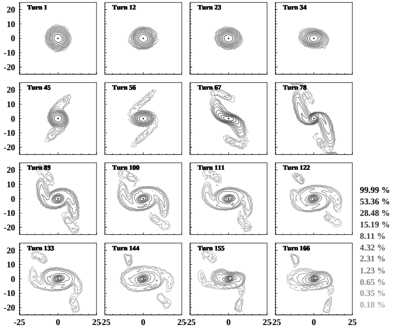
Matched beam, flat phase (black):



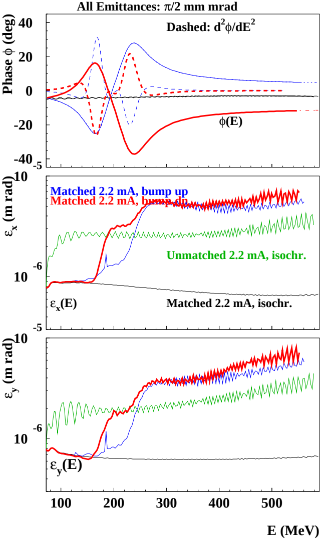
Matched Round Beam in Ideal Cyclotron II



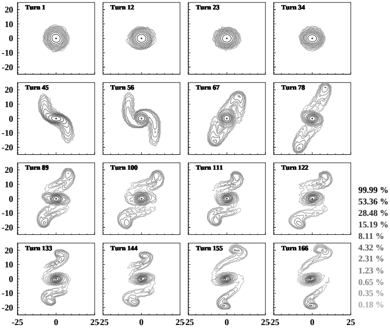
Matched beam, blue phase:



Matched Round Beam in Ideal Cyclotron III



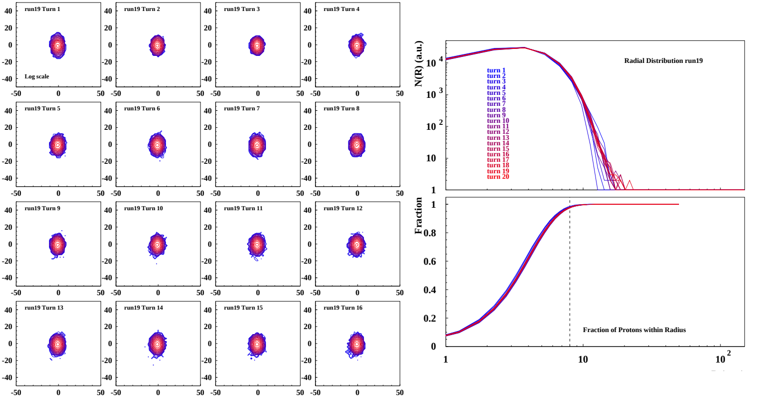
Matched beam, red phase:



Questions and Answers

- Is vortex motion stable? No, it is metastable!
  If the bunch has expanded spatially, the space charge force and the space charge tune decrease and the bunch does not "shrink" back to original size. Distortions of a matched beam induce, due to the non-linearity of the space charge force, bunch deformation followed by filamentation and emittance increase.
- Can we neglect RF? What about injection and the first turns?
- Is the adiabatic approximation valid, i.e. $\Delta E/E \ll 1$?.
  Joho's N^3-law [7] suggests to use highest possible voltage!
  In Injector 2 we have $\Delta E \approx E$ at injection!
- What, if the phase is not zero? Are there (de-) bunching effects?
- Do strong RF voltage gradients disturb the vortex effect?

Example 2: Central Region of Injector 2

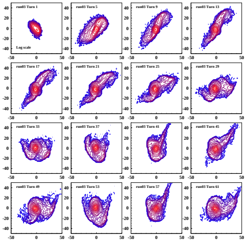


Results for a matched coasting beam of 1 MeV in Injector 2. OPAL [5, 6]: Matched gaussian *coasting* beam is stable.

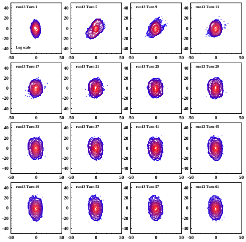
Adiabatic Approximation I

Using OPAL simulations to test the simplified linear model:
Fast acceleration versus adiabatic approximation.

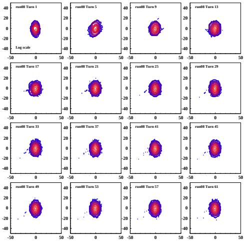
$V = 100\,\%$



$V = 50\,\%$

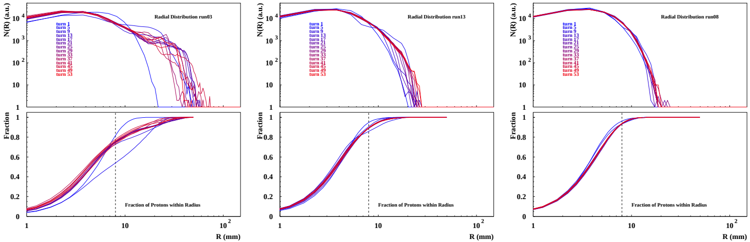


$V = 25\,\%$



Adiabatic Approximation II

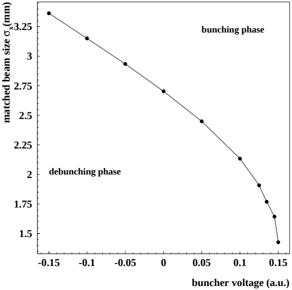
$V = 100\,\%$ $V = 50\,\%$ $V = 25\,\%$



If adiabatic condition is not fulfilled due to high accelerating voltage, the beam halo increases. The beam must be cleaned up by beam collimation [9].

(De-)Bunching by RF Voltage I

- RF-phase $\phi > 0$: Bunch lags behind. $V_{head} > V_{tail}$ ("debunching")
- RF-phase $\phi < 0$: RF lags behind. $V_{head} < V_{tail}$ ("bunching")

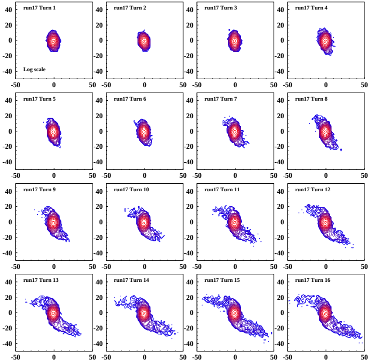


- Matched beam size increases with debunching voltage.
- Matched beam size decrease with bunching voltage.
- Compromize: Accelerate at phase $\phi = 0$.

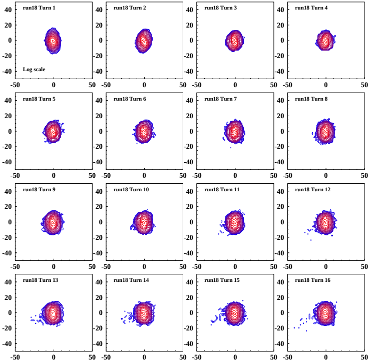
(De-)Bunching by RF Voltage II

- RF-phase $\phi = -90^\circ$: No acceleration, "bunching" phase.
- RF-phase $\phi = 90^\circ$: No acceleration, "debunching" phase.

$V = 10\,\%$, $\phi = -90^{\circ}$, bunching



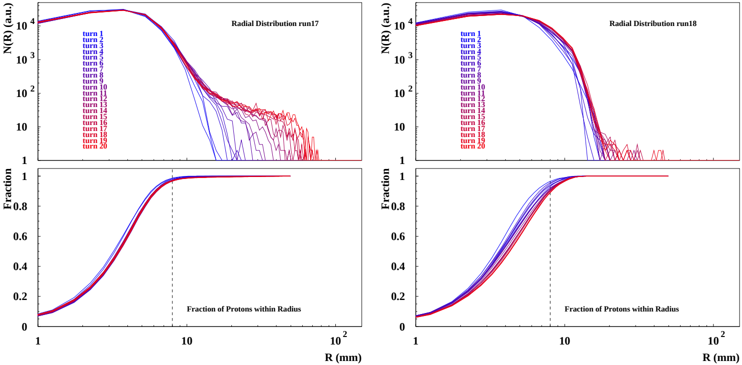
$V = 10\,\%$, $\phi = 90^{\circ}$, debunching



(De-)Bunching by RF Voltage III

$V = 10\,\%$, $\phi = -90^\circ$, "bunching".

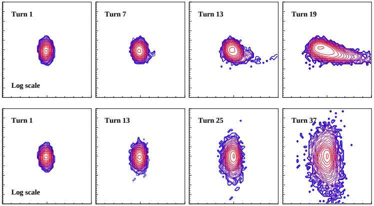
$V = 10\,\%$, $\phi = 90^\circ$, "debunching".

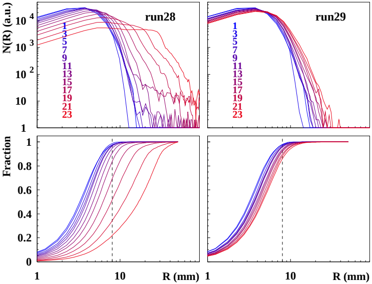


- "Bunching" RF-phase: More halo, but more compact core.
- "Debunching" RF-phase: Few halo, but larger core.
- Best compromize: RF-phase close to zero.

Strong RF Voltage Gradients

- $\langle R \rangle \approx R_0$.
- $V(R) \propto (R - R_0)$, RF-phase $\phi = 0°$: Positive $V' > 0$.
- $V(R) \propto (R - R_0)$, RF-phase $\phi = 180°$: Negative $V' < 0$.





- Positive $V' > 0$: Bunch deforms quickly.
- Negative $V' < 0$: Bunch size increases continuously.
- Best: $V' \approx 0$.

Summary

- Linear model allows to compute matching conditions.
- Vortex motion requires good isochronism.
- Adiabatic approximation not valid in center (of Injector 2).
  - $\Rightarrow$ bad conditions for smooth acc. of matched beam.
  - $\Rightarrow$ halo formation difficult to avoid.
  - $\Rightarrow$ beam collimation required.
  - $\Rightarrow$ high energy gain required (space for collimators).
  - $\Rightarrow$ self-matching of beam by filamentation unavoidable.
  - $\Rightarrow$ some emittance increase is unavoidable.
- Strong voltage gradients at low energy are potentially harmful.
- (De-) bunching by $\phi <> 0$ does not (always) help to stabilize beam.
- It is difficult to predict beam/halo formation without simulations (OPAL).

References

Thank You.

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